Approximation and classification *[recap]*...
Fujisaki's Compactness Lemma and corollaries:
finiteness of class number, Dirichlet units theorem

Classification of completions *(often attributed to Ostrowski) :*
The topologically incomparable (non-discrete) norms on $\mathbb{Q}$ are the usual $\mathbb{R}$ norm and the p-adic $\mathbb{Q}_p$'s.

Corollary: Up to topological equivalence, every norm on a number field is either $\mathfrak{p}$-adic or arises from $\mathbb{R}$ and $\mathbb{C}$.

Additive (Weak) Approximation: *(Artin-Whaples, Lang)* Let $v_1, \ldots, v_n$ index pairwise topologically inequivalent norms on a field k. The diagonal copy of k in $\prod_j k_{v_j}$ is *dense.*

Remark: When the norms are p-adic, arising from prime ideals in a Dedekind ring $\mathfrak{o}$ inside k, this is Sun-Ze's theorem.

The ring of **adeles** $\mathbb{A} = \mathbb{A}_k$ of k is

$$\mathbb{A} \;=\; \mathbb{A}_k \;=\; \mathrm{colim}_S \Big(\prod_{v \in S} k_v \times \prod_{v \notin S} \mathfrak{o}_v \Big)$$

The group of **ideles** $\mathbb{J} = \mathbb{J}_k$ is

$$\mathbb{J} \;=\; \mathbb{J}_k \;=\; \mathrm{colim}_S \Big(\prod_{v \in S} k_v^\times \times \prod_{v \notin S} \mathfrak{o}_v^\times \Big)$$

Claim: Imbedding k diagonally in $\mathbb{A}_k$, by

$$\alpha \longrightarrow (\ldots, \alpha, \ldots) \;\in\; \mathbb{A}_k$$

the image of k is *discrete*, and the quotient $\mathbb{A}/k$ is *compact*. ///

The **idele norm** is

$$|x| \;=\; \prod_{v \leq \infty} |x_v|_v$$

Let $\mathbb{J}^1 = \{x \in \mathbb{J} : |x| = 1\}$. The product formula shows $k^\times \subset \mathbb{J}^1$.

Fujisaki's lemma: $\mathbb{J}^1/k^\times$ is *compact.*

Corollary: The class number of $\mathfrak{o}$ is finite.

Let $k \otimes_{\mathbb{Q}} \mathbb{R} \approx \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. That is, k has r_1 *real* archimedean completions, and r_2 *complex* archimedean completions. These r_1, r_2 are *standard* references.

Corollary: *(Dirichlet's Units Theorem)* The unit group $\mathfrak{o}^\times$, modulo torsion, is a free $\mathbb{Z}$-module of rank $r_1 + r_2 - 1$.

Remark: It is striking that the first two big theorems of general number theory, finiteness of class number, and the Units Theorem, follow from an innocuous compactness assertion.

Also, note the contrast to *additive* approximation, which is essentially a reformulation of elementary things akin to Sun-Ze's theorem, and has no breath-taking corollaries.

Interlude: Pell's equation

Fermat considered the simplest non-trivial case of the Units Theorem, namely, *real quadratic* fields k, with $r_1 = 2$ and $r_2 = 0$. Note

$$N^k_{\mathbb{Q}}(x + y\sqrt{D}) \;=\; x^2 - Dy^2 \qquad\qquad (0 < D \in \mathbb{Z} \text{ squarefree})$$

To solve *Pell's equation* $x^2 - Dy^2 = 1$ with $x, y \in \mathbb{Z}$ is to find units in $\mathbb{Z}[\sqrt{D}]^\times$ with Galois norms 1. These are of index at most 2 in $\mathfrak{o}^\times$.

Multiplicativity of the Galois norm *also* shows that solutions of Pell's equation form a group. This can also be verified directly and cryptically: the secret multiplication

$$(x + y\sqrt{D}) \cdot (z + w\sqrt{D}) \;=\; (xz - Dyw) + (xw + yz)\sqrt{D}$$

suggests showing by elementary algebra that with $x^2 - Dy^2 = 1$ and $z^2 - Dw^2 = 1$,

$$(xz - Dyw)^2 - D(xw + yz)^2 \;=\; \ldots \;=\; 1$$

Rational solutions $x, y \in \mathbb{Q}$ to $x^2 - Dy^2 = 1$ are elementary to find. Namely, because $x^2 - Dy^2 = 1$ is a quadratic curve with at least one rational point $(1, 0)$, the straight line $y = -t(x-1)$ through $(1, 0)$ and $(0, t)$ meets the curve at a *rational* point for rational t: replacing y by $-t(x-1)$ in the quadratic,

$$x^2(1 - Dt^2) + 2Dt^2x - (1 + Dt^2) \;=\; 0$$

By arrangement, $x = 1$ is a solution, and

$$x^2 + \frac{2Dt^2}{1 - Dt^2}x - \frac{1 + Dt^2}{1 - Dt^2} \;=\; (x-1)\big(x - \frac{Dt^2+1}{Dt^2-1}\big)$$

Thus, $x = (Dt^2 + 1)/(Dt^2 - 1)$ and $y = t/(Dt^2 - 1)$ are *rational* solutions to Pell's equation. *Integer* solutions are harder to find.

A sort of *upper* bound on *integer* solutions to $x^2 - Dy^2 = 1$ follows from *topological* considerations: for example, the collection of positive-integer solutions x, y is a free group on either 1 or 0 generators.

Proof: Imbed $\mathbb{Z}[\sqrt{D}] \to \mathbb{R}^2$ by $x + y\sqrt{D} \to (x + y\sqrt{D}, x - y\sqrt{D})$. The image of $\mathfrak{o}$ is discrete. The units $x + y\sqrt{D}$ with $0 < x, y \in \mathbb{Z}$ lie on the hyperbola $u \cdot v = 1$, and are discrete there. Map the first-quadrant piece of that hyperbola to $\mathbb{R}$ by $(u, 1/u) \to \log u$. The units map to a discrete subgroup of $\mathbb{R}$.

The discrete subgroups Γ of $\mathbb{R}$ are the trivial $\{0\}$ and free groups on a single generator. This may be intuitively plausible, but also is readily provable, as follows.

Claim: The discrete subgroups Γ of $\mathbb{R}$ are $\{0\}$ and free groups on a single generator.

Proof: For $\Gamma \neq \{0\}$, since it is closed under additive inverses, it contains *positive* elements. In the case that there is a *least* positive element γ_o, claim that $\Gamma = \mathbb{Z} \cdot gam_o$. Indeed, given $0 < \gamma \in \Gamma$, by the archimedean property of $\mathbb{R}$, there is an integer ℓ such that $\ell \cdot \gamma_o \leq \gamma < (\ell + 1) \cdot \gamma_o$. Either $\gamma = \ell \cdot \gamma_o$ and $\gamma \in \mathbb{Z} \cdot \gamma_o$, or else $0 < \gamma - \ell \cdot \gamma_o < \gamma_o$, contradiction.

Now suppose that there are $\gamma_1 > \gamma_2 > \ldots > 0$ in Γ, and show that $\Gamma = \mathbb{R}$. Since Γ is *closed* (!), the infimum γ_o of the γ_j is in Γ. Since Γ is a group, $0 < \gamma_j - \gamma_o \in \Gamma$. Replacing γ_j by $\gamma_j - \gamma_o$, we can suppose that $\gamma_j \to 0$. The collection of integer multiples of $\gamma_j > 0$ contains elements within distance γ_j of any real number, by the archimedean property of $\mathbb{R}$. Since $\gamma_j \to 0$, every real number is in the closure of Γ. Since Γ is closed (!), $\Gamma = \mathbb{R}$, which is not discrete. ///

There are two classical proof mechanisms for *existence* of solutions to Pell's equation, one by a pigeon-hole principle argument, the other by *continued fractions*. Neither obviously generalizes, although the measure-theory in the proof of Fujisaki's lemma should be construed as a vastly-more-powerful version of a pigeon-hole principle.

The proof of Fujisaki's lemma uses existence and essential uniqueness of *Haar measure* on $\mathbb{A}$, that is, a translation-invariant positive regular Borel measure. In fact, *we will not integrate anything*, but will only use some structural *properties* of Haar measure...

The simplicity and brevity of the proof, and the easy derivation of the two big corollaries, are powerful advertisements for the helpfulness of Haar measure. We discuss Haar measure afterward.