Interlude/preview:

Fourier analysis on $\mathbb{Q}_p$, $\mathbb{A}$, $\mathbb{A}/k$

In complete parallel to the way Fourier transform on $\mathbb{R}$ and Fourier series on $\mathbb{R}/\mathbb{Z}$ give *Poisson summation*, which gives the *meromorphic continuation* and *functional equation* of the zeta function $\zeta(s)$, ...

... Fourier transform on archimedean *and* non-archimedean completions k_v, and on $\mathbb{A} = \mathbb{A}_k$, give *adelic Poisson summation*, then giving the Iwasawa-Tate modernization of Hecke's treatment of the most general $GL(1)$ L-functions and zeta functions.

It's not just a rewrite: everything becomes an obvious parallel to Riemann's proof. This was a prototype for Gelfand-Piatetski-Shapiro's (1963) and Jacquet-Langlands' (1971) modernization of the classical theory of L-functions attached to $GL(2)$.

Some references:

[Gelfand-Graev-PS 1969] I. Gelfand, M. Graev, I. Piatetski-Shapiro, *Representation Theory and Automorphic Functions*, W.B. Saunders Co., Philadelphia, 1969.

The latter translation from the earlier Russian edition included a then-novel discussion of adeles and of harmonic analysis and representation theory connected to $\mathbb{Q}_p$, in addition to that connected to $\mathbb{R}$.

[Gelfand-PS 1963] I.M. Gelfand, I.I. Piatetski-Shapiro, *Automorphic functions and representation theory*, Trudy Moskov. Obshch. **8** (1963), 389-412 [Trans.: Trans. Moscow Math. Soc. **12** (1963), 438-464.]

The latter was the published research paper. Selberg's 1956 paper (below) was another input to both the book and the paper.

[Iwasawa 1950/52] K. Iwasawa, [brief announcement], in Proceedings of the 1950 International Congress of Mathematicians, Vol. 1, Cambridge, MA, 1950, p. 322, Amer. Math. Soc., Providence, RI, 1952.

The latter briefly announced treatment of zeta and L-functions using the then-new ideas about representations of abelian topological groups.

[Iwasawa 1952/92] K. Iwasawa, *Letter to J. Dieudonné*, dated April 8, 1952, in *Zeta Functions in Geometry*, editors N. Kurokawa and T. Sunada, Advanced Studies in Pure Mathematics **21** (1992), 445-450.

The latter, written at the urging of A. Weil, explained in greater detail Iwasawa's ideas.

[Jacquet-Langlands 1971] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL_2, Lecture Notes in Mathematics **114**, Springer-Verlag, Berlin and New York, 1971.

The latter systematically rewrote the classical $GL(2)$ theory in Iwasawa-Tate style.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87

The latter gave yet-another impetus to the emerging viewpoint that the discussion of zeta functions and L-functions, which had appeared from 1830's through 1930's to be a conglomeration of *ad hoc* manipulations of integrals and sums, instead was a manifestation of far more structure: harmonic analysis, representation theory, and spectral theory of operators.

[Tate 1950/67] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, Ph.D. thesis, Princeton (1950), in *Algebraic Number Theory*, J. Cassels and J. Frölich, editors, Thompson Book Co., 1967.

The latter did not circulate publicly until 1967, although because it was written out in great detail, received much more publicity than Iwasawa's ICM announcement. This has resulted in these ideas often being labelled *Tate's thesis.* It is probably better to refer to these ideas as *Iwasawa-Tate theory.*

[Weil 1940/1965] A. Weil, *L'intégration dans les groupes topologiques, et ses applications*, Hermann, Paris, 1940, second edition 1965.

The latter of course portrays topological groups, especially the representation theory of *abelian* topological groups, in the style of those times, and gives extensive references. The basic duality of *compact* and *discrete* abelian topological groups is *Pontryagin* or *Pontryagin-Weil* duality.

Unitary duals of abelian topological groups: For an abelian topological group G, the unitary dual $G^\vee$ is the collection of continuous group homomorphisms of G to the unit circle in $\mathbb{C}^\times$. For example, $\mathbb{R}^\vee \approx \mathbb{R}$, by $\xi \to (x \to e^{i\xi x})$.

Claim: $\mathbb{Q}_p^\vee \approx \mathbb{Q}_p$ and $\mathbb{A}^\vee \approx \mathbb{A}$.

Since $\mathbb{C}^\times$ contains no *small subgroups* [below], and since $\mathbb{Q}_p$ is a union of *compact* subgroups, every element of $\mathbb{Q}_p^\vee$ has image in roots of unity in $\mathbb{C}^\times$, identified with $\mathbb{Q}/\mathbb{Z}$, so

$$\mathbb{Q}_p^\vee \;\approx\; \mathrm{Hom}^o(\mathbb{Q}_p, \mathbb{Q}/\mathbb{Z}) \qquad\qquad \text{(continuous homomorphisms)}$$

where $\mathbb{Q}/\mathbb{Z} \;=\; \mathrm{colim}\, \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ is *discrete*. As a topological group, $\mathbb{Z}_p \;=\; \lim \mathbb{Z}/p^\ell\mathbb{Z}$, and $\mathbb{Z}_p$ is also a limit of the corresponding quotients of *itself*, namely,

$$\mathbb{Z}_p \;\approx\; \lim \mathbb{Z}_p/p^\ell\mathbb{Z}_p$$

More generally, an abelian *totally disconnected* topological group G is such a limit of quotients:

$$G \approx \lim_K G/K \qquad\qquad (K \text{ compact open subgroup})$$

As a topological group,

$$\mathbb{Q}_p = \bigcup \frac{1}{p^\ell}\mathbb{Z}_p = \operatorname{colim} \frac{1}{p^\ell}\mathbb{Z}_p$$

Because of the *no small subgroups* property of the unit circle in $\mathbb{C}^\times$, every continuous element of $\mathbb{Z}_p^\vee$ factors through some limitand

$$\mathbb{Z}_p/p^\ell\mathbb{Z}_p \approx \mathbb{Z}/p^\ell\mathbb{Z}$$

Thus,

$$\mathbb{Z}_p^\vee \;=\; \text{colim}\Big(\mathbb{Z}_p/p^\ell\mathbb{Z}_p\Big)^\vee \;=\; \text{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$$

since $\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$ is the dual to $\mathbb{Z}_p/p^\ell\mathbb{Z}_p$ under the pairing

$$\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\times\; \mathbb{Z}_p/p^\ell\mathbb{Z}_p \;\approx\; \frac{1}{p^\ell}\mathbb{Z}/\mathbb{Z} \;\times\; \mathbb{Z}/p^\ell\mathbb{Z}$$

by

$$\big(\frac{x}{p^\ell}+\mathbb{Z}\big) \;\times\; \big(y+p^\ell\mathbb{Z}\big) \;\longrightarrow\; xy+\mathbb{Z} \;\in\; \mathbb{Q}/\mathbb{Z}$$

The transition maps in the colimit expression for $\mathbb{Z}_p^\vee$ are inclusions, so

$$\mathbb{Z}_p^\vee \;=\; \text{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\approx\; \Big(\text{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)/\mathbb{Z}_p \;\approx\; \mathbb{Q}_p/\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \Big(\mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \lim \frac{1}{p^\ell}\mathbb{Z}_p^\vee$$

As a topological group, $\frac{1}{p^\ell}\mathbb{Z}_p \;\approx\; \mathbb{Z}_p$ by multiplying by p^ℓ, so the dual of $\frac{1}{p^\ell}\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p^\vee \;\approx\; \mathbb{Q}_p/\mathbb{Z}_p$. However, the inclusions for varying ℓ are not the identity map, so for compatibility take

$$\Big(\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \mathbb{Q}_p/p^\ell\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \lim \mathbb{Q}_p/p^\ell\mathbb{Z}_p \;\approx\; \mathbb{Q}_p$$

because, $\mathbb{Q}_p$ is the projective limit of its quotients by compact open subgroups. ///

Claim: Both $\mathbb{A}^\vee \approx \mathbb{A}$ and $\mathbb{A}_{\text{fin}}^\vee \approx \mathbb{A}_{\text{fin}}$.

Proof: The same argument applies to $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/N\mathbb{Z}$ and finite adeles $\mathbb{A}_{\text{fin}} = \text{colim}\, \frac{1}{N}\widehat{\mathbb{Z}}$, proving the self-duality of $\mathbb{A}_{\text{fin}}$. Then the self-duality of $\mathbb{R}$ gives the self-duality of $\mathbb{A}$. ///

Remark: $\widehat{\mathbb{Z}}$ does also refer to $\text{Hom}^o(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, but needs to be *topologized* by the compact-open topology [later].

Remark: Nearly the same argument applies for an arbitrary finite extension k of $\mathbb{Q}$.

Corollary: Given *non-trivial* $\psi \in \mathbb{Q}_p^\vee$, every other element of $\mathbb{Q}_p^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{Q}_p$. Similarly, given *non-trivial* $\psi \in \mathbb{A}^\vee$, every other element of $\mathbb{A}^\vee$ is of the form $x \to \psi(\xi \cdot x)$ for some $\xi \in \mathbb{A}$. [Proof below]

Remark: This sort of result is already familiar from the analogue for $\mathbb{R}$, that $x \to e^{i\xi x}$ for $\xi \in \mathbb{R}$ are all the unitary characters of $\mathbb{R}$.

Compact-discrete duality

For abelian topological groups G, pointwise multiplication makes $\widehat{G}$ an abelian group. A reasonable topology on $\widehat{G}$ is the *compact-open* topology, with a sub-basis

$$U = U_{C,E} = \{f \in \widehat{G} : f(C) \subset E\}$$

for compact $C \subset G$, open $E \subset S^1$.

Remark: The reasonable-ness of this topology is functional. For a compact topological space X, $C^o(X)$ with the *sup-norm* is a *Banach space* . On non-compact X, the semi-norms given by *sups on compacts* make $C^o(X)$ a *Fréchet space*. The compact-open topology is the analogue for $C^o(X, Y)$ where the target Y is not normed. When X, Y are topological groups, the continuous functions $f : X \to Y$ consisting of *group homomorphisms* is a (locally compact, Hausdorff) topological group. [Later]

Granting (for now) that the compact-open topology makes $\widehat{G}$ an abelian (locally-compact, Hausdorf) topological group,

Theorem: The unitary dual of a *compact* abelian group is *discrete*. The unitary dual of a *discrete* abelian group is *compact*.

Proof: Let G be compact. Let E be a small-enough open in S^1 so that E contains no non-trivial subgroups of G. Using the compactness of G itself, let $U \subset \widehat{G}$ be the open

$$U = \{f \in \widehat{G} : f(G) \subset E\}$$

Since E is *small*, $f(G) = \{1\}$. That is, f is the trivial homomorphism. This proves discreteness of $\widehat{G}$ for compact G.

For G discrete, *every* group homomorphism to S^1 is continuous. The space of *all* functions $G \to S^1$ is the cartesian product of copies of S^1 indexed by G. By Tychonoff's theorem, this product is *compact*. For *discrete* X, the compact-open topology on the space $C^o(X,Y)$ of continuous functions from $X \to Y$ *is* the product topology on copies of Y indexed by X.

The set of functions f satisfying the group homomorphism condition

$$f(gh) \;=\; f(g)\cdot f(h) \qquad\qquad (\text{for } g,h\in G)$$

is *closed*, since the group multiplication $f(g)\times f(h)\to f(g)\cdot f(h)$ in S^1 is continuous. Since the product is also *Hausdorff*, $\widehat{G}$ is also compact. ///

Theorem: $(\mathbb{A}/k)\widehat{\ } \approx k$ In particular, given any non-trivial character ψ on $\mathbb{A}/k$, *all* characters on $\mathbb{A}/k$ are of the form $x \to \psi(\alpha \cdot x)$ for some $\alpha \in k$.

Proof: For a (discretely topologized) number field k with adeles $\mathbb{A}$, $\mathbb{A}/k$ is *compact*, and $\mathbb{A}$ is *self-dual*.

Because $\mathbb{A}/k$ is compact, $(\mathbb{A}/k)\widehat{\ }$ is *discrete*. Since multiplication by elements of k respects cosets $x + k$ in $\mathbb{A}/k$, the unitary dual has a k-vectorspace structure given by

$$(\alpha \cdot \psi)(x) = \psi(\alpha \cdot x) \qquad \text{(for } \alpha \in k,\ x \in \mathbb{A}/k\text{)}$$

There is no topological issue in this k-vectorspace structure, because $(\mathbb{A}/k)\widehat{\ }$ is discrete. The quotient map $\mathbb{A} \to \mathbb{A}/k$ gives a natural *injection* $(\mathbb{A}/k)\widehat{\ } \to \widehat{\mathbb{A}}$.

Given non-trivial $\psi \in (\mathbb{A}/k)^\frown$, the k-vectorspace $k \cdot \psi$ inside $(\mathbb{A}/k)^\frown$ injects to a copy of $k \cdot \psi$ inside $\widehat{\mathbb{A}} \approx \mathbb{A}$. *Assuming* for a moment that the image in $\mathbb{A}$ is essentially the same as the diagonal copy of k, $(\mathbb{A}/k)^\frown/k$ injects to $\mathbb{A}/k$. The topology of $(\mathbb{A}/k)^\frown$ is discrete, and the quotient $(\mathbb{A}/k)^\frown/k$ is still discrete. These maps are continuous group homs, so the image of $(\mathbb{A}/k)^\frown/k$ in $\mathbb{A}/k$ is a discrete subgroup of a compact group, so is *finite*. Since $(\mathbb{A}/k)^\frown$ is a k-vectorspace, $(\mathbb{A}/k)^\frown/k$ is a singleton. Thus, $(\mathbb{A}/k)^\frown \approx k$, if the image of $k \cdot \psi$ in $\mathbb{A} \approx \widehat{\mathbb{A}}$ is the usual diagonal copy.

To see how $k \cdot \psi$ is imbedded in $\mathbb{A} \approx \widehat{\mathbb{A}}$, fix non-trivial ψ on $\mathbb{A}/k$, and let ψ be the induced character on $\mathbb{A}$. The self-duality of $\mathbb{A}$ is that the action of $\mathbb{A}$ on $\widehat{\mathbb{A}}$ by $(x \cdot \psi)(y) = \psi(xy)$ gives an *isomorphism*. The subgroup $x \cdot \psi$ with $x \in k$ is certainly the usual diagonal copy. ///

Appendix: no small subgroups:

The circle group S^1 *has no small subgroups*, in the sense that there is a neighborhood U of the identity $1 \in S^1$ such that the only subgroup of S^1 inside U is the trivial group $\{1\}$.

Essentially the same proof works for *real Lie groups*.

Use the copy of S^1 inside the complex plane. We claim that taking

$$U = S^1 \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

suffices: the only subgroup G of S^1 inside this U is $G = \{1\}$. Indeed, suppose not. Let $1 \neq e^{i\theta} \in G \cap U$. We can take $0 < \theta < \pi/2$, since both $\pm\theta$ must appear. Let $0 < \ell \in \mathbb{Z}$ be the smallest such that $\ell \cdot \theta > \pi/2$. Then, since $(\ell - 1) \cdot \theta < \pi/2$ and $0 < \theta < \pi/2$,

$$\frac{\pi}{2} < \ell \cdot \theta = (\ell - 1) \cdot \theta + \theta < \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Thus, $\ell \cdot \theta$ falls outside U, contradiction. ///