Recap:

Fujisaki's lemma: $\mathbb{J}^1/k^\times$ is *compact.* (via a measure-theory *pigeon-hole* principle)

Corollary: Ideal class groups are finite.

Let $k \otimes_{\mathbb{Q}} \mathbb{R} \approx \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. That is, k has r_1 *real* archimedean completions, and r_2 *complex* archimedean completions. The global degree is the sum of the local degrees: $[k : \mathbb{Q}] = r_1 + 2r_2$.

Corollary: *(Dirichlet's Units Theorem)* The unit group $\mathfrak{o}^\times$, modulo roots of unity, is a free $\mathbb{Z}$-module of rank $r_1 + r_2 - 1$. Generally, S-units $\mathfrak{o}_S^\times$ mod roots of unity are rank $|S| - 1$.

Thm: *(Kronecker)* $\alpha \in \mathfrak{o}$ with $|\alpha|_v = 1$ at all $v|\infty$ is a root of unity.

Now: generalized *ideal* class groups are *idele* class groups, *co-compact* subgroups of $\mathbb{R}^n$, topologies, Haar measures on $\mathbb{A}$ and k_v, ...

Generalized ideal class groups are idele class groups:

The class number above is the *absolute* class number.

The *narrow* class number is ideals modulo principal ideals generated by *totally positive* elements.

For non-zero ideal $\mathfrak{a}$, the *narrow ray class group* mod $\mathfrak{a}$ is fractional ideals *prime to* $\mathfrak{a}$ modulo principal ideals $\alpha\mathfrak{o}$ generated by *totally positive* $\alpha = 1 \bmod \mathfrak{a}$.

Every generalized ideal class group is a quotient of one of these. That is, the narrow ray class groups are *cofinal* in the collection of generalized ideal class groups.

For example, $(\mathbb{Z}/N)^\times$ is the ray class group mod N for $\mathbb{Z}$ and $\mathbb{Q}$.

Lemma: Generalized ideal class groups are *idele* class groups, quotients of the compact group $\mathbb{J}^1/k^\times$ by *open* subgroups.

Corollary: Generalized ideal class groups are *finite.* [*Last time.*]

Proof of Lemma: Let i be the *ideal map* from ideles to non-zero fractional ideals:

$$i(\alpha) \;=\; \prod_{v<\infty} \mathfrak{p}_v^{\mathrm{ord}_v \alpha} \qquad\qquad (\text{for } \alpha \in \mathbb{J})$$

where $\mathfrak{p}_v$ is the prime ideal in $\mathfrak{o}$ attached to the place v. The subgroup that maps to ideals prime to $\mathfrak{a}$ is

$$G_\mathfrak{a} \;=\; \{\alpha \in \mathbb{J} \;:\; \alpha_v \in \mathfrak{o}_v^\times, \text{ for } v|\mathfrak{a}\}$$

With $k^\times$ imbedded diagonally in $\mathbb{J}$, the totally positive $\alpha \in k^\times$ congruent to 1 mod $\mathfrak{a}$ are the intersection of $k^\times$ with

$$U_\mathfrak{a} \;=\; \{\alpha \in \mathbb{J} \;:\; \alpha_v > 0 \text{ at } v \approx \mathbb{R},\; \alpha \in 1 + \mathfrak{a}\mathfrak{o}_v, \text{ for } v|\mathfrak{a}\}$$

The kernel of the ideal map on $\mathbb{J}$ is

$$K = \prod_{v|\infty} k_v^\times \times \prod_{v<\infty} \mathfrak{o}_v^\times \subset G_\mathfrak{a} \subset \mathbb{J}$$

That is, the corresponding generalized ideal class group is immediately rewrite-able as

$$C = i(G_\mathfrak{a})/i\big(U_\mathfrak{a} \cap k^\times\big) \approx G_\mathfrak{a}/\big(K \cdot (U_\mathfrak{a} \cap k^\times)\big)$$

Note that $G_\mathfrak{a} = K \cdot U_\mathfrak{a}$. The explicit claim is that

$$G_\mathfrak{a}/\big(K \cdot (U_\mathfrak{a} \cap k^\times)\big) \approx \mathbb{J}/\big((K \cap U_\mathfrak{a}) \cdot k^\times\big)$$

Subordinate to this: claim that, given an idele x there is $\alpha \in k^\times$ such that $\alpha^{-1} \cdot x$ is totally positive at $v \approx \mathbb{R}$, and $= 1 \bmod \mathfrak{a}\mathfrak{o}_v$ at $v|\mathfrak{a}$. That is, $k^\times \cdot U_\mathfrak{a} = \mathbb{J}$.

Toward the subordinate claim, consider the weaker claim that, given $x \in \mathbb{J}$, there is $\alpha \in k^\times$ with $\alpha^{-1}x \in \mathfrak{o}_v^\times$ for $v|\mathfrak{a}$. To prove this weaker claim, let $\mathfrak{o}_{(\mathfrak{a})}$ be $\mathfrak{o}$ localized at $\mathfrak{a}$: denominators prime to $\mathfrak{a}$ are allowed. This Dedekind domain has finitely-many primes, in bijection with those dividing $\mathfrak{a}$, and is a PID.

Thus, there is $\alpha \in \mathfrak{o}_{(\mathfrak{a})}$ such that $\alpha \cdot \mathfrak{o}_{(\mathfrak{a})} = i(x) \cdot \mathfrak{o}_{(\mathfrak{a})}$. Then $\alpha^{-1}x \in \mathfrak{o}_v^\times$ for all $v|\mathfrak{a}$, proving the weaker subordinate claim.

Sharpening this, Sun-Ze's theorem in $\mathfrak{o}_{(\mathfrak{a})}$ produces $\beta \in k^\times$ such that $\beta = \alpha^{-1}x_v \bmod \mathfrak{a}\mathfrak{o}_v$. Thus, $\beta^{-1}(\alpha^{-1}x) = 1 \bmod \mathfrak{a}\mathfrak{o}_v$ at $v|\mathfrak{a}$.

To prove the subordinate claim, it remains to adjust ideles at $v \approx \mathbb{R}$ without disturbing things at $v|\mathfrak{a}$.

We want $\gamma \in k^\times$ with $\gamma = 1 \bmod \mathfrak{a}\mathfrak{o}_v$ at $v|\mathfrak{a}$, and of specified *sign* at $v \approx \mathbb{R}$.

Recall that $\mathfrak{o}$ and any non-zero $\mathfrak{a}$ are *lattices* in k_∞, that is, $\mathfrak{a}$ is a *discrete* subgroup such that $k_\infty/\mathfrak{a}$ is *compact*. Thus, there is $\gamma \in 1 + \mathfrak{a}$ of specified sign at all $v \approx \mathbb{R}$. Thus, given $\beta^{-1}\alpha^{-1}x$, there exists $\gamma \in 1 + \mathfrak{a}$ such that $\gamma \cdot \beta^{-1}\alpha^{-1}x > 0$ at $v \approx \mathbb{R}$ and $= 1 \bmod \mathfrak{a}\mathfrak{o}_v$ at $v|\mathfrak{a}$. This proves the subordinate claim.

From the subordinate claim, the canonical injection

$$U_\mathfrak{a}/(U_\mathfrak{a} \cap k^\times) \;\approx\; (U_\mathfrak{a} \cdot k^\times)/k^\times \;\longrightarrow\; \mathbb{J}/k^\times$$

is an *isomorphism*. Recalling that $G_\mathfrak{a} = K \cdot U_\mathfrak{a}$, we obtain an isomorphism

$$G_\mathfrak{a}/\big(K \cdot (U_\mathfrak{a} \cap k^\times)\big) \;\approx\; U_\mathfrak{a}/\big((K \cap U_\mathfrak{a}) \cdot (U_\mathfrak{a} \cap k^\times)\big)$$

$$\approx\; (U_\mathfrak{a} \cdot k^\times)/\big((K \cap U_\mathfrak{a}) \cdot k^\times\big) \;\approx\; \mathbb{J}/\big((K \cap U_\mathfrak{a}) \cdot k^\times\big)$$

Thus, generalized ideal class groups are quotients of $\mathbb{J}/k^\times$ by open subgroups, so are finite. ///

Closed subgroups of $\mathbb{R}^n$: The closed topological subgroups H of $V \approx \mathbb{R}^n$ are the following: for a *vector subspace* W of V, and for a *discrete* subgroup Γ of V/W,

$$H \;=\; q^{-1}(\Gamma) \qquad\qquad \text{(with } q : V \to V/W \text{ the quotient map)}$$

The *discrete* subgroups Γ of $V \approx \mathbb{R}^n$ are free $\mathbb{Z}$-modules $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ on $\mathbb{R}$-linearly-independent vectors $v_j \in V$, with $m \le n$.

Proof: Induction on $n = \dim_{\mathbb{R}} V$. We already treated $n = 1$.

When H contains a *line* L, reduce to a lower-dimensional question, as follows. Let $q : V \to V/L$ be the quotient map. Then $H = q^{-1}(q(H))$. With $H' = q(H)$, by induction, there is a vector subspace W' of V/L and discrete subgroup Γ' of $(V/L)/W'$ such that

$$H' \;=\; q'^{-1}(q'(\Gamma')) \qquad\qquad \text{(quotient } q' : V/L \to (V/L)/W'\text{)}$$

Then
$$H \;=\; q^{-1}\Big(q(H)\Big) \;=\; q^{-1}\Big(q'^{-1}(\Gamma')\Big) \;=\; (q'\circ q)^{-1}(\Gamma')$$

The kernel of $q' \circ q$ is the vector subspace $N = q^{-1}(W')$ of V. It is necessary to check that $q(H) = H/N$ is a *closed* subgroup of V/N. It suffices to prove that $q^{-1}(V/N - q(H))$ is *open*. Since H contains N, $q^{-1}\big(q(H)\big) = H$, and

$$q^{-1}(V/N - qH) = V - q^{-1}(qH) \;=\; V - H \;=\; V - (\text{closed}) \;=\; \text{open}$$

This shows that $q(H)$ is closed, and completes the induction step when $\mathbb{R}\cdot h \subset H$.

Next show that H containing *no* lines is *discrete.* If not, then there are distinct h_i in H with an accumulation point h_o. Since H is closed, $h_o \in H$, and replace h_i by $h_i - h_o$ so that, without loss of generality, the accumulation point is 0. Without loss of generality, remove any 0s from the sequence. The sequence $h_i/|h_i|$ has an accumulation point e on the *unit sphere*, since the sphere is *compact.* Replace the sequence by a subsequence so that the $h_i/|h_i|$ *converge* to e. Given real $t \neq 0$, let $n \neq 0$ be an integer so that $|n - \frac{t}{|h_i|}| \leq 1$. Then

$$|n \cdot h_i - te| \;\leq\; \big|(n - \frac{t}{|h_i|})h_i\big| + \big|\frac{th_i}{|h_i|} - te\big| \;\leq\; 1 \cdot |h_i| + |t| \cdot \big|\frac{h_i}{|h_i|} - e\big|$$

Since $|h_i| \to 0$ and $h_i/|h_i| \to e$, this goes to 0. Thus, te is in the closure of $\bigcup_i \mathbb{Z} \cdot h_i$. Thus, H contains the line $\mathbb{R} \cdot e$, *contradiction.* That is, H is discrete.

We claim that *discrete* H is generated as a $\mathbb{Z}$-module by at most n elements, and that these are $\mathbb{R}$-linearly independent. For $h_1, \ldots, h_m$ in H linearly *dependent* over $\mathbb{R}$, there are real numbers r_i so that

$$r_1 h_1 + \ldots + r_m h_m \;=\; 0$$

Re-ordering if necessary, suppose that $r_1 \neq 0$. Given a large integer N, let $a_i^{(N)}$ be integers so that $|r_i - a_i^{(N)}/N| < 1/N$. Then

$$\sum_i a_i^{(N)} h_i \;=\; N \sum_i \Big(\frac{a_i^{(N)}}{N} - r_i\Big) h_i + N \sum_i r_i h_i$$

$$=\; N \sum_i \Big(\frac{a_i^{(N)}}{N} - r_i\Big) h_i + 0$$

Then

$$\Big|\sum_i a_i^{(N)} h_i\Big| \;\leq\; N \sum_i \frac{1}{N}\, |h_i| \;\leq\; \sum_i |h_i|$$

That is, for every N, the $\mathbb{Z}$-linear combination $\sum_i a_i^{(N)} h_i \in H$ is inside the ball of radius $\sum_i |h_i|$ centered at 0. Since H is discrete, there are only finitely-many *different* points of this form. Since $r_1 \neq 0$ and $|Nr_1 - a_1^{(N)}| < 1$, for large varying N the corresponding integers $a_1^{(N)}$ are *distinct*. Thus, for some large $N < N'$,

$$\sum_i a_i^{(N)} h_i \;=\; \sum_i a_i^{(N')} h_i$$

Subtracting,

$$\sum_i \big(a_i^{(N)} - a_i^{(N')}\big) h_i \;=\; 0 \qquad\qquad (\text{with } a_1^{(N)} - a_1^{(N')} \neq 0)$$

This is a non-trivial $\mathbb{Z}$-linear dependence relation among the h_i. Thus, $\mathbb{R}$-linear dependence implies $\mathbb{Z}$-linear dependence of the h_i in a *discrete* subgroup H. ///

Topology on $\mathbb{J}$ versus subspace topology from $\mathbb{A}$:

Claim that the topology on $\mathbb{J}$ is strictly finer than the subspace topology from $\mathbb{J} \subset \mathbb{A}$. In particular, it is obtained from the inclusion

$$\mathbb{J} \subset \mathbb{A} \times \mathbb{A} \qquad \text{by} \qquad \alpha \longrightarrow (\alpha, \alpha^{-1})$$

Proof: The crucial idea is that

$$\prod_{v<\infty} \mathfrak{o}_v \cap \Big(\prod_{v<\infty} \mathfrak{o}_v \Big)^{-1} \;=\; \prod_{v<\infty} \mathfrak{o}_v^\times$$

That is, a typical open in $\mathbb{J}_{\mathrm{fin}}$ is the intersection of a typical open from $\mathbb{A}$ and its image under inversion.

The archimedean and finite-prime components truly are factors in $\mathbb{A} = k_\infty \times \mathbb{A}_{\mathrm{fin}}$ and $\mathbb{J} = k_\infty^\times \times \mathbb{J}_{\mathrm{fin}}$. The topology on $k_\infty^\times$ is *both* the subspace topology from $k_\infty^\times \subset k_\infty$, *and* from $k_\infty^\times \to k_\infty \times k_\infty$ by $\alpha \to (\alpha, \alpha^{-1})$. Thus, it suffices to prove the claim for the finite-prime parts. [*cont'd*]