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Moments for L-functions for $GL_r \times GL_{r-1}$

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Introduction

We exhibit elementary kernels $\mathfrak{P}$ which produce sums of integral moments for cuspforms f on GL_r by

$$\int_{Z_{\mathbb{A}} GL_r(k)\backslash GL_r(\mathbb{A})} \mathfrak{P}\cdot|f|^2 = \sum_{F \text{ on } GL_{r-1}} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{|L(s, f\otimes F)|^2}{\langle F,F\rangle} M(s)\,ds \;+\; (\text{continuous part})$$

over number fields k, with certain weights $M(s)$. Here F runs over an orthogonal basis for cuspforms on GL_{r-1}. There are further continuous-spectrum terms analogous to the discrete-spectrum sum over cuspforms. The kernel (Poincaré series) $\mathfrak{P}$ admits a spectral decomposition, surprisingly consisting of only three parts: a leading term, a sum arising from cuspforms on GL_2, and a continuous part from GL_2. That is, no cuspforms on GL_ℓ with $2 < \ell \le r$ contribute. This spectral decomposition makes possible the meromorphic continuation of $\mathfrak{P}$ in auxiliary parameters.

Moments of level-one holomorphic elliptic modular forms were treated in [Good 1983] and [Good 1986], the latter using an idea that is a precursor of part of the present approach. Level-one waveforms over $\mathbb{Q}$ appear in [Diaconu-Goldfeld 2006a], over $\mathbb{Q}(i)$ in [Diaconu-Goldfeld 2006b]. Arbitrary level, groundfield, and ∞-type for GL_2 are in [Diaconu-Garrett 2009a] and [Diaconu-Garrett 2009b].

We do have in mind application not only to cuspforms, but also to truncated Eisenstein series (with cuspidal data) or wave packets of Eisenstein series, giving a non-trivial application of harmonic analysis on larger groups GL_r to L-functions attached to smaller groups, for example, on GL_1, giving high integral moments of $\zeta_k(s)$.

For context, we review the [Diaconu-Goldfeld 2006a] treatment of spherical waveforms f for $GL_2(\mathbb{Q})$. In that case, the sum of moments is a single term

$$\int_{Z_{\mathbb{A}} GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})} \mathfrak{P}(g)\,|f(g)|^2\,dg = \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} L(s'+s, f)\cdot \overline{L}(s,f)\cdot\Gamma(s,s',s'',f_\infty)\,ds$$

where $\Gamma(s,s',s'',f_\infty)$ is a ratios of products of gammas, with arguments depending upon the archimedean data attached to f. Here the Poincaré series $\mathfrak{P}(g) = \mathfrak{P}(g,s',s'')$ has a *spectral expansion*

$$\mathfrak{P}(s',s'') \;=\; \frac{\pi^{\frac{1-s''}{2}}\Gamma(\frac{s''-1}{2})}{\pi^{-\frac{s''}{2}}\Gamma(\frac{s''}{2})}\cdot E_{1+s'} \;+\; \frac{1}{2}\sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2}+s',\overline{F})}{\langle F,F\rangle}\cdot\mathcal{G}(\tfrac{1}{2}-it_F,s',s'')\cdot F$$

$$+\frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\zeta(s'+s)\,\zeta(s'+1-s)}{\xi(2-2s)}\,\mathcal{G}(1-s,s',s'')\cdot E_s\,ds \qquad (\text{for } \mathrm{Re}\,(s')\gg\tfrac{1}{2},\ \mathrm{Re}\,(s'')\gg 0)$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\mathcal{G}$ is essentially a product of gamma function values

$$\mathcal{G}(s,s',s'') = \pi^{-(s'+\frac{s''}{2})}\frac{\Gamma(\frac{s'+1-s}{2})\,\Gamma(\frac{s'+s}{2})\,\Gamma(\frac{s'-s+s''}{2})\,\Gamma(\frac{s'+s-1+s''}{2})}{\Gamma(s'+\frac{s''}{2})}$$

and F is summed over (an orthogonal basis for) spherical (at finite primes) cuspforms on GL_2 with Laplacian eigenvalues $\frac{1}{4}+t_F^2$, and E_s is the usual spherical Eisenstein series

$$E_s \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = |y|^s + \frac{\xi(2-2s)}{\xi(2s)}\,|y|^{1-s} + \ldots$$

It is not obvious, but the continuous part (the *integral* of Eisenstein series) cancels the pole at $s'=1$ of the leading term, and when evaluated at $s'=0$ is

$$\mathfrak{P}(g,0,s'') \;=\; (\text{holomorphic at } s'\!=\!0) \;+\; \frac{1}{2}\sum_{F \text{ on } GL_2} \frac{L(\frac{1}{2},\overline{F})}{\langle F,F\rangle}\cdot \mathcal{G}(\tfrac{1}{2}-it_F,0,s'')\cdot F$$

$$+\frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\zeta(s)\,\zeta(1-s)}{\xi(2-2s)}\,\mathcal{G}(1-s,0,s'')\cdot E_s\,ds$$

In this spectral expansion, the coefficient in front of a cuspform F includes $\mathcal{G}$ evaluated at $s'=0$ and $s=\frac{1}{2}\pm it_F$, namely

$$\mathcal{G}(\tfrac{1}{2}-it_F,0,s'') = \pi^{-\frac{s''}{2}}\,\frac{\Gamma(\frac{\frac{1}{2}-it_F}{2})\,\Gamma(\frac{\frac{1}{2}+it_F}{2})\,\Gamma(\frac{s''-\frac{1}{2}-it_F}{2})\,\Gamma(\frac{s''-\frac{1}{2}+it_F}{2})}{\Gamma(\frac{s''}{2})}$$

The gamma function has poles at $0,-1,-2,\ldots$, so this coefficient has poles at $s''=\frac{1}{2}\pm it_F$, $-\frac{3}{2}\pm it_F$, $\ldots$. Over $\mathbb{Q}$, among spherical cuspforms (or for any fixed level) these values have no accumulation point. The continuous part of the spectral side at $s'=0$ is

$$\frac{1}{4\pi i}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{\xi(s)\,\xi(1-s)}{\xi(2-2s)}\;\frac{\Gamma(\frac{s''-s}{2})\,\Gamma(\frac{s''-1+s}{2})}{\Gamma(\frac{s''}{2})}\cdot E_s\,ds$$

with gamma factors grouped with corresponding zeta functions, to form the completed L-functions ξ. Thus, the evident pole of the leading term at $s''=1$ can be exploited, using the continuation to $\mathrm{Re}\,(s'')>1/2$.

Further, a subtle contour-shifting argument shows that the continuous part of this spectral decomposition has a meromorphic continuation to $\mathbb{C}$ with poles at $\rho/2$ for zeros ρ of ζ, in addition to the poles from the gamma functions.

Already for GL_2, over general groundfields k, infinitely many Hecke characters enter both the spectral decomposition of the Poincaré series and the moment expression. This naturally complicates isolation of literal moments, and complicates analysis of poles via the spectral expansion. Suppressing constants, the moment expansion is a sum of twists by χ's

$$\int_{Z_{\mathbb{A}}GL_2(k)\backslash GL_2(\mathbb{A})} \mathfrak{P}\cdot|f|^2 = \sum_{\chi}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} L(s'+s,f\otimes\chi)\cdot L(1-s,\overline{f}\otimes\overline{\chi})\cdot M_\chi(s)\,ds$$

And, suppressing constants, the spectral expansion is

$$\mathfrak{P} \;=\; (\infty-\text{part})\cdot E_{1+s'} \;+\; \sum_{F \text{ on } GL_2} (\infty-\text{part})\cdot \frac{L(\frac{1}{2}+s',\overline{F})}{\langle F,F\rangle}\cdot F$$

$$+\;\sum_{\chi}\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{L(s'+s,\overline{\chi})\,L(s'+1-s,\chi)}{L(2-2s,\overline{\chi}^2)}\,\mathcal{G}_\chi(s)\cdot E_{s,\chi}\,ds$$

In the simplest case beyond GL_2, take f a spherical cuspform on GL_3 over $\mathbb{Q}$. We construct a weight function $\Gamma(s,s',s'',f_\infty,F_\infty)$ depending upon complex parameters s, s', and s'', *and* upon the *archimedean* data for both f and cuspforms F on GL_2, such that $\Gamma(s, s', s'', f_\infty, F_\infty)$ has explicit asymptotic behavior, and such that the *moment expansion* is

$$\int_{Z_{\mathbb{A}} GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \mathfrak{P}(s', s'') \cdot |f|^2 \, dg = \sum_{F \text{ on } GL_2} \frac{1}{2\pi i} \int_{\mathrm{Re}\,(s)=\frac{1}{2}} \frac{|L(s, f \otimes F)|^2}{\langle F, F \rangle} \cdot \Gamma(s, 0, s'', f_\infty, F_\infty) \, ds$$

$$+ \frac{1}{4\pi i} \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \int_{\mathrm{Re}\,(s_1)=\frac{1}{2}} \int_{\mathrm{Re}\,(s_2)=\frac{1}{2}} \frac{|L(s_1, f \otimes E^{(k)}_{1-s_2})|^2}{|\xi(1-2it_2)|^2} \cdot \Gamma(s_1, 0, s'', f_\infty, E^{(k)}_{1-s_2,\infty}) \, ds_1 \, ds_2$$

where F runs over (an orthogonal basis for) all level-one cuspforms on GL_2, with *no* restriction on the right K_∞-type, and $E^{(k)}_s$ is the usual level-one Eisenstein series of K_∞-type k. Here and throughout, for $\mathrm{Re}\,(s) = 1/2$, write $1 - s$ in place of $\overline{s}$, to maintain holomorphy in complex-conjugated parameters. In this vein, over $\mathbb{Q}$, it is reasonable to put

$$L(s_1, f \otimes \overline{E}^{(k)}_{s_2}) = L(s_1, f \otimes E^{(k)}_{1-s_2}) = \frac{L(s_1 + \frac{1}{2} - s_2, f) \cdot L(s_1 - \frac{1}{2} + s_2, f)}{\zeta(2-2s_2)} \qquad \text{(finite-prime parts only)}$$

since the natural normalization of the Eisenstein series $E^{(k)}_{s_2}$ on GL_2 contributes the denominator $\zeta(2-2s_2)$. Meromorphic continuation in s' and evaluation at $s' = 0$ gives the desired specialization of the *moment expansion*. There is also a meromorphic continuation in the parameter s'' in the archimedean data.

More generally, for a cuspform f on GL_r with $r \geq 3$, whether over $\mathbb{Q}$ or over a numberfield, the *moment expansion* includes an infinite sum of $|L(s, f \otimes F)|^2 / \langle F, F \rangle$ over an orthogonal basis for cuspforms F on GL_{r-1}, as well as *integrals* of products of L-functions $L(s, f \otimes F)$ for F ranging over cuspforms on $GL_{r_1} \times \ldots \times GL_{r_\ell}$ for all partitions $(r_1, \ldots, r_\ell)$ of r. Correspondingly, the natural normalization of the cuspidal-data Eisenstein series gives products of convolution L-functions $L(*, F_i \otimes F_j)$ in the denominators of these terms, as well as factors $\langle F_i, F_i \rangle^{1/2} \cdot \langle F_j, F_j \rangle^{1/2}$.

Generally, the spectral expansion for GL_r is an induced-up version of that for GL_2. Suppressing constants, using groundfield $\mathbb{Q}$ to skirt Hecke characters,

$$\mathfrak{P} \quad = \quad (\infty - \text{part}) \cdot E^{r-1,1}_{s'+1} \quad + \sum_{F \text{ on } GL_2} (\infty - \text{part}) \cdot \frac{L(\frac{rs'+r-2}{2} + \frac{1}{2}, \overline{F})}{\langle F, F \rangle} \cdot E^{r-2,2}_{\frac{s'+1}{2}, F}$$

$$+ \quad \int_{\mathrm{Re}\,(s)=\frac{1}{2}} (\infty - \text{part}) \cdot \frac{\zeta(\frac{rs'+r-2}{2} + \frac{1}{2} - s) \cdot \zeta(\frac{rs'+r-2}{2} + \frac{1}{2} + s)}{\zeta(2-2s)} \quad \cdot \quad E^{r-2,1,1}_{s'+1,\, s - \frac{s'+1}{2},\, -s - \frac{s'+1}{2}} \, ds$$

where the Eisenstein series are normalized naively. The continuous part has a pole that cancels the pole of the leading term at $s' = 0$.

Again over $\mathbb{Q}$, the *most-continuous* part of the moment expansion for GL_r is of the form

$$\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \int_{t \in \Lambda} |L(s, f \otimes E^{\min}_{\frac{1}{2}+it})|^2 \, M_t(s) \, ds \, dt = \int \int_{\Lambda} \left| \frac{\Pi_{1 \leq \ell \leq r-1} \, L(s + it_\ell, f)}{\Pi_{1 \leq j < \ell < n} \, \zeta(1 - it_j + it_\ell)} \right|^2 M_t(s) \, ds \, dt$$

where

$$\Lambda = \{t \in \mathbb{R}^{r-1} \; : \; t_1 + \ldots + t_{r-1} = 0\}$$

and where M is a weight function depending upon f and F. More generally, let $r - 1 = m \cdot b$. For F on GL_m, let

$$F^\Delta = F \otimes \ldots \otimes F$$

on $GL_m \times \ldots \times GL_m$. Inside the moment expansion we have (recall Langlands-Shahidi)

$$\int_{\mathrm{Re}\,(s)=\frac{1}{2}} \int_{\Lambda} |L(s, f \otimes E_{F^\Delta, \frac{1}{2}+it})|^2 \, M_{F,t}(s) \, ds \, dt = \int \int \left| \frac{\Pi_{1 \leq \ell \leq b} \, L(s + it_\ell, f \otimes F)}{\Pi_{1 \leq j < \ell \leq b} \, L(1 - it_j + it_\ell, F \otimes F^\vee)} \right|^2 M \, ds \, dt$$

If we replace the cuspform f on $GL_r(\mathbb{Q})$ by a (truncated) minimal-parabolic Eisenstein series E_α with $\alpha \in \mathbb{C}^{n-1}$, the most-continuous part of the moment expansion contains a term

$$\int \int_\Lambda \left| \frac{\Pi_{1\le\mu\le n,\ 1\le\ell\le r-1}\ \zeta(\alpha_\mu + s + it_\ell)}{\Pi_{1\le j<\ell<r}\ |\zeta(1 - it_j + it_\ell)} \right|^2 ds\, dt$$

Taking $\alpha = 0 \in \mathbb{C}^{r-1}$ gives

$$\int \int_\Lambda \left| \frac{\Pi_{1\le\ell\le r-1}\ \ \zeta(s + it_\ell)^r}{\Pi_{1\le j<\ell<r}\ \zeta(1 - it_j + it_\ell)} \right|^2 M\, ds\, dt$$

For example, for GL_3, where $\Lambda = \{(t, -t)\} \approx \mathbb{R}$,

$$\int \int_{\mathbb{R}} \left| \frac{\zeta(s + it)^3 \cdot \zeta(s - it)^3}{\zeta(1 - 2it)} \right|^2 M\ ds\, dt$$

and for GL_4

$$\int_{(s)} \int_\Lambda \left| \frac{\zeta(s + it_1)^4 \cdot \zeta(s + it_2)^4 \cdot \zeta(s + it_3)^4}{\zeta(1{-}it_1{+}it_2)\, \zeta(1{-}it_1{+}it_3)\, \zeta(1{-}it_2{+}it_3)} \right|^2 M\, ds\, dt$$

1. *The moment expansion*

Let $G = GL_r$ over a number field k. Let P be the standard maximal proper parabolic

$$P = P^{r-1,1} = \{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & * \\ 0 & 1\text{-by-}1 \end{pmatrix} \}$$

Let

$$U = \{ \begin{pmatrix} 1_{r-1} & * \\ 0 & 1 \end{pmatrix} \} \qquad H = \{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & 0 \\ 0 & 1 \end{pmatrix} \}$$

and

$$N = \{\text{upper triangular unipotent elements in } H\}$$

$$= (\text{unipotent radical of standard minimal parabolic in } H)$$

Let Z be the center of G. Let K_v be the standard maximal compact in the k_v-valued points G_v of G. Thus, for $v < \infty$, $K_v = GL_r(\mathfrak{o}_v)$. For $v \approx \mathbb{R}$, take $K_v = O_r(\mathbb{R})$. For $v \approx \mathbb{C}$ take $K_v = U(r)$.

The standard choice of non-degenerate character on $N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}$ is

$$\psi(n \cdot u) = \psi_0(n_{12} + n_{23} + \ldots + n_{r-2,r-1}) \cdot \psi_0(u_{r-1,r})$$

where ψ_0 is a fixed non-trivial character on $\mathbb{A}/k$. A cuspform f has a Fourier expansion along NU

$$f(g) = \sum_{\xi \in N_k \backslash H_k} W_f(\xi g) \quad \text{where} \quad W_f(g) = \int_{N_k U_k \backslash N_{\mathbb{A}} U_{\mathbb{A}}} \overline{\psi}(nu)\, f(nug)\, dn\, du$$

The (Whittaker) function $W_f(g)$ factors over primes.

Poincaré series: For $s' \in \mathbb{C}$, let

$$\varphi = \bigotimes_v \varphi_v$$

where for v finite

$$\varphi_v(g) = \begin{cases} \left|(\det A)/d^{r-1}\right|_v^{s'} & \text{(for } g = mk \text{ with } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \text{ in } Z_v H_v \text{ and } k \in K_v) \\ 0 & \text{(otherwise)} \end{cases}$$

and for v archimedean require right K_v-invariance and left equivariance

$$\varphi_v(mg) = \left|\frac{\det A}{d^{r-1}}\right|_v^{s'} \cdot \varphi_v(g) \qquad \text{(for } g \in G_v, \text{ for } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_v H_v)$$

Thus, for $v|\infty$, the further data determining φ_v consists of its values on U_v. The simplest useful choice is

$$\varphi_v \begin{pmatrix} 1_{r-1} & x \\ 0 & 1 \end{pmatrix} = (1 + |x_1|^2 + \ldots + |x_{r-1}|^2)^{-s''/2} \qquad \text{(where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix}, \text{ and } s'' \in \mathbb{C})$$

and where the norm $|x_1|^2 + \ldots + |x_{r-1}|^2$ is normalized to be invariant under K_v. Thus, φ is left $Z_{\mathbb{A}} H_k$-invariant. We attach to φ a **Poincaré series**

$$\mathfrak{P}(g) = \sum_{\gamma \in Z_k H_k \backslash G_k} \varphi(\gamma g)$$

Two unwindings: Integrate the norm-squared $|f|^2$ of a cuspform f against $\mathfrak{P}$. The typical first unwinding is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2\, dg = \int_{Z_{\mathbb{A}} H_k \backslash G_{\mathbb{A}}} \varphi(g)\, |f(g)|^2\, dg$$

Next, express f in its Fourier-Whittaker expansion, and unwind further:

$$\int_{Z_{\mathbb{A}} H_k \backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi \in N_k \backslash H_k} W_f(\xi g)\, \overline{f}(g)\, dg = \int_{Z_{\mathbb{A}} N_k \backslash G_{\mathbb{A}}} \varphi(g)\, W_f(g)\, \overline{f}(g)\, dg$$

Iwasawa decomposition, simplification of integral: Suppose for simplicity that f is right $K_{\mathbb{A}}$-invariant, so we can use an Iwasawa decomposition $G = (HZ)UK$ (everywhere locally) to rewrite the whole integral as

$$\int_{N_k \backslash H_{\mathbb{A}} \times U_{\mathbb{A}}} \varphi(hu)\, W_f(hu)\, \overline{f}(hu)\, dh\, du$$

Spectral decomposition on GL_{r-1}: Use a spectral decomposition for $F \in L^2(H_k \backslash H_{\mathbb{A}})$, inexplicitly

$$F = \int_{(\eta)} \langle F, \eta \rangle \cdot \eta\, d\eta$$

where each η generates an irreducible representation of $H_{\mathbb{A}}$.

Expand $\overline{f}(hu)$: Since $\overline{f}$ is left H_k-invariant, it decomposes along $H_k \backslash H_{\mathbb{A}}$ as

$$\overline{f}(hu) = \int_{(\eta)} \eta(h) \int_{H_k \backslash H_{\mathbb{A}}} \overline{\eta}(m)\, \overline{f}(mu)\, dm\, d\eta$$

Unwind the Fourier-Whittaker expansion of $\overline{f}$

$$\overline{f}(hu) = \int_{(\eta)} \eta(h) \int_{H_k \backslash H_{\mathbb{A}}} \overline{\eta}(m) \sum_{\xi \in N_k \backslash H_k} \overline{W}_f(\xi m u)\, dm\, d\eta$$

$$= \int_{(\eta)} \eta(h) \int_{N_k\backslash H_{\mathbb{A}}} \overline{\eta}(m)\, \overline{W}_f(mu)\, dm\, d\eta$$

Then the whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2\, dg$$

$$= \int_{(\eta)} \int_{N_k\backslash H_{\mathbb{A}} \times U_{\mathbb{A}}} \varphi(hu)\, \eta(h)\, W_f(hu) \int_{N_k\backslash H_{\mathbb{A}}} \overline{W}_f(mu)\, \overline{\eta}(m)\, dm\, dh\, du\, d\eta$$

Decoupling at finite primes: The part of the integrand that depends upon $u \in U$ is

$$\int_{U_{\mathbb{A}}} \varphi(hu)\, W_f(hu)\, \overline{W}_f(mu)\, du = \varphi(h)\, W_f(h)\, \overline{W}_f(m) \cdot \int_{U_{\mathbb{A}}} \varphi(u)\, \psi(huh^{-1})\, \overline{\psi}(mum^{-1})\, du$$

The latter integrand visibly factors over primes.

1.1 Lemma: Let v be a finite prime. For $h, m \in H_v$ such that $W_{f,v}(h) \neq 0$ and $\overline{W}_{f,v}(m) \neq 0$,

$$\int_{U_v} \varphi_v(h)\, \psi_v(huh^{-1})\, \overline{\psi}_v(mum^{-1})\, du = \int_{U_v \cap K_v} 1\, du$$

Proof: At a finite place v, $\varphi_v(u) \neq 0$ if and only if $u \in U_v \cap K_v$, and for such u

$$\psi_v(huh^{-1}) \cdot W_{f,v}(h) = W_{f,v}(huh^{-1} \cdot h) = W_{f,v}(hu) = W_{f,v}(h) \cdot 1$$

by the right K_v-invariance. Thus, for $W_{f,v}(h) \neq 0$, $\psi_v(huh^{-1}) = 1$, and similarly for $\psi_v(mum^{-1})$. Thus, the finite-prime part of the integral over U_v is just the integral of 1 over $U_v \cap K_v$, as indicated. ///

Archimedean kernel: The archimedean part of the integral does not necessarily decouple. Thus, with subscripts ∞ denoting the infinite-adele part of various objects, for $h, m \in H_\infty$, define

$$\mathcal{K}(h, m) = \int_{U_\infty} \varphi_\infty(u)\, \psi_\infty(huh^{-1})\, \overline{\psi}_\infty(mum^{-1})\, du$$

The whole integral is

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mathfrak{P}(g)\, |f(g)|^2\, dg = \int_{(\eta)} \int_{N_k\backslash H_{\mathbb{A}}} \int_{N_k\backslash H_{\mathbb{A}}} \mathcal{K}(h, m)\, \varphi(h) \Big(W_f(h)\, \eta(h)\Big) \Big(\overline{W}_f(m)\, \overline{\eta}(m)\Big)\, dm\, dh\, d\eta$$

Fourier expansion of η: Normalize the volume of $N_k\backslash N_{\mathbb{A}}$ to 1. Thus, for a left N_k-invariant function F on $H_{\mathbb{A}}$

$$\int_{N_k\backslash H_{\mathbb{A}}} F(h)\, dh = \int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \int_{N_k\backslash N_{\mathbb{A}}} F(nh)\, dn\, dh$$

Using the left $N_{\mathbb{A}}$-equivariance of W by ψ, and the left $N_{\mathbb{A}}$-invariance of φ,

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(nh)\, \eta(nh)\, W_f(nh)\, dn = \varphi(h)\, W_f(h) \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\, \eta(nh)\, dn = \varphi(h)\, W_f(h)\, W_\eta(h)$$

where

$$W_\eta(h) = \int_{N_k\backslash N_{\mathbb{A}}} \psi(n)\, \eta(nh)\, dn$$

(The integral is not against $\overline{\psi}(n)$, but $\psi(n)$.) That is, the integral over $N_k\backslash H_{\mathbb{A}}$ is equal to an integral against (up to an alteration of the character) the Whittaker function W_η of η, which factors over primes. The whole integral is

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \mathfrak{P}(g)\,|f(g)|^2\,dg = \int_{(\eta)}\int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}}\int_{N_{\mathbb{A}}\backslash H_{\mathbb{A}}} \mathcal{K}(h,m)\,\varphi(h)\,\Big(W_f(h)\,W_\eta(h)\Big)\Big(\overline{W}_f(m)\,\overline{W}_\eta(m)\Big)\,dm\,dh\,d\eta$$

And the η^{th} part is a product of two Euler products. It is evident that for f right K_{fin}-invariant only right $(K_{\text{fin}} \cap H_{\text{fin}})$-invariant η's will appear, due to the decoupling. However, at archimedean places v right K_v-invariance of f does *not* allows us to restrict our attention to right $(K_v \cap H_v)$-invariant η.

Appearance of the parameter s: In fact, as usual,

$$H_k\backslash H_{\mathbb{A}} \approx GL_{r-1}(k)\backslash GL_{r-1}(\mathbb{A}) \approx \mathbb{R}^+ \times H_k\backslash H^1$$

where $\mathbb{R}^+$ is positive real numbers, and

$$H^1 = \{\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} \;:\; a \in GL_{r-1}(\mathbb{A}),\ |\det a| = 1\}$$

The quotient $H_k\backslash H^1$ has finite volume. Thus, the spectral decomposition uses functions

$$\eta\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} \;=\; |\det a|^s \cdot F(a) \quad \text{with} \quad F \in L^2(H_k\backslash H^1),\ \ s \in i\mathbb{R}$$

The real part of the parameter s will necessarily be shifted in the subsequent discussion. Thus, the functions η above are of the form $|\det|^s \otimes F$, and the Whittaker function W_η of $\eta = |\det|^s \otimes F$ is

$$W_\eta\begin{pmatrix} a & \\ & 1\end{pmatrix} = |\det a|^s \cdot W_F(a)$$

where W_F is the Whittaker function of F, normalized here by

$$W_F(g) = \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\,F(ng)\,dn$$

where N is the unipotent radical of the standard minimal parabolic in GL_{r-1}.

Non-archimedean local factors: In terms of s and F, the non-archimedean local factors are

$$\int_{N_v\backslash H_v} |\det a|^{s+s'}\,W_{f,v}\begin{pmatrix} a & \\ & 1\end{pmatrix} W_{F,v}(a)\,da = \frac{L_v(s+s'+\frac{1}{2}, f\otimes F)}{\langle F,F\rangle^{1/2}} \qquad (\text{for } \mathrm{Re}\,(s+s') \gg 0)$$

The second Euler product is the complex conjugate of this, but lacking the shift by s', namely, the *complex conjugate* of

$$\int_{N_v\backslash H_v} |\det a'|^{s}\,W_{f,v}\begin{pmatrix} a' & \\ & 1\end{pmatrix} W_{F,v}(a')\,da' = \frac{L_v(s+\frac{1}{2}, f\otimes F)}{\langle F,F\rangle^{1/2}} \qquad (\text{for } \mathrm{Re}\,(s+s') \gg 0 \text{ and } \mathrm{Re}\,(s) \gg 0)$$

When $\eta = |\det|^s \otimes F$ is not cuspidal, but, instead, is an Eisenstein series with cuspidal data, it still does generate an irreducible representation of $G_{\mathbb{A}}$. At a place v where η generates a spherical representation, the Euler product expansion of degree $r\cdot(r-1)$ falls apart into smaller factors, *and* has a *denominator* arising from the (natural) normalization of the cuspidal-data Eisenstein series entering. Discussion of these terms and their normalizations is postponed.

Replace $\overline{s}$ by $1-s$ on $\mathrm{Re}(s)=1/2$*:* The global integrals for the L-functions $L(s'+s+\frac{1}{2}, f\otimes F)$ and $L(s+\frac{1}{2}, f\otimes F)$ only converge for $\mathrm{Re}(s'+s)\gg 0$ and $\mathrm{Re}(s)\gg 0$, so we will need to meromorphically continue. To this end, it is most convenient for the whole integral to be *holomorphic* in s, rather than having both s and $\overline{s}$ appear.

To these ends, first absorb the $1/2$ into s by replacing s by $s+\frac{1}{2}$, so we have

$$L(s'+s, f\otimes F)\cdot \overline{L(s, f\otimes F)}$$

and want to eventually move to the line $\mathrm{Re}(s)=1/2$. To avoid the anti-holomorphy in the second factor, since $\overline{s}=1-s$ on the line $\mathrm{Re}(s)=1/2$, we can rewrite this as

$$L(s'+s, f\otimes F)\cdot L(1-s, \overline{f}\otimes\overline{F}) \qquad (\text{for } \mathrm{Re}(1-s)\gg 0 \text{ and } \mathrm{Re}(s'+s)\gg 0)$$

The vertical integral(s): Keep in mind that we have absorbed a $1/2$ into s, and have replaced $\overline{s}$ by $1-s$. The archimedean part of the whole integral is the function $\Gamma_{\varphi_\infty}(s,s',f,F)$ defined by

$$\Gamma_{\varphi_\infty}(s,s',f,F) =$$

$$\int_{N_\infty\backslash H_\infty}\int_{N_\infty\backslash H_\infty} \mathcal{K}(h,m)\,|\det a|^{s'+s-\frac{1}{2}}\,|\det a'|^{\frac{1}{2}-s}\Big(W_{f,\infty}\begin{pmatrix} a & \\ & 1\end{pmatrix} W_{F,\infty}(a)\Big)$$

$$\times\ \Big(\overline{W}_{f,\infty}\begin{pmatrix} a' & \\ & 1\end{pmatrix}\overline{W}_{F,\infty}(a')\Big)\,da\,da' \qquad \Big(\text{with } h=\begin{pmatrix} a & \\ & 1\end{pmatrix} \text{ and } m=\begin{pmatrix} a' & \\ & 1\end{pmatrix}\Big)$$

since $\varphi_\infty(h)=|\det a|^{s'}$. Note that this depends only upon the archimedean data attached to f and F. Thus, so far, the whole is

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \mathfrak{P}(g)\,|f(g)|^2\,dg$$

$$= \sum_{F \text{ on } GL(r-1)} \int_{\mathrm{Re}(s)=\frac{1}{2}} \Gamma_{\varphi_\infty}(s,s',f,F)\,\frac{L(s'+s, f\otimes F)\,L(1-s,\overline{f}\otimes\overline{F})}{\langle F,F\rangle}\,dt$$

$$+\ (\text{continuous part}) \qquad (\text{with } \mathrm{Re}(s')\gg 0)$$

Again, we want to meromorphically continue to $s'=0$.

1.2 Remark: With or without detailed knowledge of the *residual* part of L^2 (meaning that consisting of square-integrable iterated residues of cuspidal-data Eisenstein series), automorphic forms in the residual spectrum not admitting Whittaker models do not enter in this expansion.

2. *Spectral expansion: reduction to GL_2*

The Poincaré series admits a spectral expansion in terms of Eisenstein series, cuspforms, and L-functions, preparing for its meromorphic continuation. This section reduces the general spectral expansion to the case $r=2$.

Poisson summation: Form the Poincaré series in two stages to allow application of Poisson summation, namely

$$\mathfrak{P}(g) = \sum_{Z_kH_k\backslash G_k}\varphi(\gamma g) = \sum_{Z_kH_kU_k\backslash G_k}\ \sum_{\beta\in U_k}\varphi(\beta\gamma g) = \sum_{Z_kH_kU_k\backslash G_k}\ \sum_{\psi\in(U_k\backslash U_{\mathbb{A}})^\frown}\widehat{\varphi}_{\gamma g}(\psi)$$

where

$$\widehat{\varphi}_g(\psi) = \int_{U_{\mathbb{A}}}\overline{\psi}(u)\,\varphi(ug)\,du \qquad (\text{for } g\in G_{\mathbb{A}})$$

The leading term: The inner summand for $\psi = 1$ gives a vector from which an extremely degenerate Eisenstein series for the $(r-1,1)$ parabolic $P^{r-1,1} = ZHU$ is formed by the outer sum. That is,

$$g \to \int_{U_\mathbb{A}} \varphi(ug)\, du$$

is left equivariant by a character on $P_\mathbb{A}^{r-1,1}$, and is left invariant by $P_k^{r-1,1}$, namely,

$$\int_{U_\mathbb{A}} \varphi(upg)\, du = \int_{U_\mathbb{A}} \varphi(p \cdot p^{-1}up \cdot g)\, du = \delta_{P^{r-1,1}}(m) \cdot \int_{U_\mathbb{A}} \varphi(m \cdot u \cdot g)\, du$$

$$= \left|\frac{\det A}{d^{r-1}}\right|^{s'+1} \int_{U_\mathbb{A}} \varphi(ug)\, du \qquad \left(\text{where } p = \begin{pmatrix} A & * \\ 0 & d \end{pmatrix},\ A \in GL_{r-1},\ d \in GL_1\right)$$

The normalization is explicated by setting $g = 1$:

$$\int_{U_\mathbb{A}} \varphi(u)\, du = \int_{U_\infty} \varphi_\infty \cdot \int_{U_{\text{fin}}} \varphi_{\text{fin}} = \int_{U_\infty} \varphi_\infty \cdot \text{meas}\,(U_{\text{fin}} \cap K_{\text{fin}}) = \int_{U_\infty} \varphi_\infty$$

A natural normalization would have been that this value be 1, so the Eisenstein series here implicitly includes the archimedean integral and finite-prime measure constant as factors:

$$\int_{U_\infty} \varphi_\infty \quad \cdot \quad E_{s'+1}^{r-1,1}(g) \;=\; \sum_{\gamma \in P_k^{r-1,1} \backslash G_k} \left(\int_{U_\mathbb{A}} \varphi(u\gamma g)\, du \right)$$

As advance warning: the pole at $s' = 0$ of this leading term will be *cancelled* by a contribution from the continuous part of the spectral decomposition, below.

Main terms: appearance of $\mathfrak{Q}$ from GL_2: The group H_k is transitive on non-trivial characters on $U_k \backslash U_\mathbb{A}$. As usual, for fixed non-trivial character ψ_0 on $k \backslash A$, let

$$\psi^\xi(u) = \psi_0(\xi \cdot u_{r-1,r}) \qquad (\text{for } \xi \in k^\times)$$

The spectral expansion of $\mathfrak{P}$ with the leading term removed, is

$$\sum_{\gamma \in P_k^{r-1,1} \backslash G_k} \; \sum_{\alpha \in P_k^{r-2,1} \backslash H_k} \left(\sum_{\xi \in k^\times} \widehat{\varphi}_{\alpha\gamma g}(\psi^\xi) \right)$$

where $P^{r-2,1}$ is the parabolic subgroup of $H \approx GL_{r-1}$. Let

$$U' \;=\; \{\begin{pmatrix} 1_{r-2} & & * \\ & 1 & \\ & & 1 \end{pmatrix}\} \qquad U'' \;=\; \{\begin{pmatrix} 1_{r-2} & & \\ & 1 & * \\ & & 1 \end{pmatrix}\}$$

Let

$$\Theta \;=\; \{\begin{pmatrix} 1_{r-2} & & \\ & * & * \\ & * & * \end{pmatrix}\}$$

Then the expansion of the Poincaré series with leading term removed is

$$\sum_{\gamma \in P_k^{r-2,1,1} \backslash G_k} \left(\sum_{\xi \in k^\times} \int_{U''_\mathbb{A}} \overline{\psi}^\xi(u'') \int_{U'_\mathbb{A}} \varphi(u'u''\gamma g)\, du'\, du'' \right)$$

$$= \sum_{\gamma \in P_k^{r-2,2} \backslash G_k} \; \sum_{\alpha \in P^{1,1} \backslash \Theta_k} \left(\sum_{\xi \in k^\times} \int_{U''_\mathbb{A}} \overline{\psi}^\xi(u'') \int_{U'_\mathbb{A}} \varphi(u'u''\alpha\gamma g)\, du'\, du'' \right)$$

Letting

$$\widetilde{\varphi}(g) = \int_{U'_{\mathbb{A}}} \varphi(u'g)\,du'$$

the expansion becomes

$$\sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \;\sum_{\alpha \in P^{1,1}\backslash \Theta_k} \;\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\,\widetilde{\varphi}(u''\alpha\gamma g)\,du''$$

We claim the equivariance

$$\widetilde{\varphi}(pg) = |\det A|^{s'+1} \cdot |a|^{s'} \cdot |d|^{-(r-1)s'-(r-2)} \cdot \widetilde{\varphi}(g) \qquad (\text{for } p = \begin{pmatrix} A & * & * \\ & a & \\ & & d \end{pmatrix} \in G_{\mathbb{A}}, \text{ with } A \in GL_{r-2})$$

This is verified by changing variables in the defining integral: let $x \in \mathbb{A}^{r-1}$ and compute

$$\begin{pmatrix} 1_{r-2} & & x \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c+xd \\ & a & \\ & & d \end{pmatrix} = \begin{pmatrix} A & b & c \\ & a & \\ & & d \end{pmatrix} \begin{pmatrix} 1_{r-2} & & A^{-1}xd \\ & 1 & \\ & & 1 \end{pmatrix}$$

Thus, $|\det A|^{s'} \cdot |a|^{s'} \cdot |d|^{-(r-1)s'}$ comes out of the definition of φ, and another $|\det A| \cdot |d|^{2-r}$ from the change-of-measure in the change of variables replacing x by Ax/d in the integral defining $\widetilde{\varphi}$ from φ. Note that

$$|a|^{s'} \cdot |d|^{-(r-1)s'-(r-2)} = |\det \begin{pmatrix} a & \\ & d \end{pmatrix}|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot |a/d|^{\frac{rs'+(r-2)}{2}}$$

Thus, letting

$$\Phi(g) = \sum_{\alpha \in P_k^{1,1}\backslash \Theta_k} \;\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\,\widetilde{\varphi}(u''\alpha g)\,du''$$

we can write

$$\mathfrak{P}(g) \;-\; \sum_{\gamma \in P_k^{r-1,1}\backslash G_k} \int_{U_{\mathbb{A}}} \varphi(u\gamma g)\,du = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi(\gamma g)$$

This is not an Eisenstein series for $P^{r-2,2}$ in the strictest sense. An expression in terms of genuine Eisenstein series is helpful in understanding meromorphic continuations.

Define a GL_2 kernel $\varphi^{(2)}$ for a Poincaré series as follows. We require right invariance by the maximal compact subgroups locally everywhere, and left equivariance

$$\varphi^{(2)}(\begin{pmatrix} a & \\ & d \end{pmatrix} \cdot D) \;=\; |a/d|^{s} \cdot \varphi^{(2)}(D)$$

Then the the archimedean data $\varphi^{(2)}_\infty$ is completely specified by

$$\varphi^{(2)}_\infty \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;=\; \widetilde{\varphi} \begin{pmatrix} 1_{r-2} & & \\ & 1 & x \\ & & 1 \end{pmatrix} \qquad (\text{with } \widetilde{\varphi} \text{ as above})$$

Then put

$$\varphi^*(s, \widetilde{\varphi}, D) \;=\; \sum_{\xi \in k^\times} \int_{U_{\mathbb{A}}} \overline{\psi}^{\xi}(u)\,\varphi^{(2)}(s, uD)\,du \qquad (\text{with } U \text{ now the unipotent radical of } P^{1,1} \text{ in } GL_2)$$

The corresponding GL_2 Poincaré series with leading term removed is

$$\mathfrak{Q}(s, D) = \sum_{\alpha \in P_k^{1,1} \backslash GL_2(k)} \varphi^*(s, \alpha D)$$

Thus, for

$$g = \begin{pmatrix} A & * \\ & D \end{pmatrix} \qquad \text{(with } A \in GL_{r-2}(\mathbb{A}) \text{ and } D \in GL_2(\mathbb{A}))$$

the inner integral

$$g \to \int_{U''_{\mathbb{A}}} \overline{\psi}(u'')\, \widetilde{\varphi}(u''g)\, du''$$

is expressible in terms of the kernel φ^* for $\mathfrak{Q}$, namely,

$$\sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\, \widetilde{\varphi}(u''g)\, du'' \;=\; |\det A|^{s'+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot \varphi^*\left(\frac{rs'+r-2}{2}, D\right)$$

Thus,

$$\sum_{\alpha \in P_k^{1,1} \backslash \Theta_k} \sum_{\xi \in k^\times} \int_{U''_{\mathbb{A}}} \overline{\psi}^{\xi}(u'')\, \widetilde{\varphi}(u''\alpha g)\, du'' \;=\; |\det A|^{s'+1} \cdot |\det D|^{-\frac{(r-2)}{2}\cdot(s'+1)} \cdot \mathfrak{Q}\left(\frac{rs'+r-2}{2}, D\right)$$

Thus, to obtain a (not necessarily L^2) spectral decomposition of the Poincaré series $\mathfrak{P}$ (with main term removed) we first determine the (L^2) spectral decomposition of $\mathfrak{Q}$ for $r = 2$, and then form $P^{r-2,2}$ Eisenstein series from the spectral fragments.

3. *Spectral expansion for* GL_r

The spectral decomposition of the Poincaré series for $r = 2$ yields that for $r > 2$ by inducing. For $r > 2$, nothing remains after all the non-L^2 terms are removed. The non-L^2 terms are induced from the genuinely L^2 spectral expansion of the Poincaré series on GL_2.

Before carrying out the spectral expansion for $r = 2$, we had found that

$$\mathfrak{P}(g) \;=\; \left(\int_{U_\infty} \varphi_\infty\right) \cdot E^{r+1,1}_{s'+1}(g) \;+\; \sum_{\gamma \in P_k^{r-2,2} \backslash G_k} \Phi(\gamma g)$$

where

$$\Phi\begin{pmatrix} A & * \\ & D \end{pmatrix} = |\det A|^{s'+1} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}} \cdot \mathfrak{Q}\left(\frac{rs'+r-2}{2}, \widetilde{\varphi}, D\right) \qquad \text{(with } A \in GL_{r-2} \text{ and } D \in GL_2)$$

with $\mathfrak{Q}$ from GL_2, and

$$\widetilde{\varphi}(g) = \int_{U'_k \backslash U'_{\mathbb{A}}} \varphi(u'g)\, du'$$

Thus, formation of $\mathfrak{P}$ (with its leading term removed) amounts to forming an Eisenstein series from Φ, with analytical properties explicated by expressing Φ as a superposition of vectors generating irreducibles.

Decomposition into irreducibles:

From the decomposition of $\mathfrak{Q}$ on GL_2 for $\mathrm{Re}(s') \gg 0$

$$\Phi\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \;=\; |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}} \cdot \mathfrak{Q}(\frac{rs'+r-2}{2}, \widetilde{\varphi}, D)$$

$$= |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}} \sum_F \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W_{\overline{F},\infty}\right) \cdot L(\frac{rs'+r-2}{2} + \tfrac{1}{2}, \overline{F}) \cdot \frac{F}{\langle F, F\rangle}$$

$$+ \; |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}} \sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i\kappa}$$

$$\int_{\mathrm{Re}(s)=\frac{1}{2}} \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W^E_{1-s,\overline{\chi},\infty}\right) \cdot \frac{L(\frac{rs'+r-2}{2} + 1 - s, \overline{\chi}) \cdot L(\frac{rs'+r-2}{2} + s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)} \cdot E_{s,\chi}(D)\, ds$$

Least continuous part of Poincaré series:
Let

$$\Phi_{\frac{s'+1}{2},F}(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \cdot \theta) = |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\, F(D) \qquad (\text{for } \theta \in K_{\mathbb{A}})$$

and define a half-degenerate Eisenstein series

$$E^{r-2,2}_{\frac{s'+1}{2},F}(g) = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{s'+1}{2},F}(\gamma g)$$

Then the *most-cuspidal* (or *least continuous*) part of the Poincaré series is

$$\sum_F \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi} \cdot W_{\overline{F},\infty}\right) \cdot \frac{L(\frac{rs'+r-2}{2} + \frac{1}{2}, \overline{F})}{\langle F, F\rangle} \cdot E^{r-2,2}_{\frac{s'+1}{2},F} \qquad (\text{cuspforms } F \text{ on } GL_2)$$

The *summands* of this expression have relatively well understood meromorphic continuations. As discussed in an appendix, the half-degenerate Eisenstein series $E^{r-2,2}_{s,F}$ has *no poles* in $\mathrm{Re}(s) \geq 1/2$. With $s = (s'+1)/2$ this assures absence of poles in $\mathrm{Re}(s') \geq 0$.

Continuous part of the Poincare series: The Eisenstein series integral part of $\mathfrak{Q}$ on GL_2 gives degenerate Eisenstein series attached to the $(r-2,1,1)$-parabolic in GL_r. This arises from a similar consideration as for GL_2 *cuspforms*, but for GL_2 *Eisenstein series*, as follows.

Let $E_{s,\chi}$ be the usual Eisenstein series for GL_2, and let

$$\Phi_{\frac{s'+1}{2},E_{s,\chi}}(\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \cdot \theta) = |\det A|^{2\cdot\frac{s'+1}{2}} \cdot |\det D|^{-(r-2)\cdot\frac{s'+1}{2}}\, E_{s,\chi}(D) \qquad (\text{for } \theta \in K_{\mathbb{A}})$$

and define an Eisenstein series

$$E^{r-2,2}_{\frac{s'+1}{2},E_{s,\chi}}(g) = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \Phi_{\frac{s'+1}{2},E_{s,\chi}}(\gamma g)$$

For given $s \in \mathbb{C}$, an easy variant of Godement's criterion proves convergence for sufficiently large $\mathrm{Re}(s')$.

Then, ignoring the issue of interchange of sums and integrals, the Poincaré series has *most continuous* part

$$\sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i\kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W^E_{1-s,\overline{\chi},\infty}\right)$$

$$\times \; \frac{L(\frac{rs'+r-2}{2} + 1 - s, \overline{\chi}) \cdot L(\frac{rs'+r-2}{2} + s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2}+s-1/2)} \cdot E^{r-2,2}_{\frac{s'+1}{2},E_{s,\chi}}\; ds$$

That is, it is the *analytically continued* E_s on the line $\mathrm{Re}(s) = \frac{1}{2}$ that enters. However, as usual, for $\mathrm{Re}(s) \gg 0$ and $\mathrm{Re}(s') \gg 0$ this iterated formation of Eisenstein series is equal to a single-step Eisenstein series. The equality persists after analytic continuation.

Thus, let

$$\Phi_{s_1,s_2,s_3,\chi}(\begin{pmatrix} A & * & * \\ 0 & m_2 & * \\ 0 & 0 & m_3 \end{pmatrix} \cdot \theta)$$

$$= \;|\det A|^{s_1} \cdot |m_2|^{s_2} \chi(m_2) \cdot |m_3|^{s_3} \overline{\chi}(m_3) \qquad (\text{for } \theta \in K_{\mathbb{A}} \text{ and } A \in GL_{r-2})$$

and

$$E^{r-2,1,1}_{s_1,s_2,s_3,\chi} = \sum_{\gamma \in P_k^{r-2,1,1} \backslash G_k} \Phi_{s_1,s_2,s_3,\chi}(\gamma g)$$

Then, ignoring the issue of interchange of sums and integrals, the Poincaré series has *most continuous* part

$$\sum_\chi \frac{\chi(\mathfrak{d})}{4\pi i \kappa} \int_{\mathrm{Re}(s)=\frac{1}{2}} \left(\int_{PGL_2(k_\infty)} \widetilde{\varphi}_\infty \cdot W^E_{1-s,\overline{\chi},\infty} \right)$$

$$\times \; \frac{L(\frac{rs'+r-2}{2} + 1 - s, \overline{\chi}) \cdot L(\frac{rs'+r-2}{2} + s, \chi)}{L(2-2s, \overline{\chi}^2)} \cdot |\mathfrak{d}|^{-(\frac{rs'+r-2}{2} + s - 1/2)} \cdot E^{r-2,1,1}_{2\cdot\frac{s'+1}{2},\; s-(r-2)\cdot\frac{s'+1}{2},\; -s-(r-2)\cdot\frac{s'+1}{2},\; \chi} \; ds$$

Comment: It is remarkable that there are no further terms in the spectral expansion of $\mathfrak{P}$, beyond the main term, the cuspidal GL_2 part induced up to GL_r, and the continuous GL_2 part induced up to GL_r.

4. *Appendix: half-degenerate Eisenstein series*

Take $q > 1$, and let f be a cuspform on $GL_q(\mathbb{A})$, in the strong sense that f is in $L^2(GL_q(k)\backslash GL_q(\mathbb{A})^1)$, and f meets the Gelfand-Fomin-Graev conditions

$$\int_{N_k \backslash N_{\mathbb{A}}} f(ng)\, dn = 0 \qquad (\text{for almost all } g)$$

and f generates an irreducible representation of $GL_q(k_\nu)$ locally at all places ν of k. For a Schwartz function Φ on $\mathbb{A}^{q \times r}$ and Hecke character χ, let

$$\varphi(g) = \varphi_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(\mathbb{A})} f(h^{-1})\, \chi(\det h)^r\, \Phi(h \cdot [0_{q\times(r-q)}\ 1_q] \cdot g)\, dh$$

This function φ has the same central character as f. It is left invariant by the adele points of the unipotent radical

$$N = \{\begin{pmatrix} 1_{r-q} & * \\ & 1_r \end{pmatrix}\} \qquad (\text{unipotent radical of } P = P^{r-q,q})$$

The function φ is left invariant under the *k-rational* points M_k of the standard Levi component of P,

$$M = \{\begin{pmatrix} a & \\ & d \end{pmatrix} \;:\; a \in GL_{r-q},\ d \in GL_r\}$$

To understand the normalization, observe that

$$\xi(\chi^r, f, \Phi(0,*)) = \varphi(1) = \int_{GL_q(\mathbb{A})} f(h^{-1})\, \chi(\det h)^r\, \Phi(h \cdot [0_{q\times(r-q)}\ 1_q])\, dh$$

is a zeta integral as in [Godement-Jacquet 1972] for the standard L-function attached to the cuspform f (or perhaps a contragredient). Thus, the Eisenstein series formed from φ includes this zeta integral as a factor, so write

$$\xi(\chi^r, f, \Phi(0,*)) \cdot E^P_{\chi,f,\Phi}(g) = \sum_{\gamma \in P_k \backslash GL_r(k)} \varphi(\gamma\, g) \qquad \text{(convergent for } \mathrm{Re}(\chi) >> 0)$$

Now prove the meromorphic continuation via Poisson summation:

$$\xi(\chi^r, f, \Phi(0,*)) \cdot E^P_{\chi,f,\Phi}(g)$$

$$= \chi(\det g)^q \sum_{\gamma \in P_k \backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{\alpha \in GL_q(k)} \Phi(h^{-1} \cdot [0\ \alpha] \cdot g)\, dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y \in k^{q\times r},\ \text{full rank}} \Phi(h^{-1} \cdot y \cdot g)\, dh$$

The Gelfand-Fomin-Graev condition on f will compensate for the otherwise-irksome full-rank constraint. Anticipating that we can drop the rank condition suggests that we define

$$\Theta_\Phi(h,g) = \sum_{y \in k^{q\times r}} \Phi(h^{-1} \cdot y \cdot g)$$

As in [Godement-Jacquet 1972], the non-full-rank terms integrate to 0:

4.1 Proposition: For f a cuspform, less-than-full-rank terms integrate to 0, that is,

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y \in k^{q\times r},\ \text{rank} < q} \Phi(h^{-1} \cdot y \cdot g)\, dh = 0$$

Proof: Since this is asserted for arbitrary Schwartz functions Φ, we can take $g = 1$. By linear algebra, given $y_0 \in k^{q\times r}$ of rank ℓ, there is $\alpha \in GL_q(k)$ such that

$$\alpha \cdot y_0 = \begin{pmatrix} y_{\ell\times r} \\ 0_{(q-\ell)\times r} \end{pmatrix} \qquad \text{(with } \ell\text{-by-}r \text{ block } y_{\ell\times r} \text{ of rank } \ell)$$

Thus, without loss of generality fix y_0 of the latter shape. Let Y be the orbit of y_0 under left multiplication by the rational points of the parabolic

$$P^{\ell,q-\ell} = \{ \begin{pmatrix} \ell\text{-by-}\ell & * \\ 0 & (q-\ell)\text{-by-}(q-\ell) \end{pmatrix} \} \subset GL_q$$

This is some set of matrices of the same shape as y_0. Then the subsum over $GL_q(k) \cdot y_0$ is

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y \in GL_q(k)\cdot y_0} \Phi(h^{-1} \cdot y)\, dh = \int_{P_k^{\ell,q-\ell}\backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y \in Y} \Phi(h^{-1} \cdot y)\, dh$$

Let N and M be the unipotent radical and standard Levi component of $P^{\ell,q-\ell}$,

$$N = \begin{pmatrix} 1_\ell & * \\ 0 & 1_{q-\ell} \end{pmatrix} \qquad M = \begin{pmatrix} \ell\text{-by-}\ell & 0 \\ 0 & (q-\ell)\text{-by-}(q-\ell) \end{pmatrix}$$

Then the integral can be rewritten as an iterated integral

$$\int_{N_k M_k \backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r} \sum_{y \in Y} \Phi(h^{-1} \cdot y)\, dh$$

$$= \int_{N_{\mathbb{A}} M_k \backslash GL_q(\mathbb{A})} \sum_{y \in Y} \int_{N_k \backslash N_{\mathbb{A}}} f(nh)\, \chi(\det nh)^{-r}\, \Phi((nh)^{-1} \cdot y)\, dn\, dh$$

$$= \int_{N_{\mathbb{A}} M_k \backslash GL_q(\mathbb{A})} \sum_{y \in Y} \chi(\det h)^{-r}\, \Phi(h^{-1} \cdot y) \left(\int_{N_k \backslash N_{\mathbb{A}}} f(nh)\, dn \right) dh$$

since all fragments but $f(nh)$ in the integrand are left invariant by $N_{\mathbb{A}}$. But the inner integral of $f(nh)$ is 0, by the Gelfand-Fomin-Graev condition, so the whole is 0. ///

Let ι denote the transpose-inverse involution(s). Poisson summation gives

$$\Theta_\Phi(h, g) = \sum_{y \in k^{q \times r}} \Phi(h^{-1} \cdot y \cdot g)$$

$$= |\det(h^{-1})^\iota|^r\, |\det g^\iota|^q \sum_{y \in k^{q \times r}} \widehat{\Phi}((h^\iota)^{-1} \cdot y \cdot g^\iota) \;=\; |\det(h^{-1})^\iota|^r\, |\det g^\iota|^q\, \Theta_{\widehat{\Phi}}(h^\iota, g^\iota)$$

As with Θ_Φ, the not-full-rank summands in $\Theta_{\widehat{\Phi}}$ integrate to 0 against cuspforms. Thus, letting

$$GL_q^+ = \{h \in GL_q(\mathbb{A}) \;:\; |\det h| \geq 1\} \qquad\qquad GL_q^- = \{h \in GL_q(\mathbb{A}) \;:\; |\det h| \leq 1\}$$

$$\xi(\chi^r, f, \Phi(0, *)) \cdot E^P_{\chi, f, \Phi}(g) = \chi(\det g)^q \int_{GL_q(k) \backslash GL_q(\mathbb{A})} f(h)\, \chi(\det h)^{-r}\, \Theta_\Phi(h, g)\, dh$$

$$= \chi(\det g)^q \int_{GL_q(k) \backslash GL_q^+} f(h)\, \chi(\det h)^{-r}\, \Theta_\Phi(h, g)\, dh + \chi(\det g)^q \int_{GL_q(k) \backslash GL_q^-} f(h)\, \chi(\det h)^{-r}\, \Theta_\Phi(h, g)\, dh$$

$$= \quad \chi(\det g)^q \int_{GL_q(k) \backslash GL_q^+} f(h)\, \chi(\det h)^{-r}\, \Theta_\Phi(h, g)\, dh$$

$$+\, \chi(\det g)^q \int_{GL_q(k) \backslash GL_q^-} |\det(h^{-1})^\iota|^r\, |\det g^\iota|^q\, f(h)\, \chi(\det h)^{-r}\, \Theta_{\widehat{\Phi}}(h^\iota, g^\iota)\, dh$$

By replacing h by h^ι in the second integral, convert it to an integral over $GL_q(k) \backslash GL_q^+$, and the whole is

$$\xi(\chi^r, f, \Phi(0, *)) \cdot E^P_{\chi, f, \Phi}(g) \;=\; \chi(\det g)^q \int_{GL_q(k) \backslash GL_q^+} f(h)\, \chi(\det h)^{-r}\, \Theta_\Phi(h, g)\, dh$$

$$+\, \nu\chi^{-1}(\det g^\iota)^q \int_{GL_q(k) \backslash GL_q^+} f(h^\iota)\;\; \nu\chi^{-1}(\det h^\iota)^{-r}\, \Theta_{\widehat{\Phi}}(h, g^\iota)\, dh$$

Since $f \circ \iota$ is a cuspform, the second integral is entire in χ. Thus, we have proven

$$\xi(\chi^r, f, \Phi(0, *)) \;\cdot E^P_{\chi, f, \Phi} \quad \text{is entire}$$

Remark: Except for the extreme case $q = r - 1$, these Eisenstein series are degenerate, so occur only as (iterated) *residues* of cuspidal-data Eisenstein series. Assessing poles of residues is less effective in the present special circumstances than the above argument.

5. *Appendix: residues of degenerate Eisenstein series for $P^{n-1,1}$*

We prove meromorphic continuation and determine some residues of some very degenerate Eisenstein series (sometimes called *Epstein zeta functions*). We need to recall some specifics about these well-known examples. Let

$$P = P^{r-1,1} = \{\begin{pmatrix} (r-1)\text{-by-}(r-1) & * \\ 0 & 1\text{-by-}1 \end{pmatrix}\}$$

View $\mathbb{A}^r$ and k^r as *row* vectors. Let $e_1, \ldots, e_r$ the standard basis for k^r. The parabolic P is the stabilizer in GL_r of the line ke_r. Given a Hecke character of the form $\chi(\alpha) = |\alpha|^s$ and a Schwartz function Φ on $\mathbb{A}^r$, let

$$\varphi(g) = |\det g|^s \int_{\mathbb{J}} |t|^{rs}\, \Phi(t \cdot e_r \cdot g)\, dt$$

The factor $|t|^{rs}$ in the integrand and the leading factor $|\det g|^s$ combine to give the invariance $\varphi(zg) = \varphi(g)$ for z in the center $Z_{\mathbb{A}}$ of $G = GL_r$. By changing variables in the integral observe the left equivariance

$$\varphi(pg) = |\det pg|^s \int_{\mathbb{J}} |t|^{rs}\, \Phi(t \cdot e_r \cdot pg)\, dt = |\det A|^s |d|^{-(r-1)s} \cdot \varphi(g) \qquad (\text{for } p = \begin{pmatrix} A & * \\ & d \end{pmatrix} \in P_{\mathbb{A}})$$

The normalization is *not* $\varphi(1) = 1$ but

$$\varphi(1) = \int_{\mathbb{J}} |t|^{rs}\, \Phi(t \cdot e_r)\, dt \qquad (\text{Tate-Iwasawa zeta integral at } rs)$$

Denote this zeta integral by $\xi = \xi(rs, \Phi(0,*))$, indicating that it only depends upon the values of Φ along the last coordinate axis. Thus, by comparison to the standard spherical Eisenstein series $E_s(g)$ corresponding to this s^{th} degenerate principal series, the Eisenstein series associated to φ has a factor of $\xi(rs, \Phi(0,*))$ included, namely

$$\xi(ns, \Phi(0,*)) \cdot E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)$$

Poisson summation proves the meromorphic continuation of this Eisenstein series, as follows. Let

$$\mathbb{J}^+ = \{t \in \mathbb{J} \,:\, |t| \geq 1\} \qquad \mathbb{J}^- = \{t \in \mathbb{J} \,:\, |t| \leq 1\}$$

and $g^\iota = (g^\top)^{-1}$ (transpose inverse)

$$\xi(rs, \Phi(0,*)) \cdot E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g) = |\det g|^s \sum_{\gamma \in P_k \backslash G_k} \int_{\mathbb{J}} |t|^{rs}\, \Phi(t \cdot x \cdot \gamma g)\, dt$$

$$= |\det g|^s \sum_{\gamma \in P_k \backslash G_k} \int_{k^\times \backslash \mathbb{J}} |t|^{rs} \sum_{\lambda \in k^\times} \Phi(t \cdot \lambda e_r \cdot \gamma g)\, dt = |\det g|^s \int_{k^\times \backslash \mathbb{J}} |t|^{rs} \sum_{x \in k^r - 0} \Phi(t \cdot x \cdot g)\, dt$$

Let

$$\Theta(g) = \sum_{x \in k^r} \Phi(t \cdot x \cdot g)$$

Then

$$\xi(rs, \Phi(0,*)) \cdot E_s(g) = \; |\det g|^s \int_{k^\times \backslash \mathbb{J}^+} |t|^{rs}\, [\Theta(g) - \Phi(0)]\, dt \;+\; |\det g|^s \int_{k^\times \backslash \mathbb{J}^-} |t|^{rs}\, [\Theta(g) - \Phi(0)]\, dt$$

The usual estimate shows that the integral over $k^\times \backslash \mathbb{J}^+$ converges absolutely for all $s \in \mathbb{C}$. Rewrite the second part of the integral as an analogous integral over $k^\times \backslash \mathbb{J}^+$. Poisson summation gives

$$\sum_{x \in k^r - 0} \Phi(t \cdot x \cdot g) + \Phi(0) = |t|^{-r} |\det g|^{-1} \sum_{x \in k^r - 0} \widehat{\Phi}(t^{-1} \cdot x \cdot g^\iota) + |t|^{-r} |\det g|^{-1} \widehat{\Phi}(0)$$

Let
$$\Theta'(g^\iota) = \sum_{x \in k^r} \widehat{\Phi}(t \cdot x \cdot g^\iota)\cdot$$

Removing the $\Phi(0)$ and $\widehat{\Phi}(0)$ terms and replacing t by t^{-1} in the integral over $k^\times \backslash \mathbb{J}^-$ turns this integral into

$$|\det g|^{s-1} \int_{k^\times \backslash \mathbb{J}^+} |t|^{r(1-s)} \, [\Theta'(g^\iota) - \widehat{\Phi}(0)] \, dt$$

$$- \; |\det g|^s \, \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{rs} \, dt \;\; + \;\; |\det g|^{s-1} \, \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{r(s-1)} \, dt$$

The integral over $k^\times \backslash \mathbb{J}^+$ is entire. Thus, the non-elementary part of the integral is converted into two *entire* integrals over $k^\times \backslash \mathbb{J}^+$ together with two elementary integrals that give the only possible poles:

$$\xi \cdot E_s(g) = (\text{entire}) - |\det g|^s \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^r \, dt + |\det g|^{s-1} \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} |t|^{r(s-1)} \, dt$$

With
$$\kappa = \int_{k^\times \backslash \mathbb{J}^1} 1 \; dt$$

the relatively elementary integrals can be evaluated

$$\int_{k^\times \backslash \mathbb{J}^-} |t|^{rs} \, dt = \left(\int_{k^\times \backslash \mathbb{J}^1} 1 \, dt \right) \cdot \left(\int_0^1 t^{rs} \, dt \right) = \frac{\kappa}{ns}$$

Similarly,
$$\int_{k^\times \backslash \mathbb{J}^-} |t|^{r(s-1)} \, dt = \frac{\kappa}{r(s-1)}$$

That is,
$$\xi(rs, \Phi(0, *)) \;\cdot\; E_s \;\; = \;\; (\text{entire}) \;\; - \;\; \frac{\kappa \, \Phi(0)}{rs} \;\; + \;\; \frac{\kappa \, \widehat{\Phi}(0)}{r(s-1)}$$

Thus, the residue at $s = 1$ of E_s is

$$\text{Res}_{s=1} E_s = \frac{\kappa \, \widehat{\Phi}(0)}{r \cdot \xi(r, \Phi(0, *))}$$

Let $\mathfrak{d}$ be an idele such that $\mathfrak{d}_v$ generates the local different at a finite place v, and is trivial at archimedean places. Let Φ be the standard Gaussian at archimedean places (so its integral is 1), and the characteristic function of $\mathfrak{o}_v^n$ at finite places v. With the standard measure on $\mathbb{A}$ we have

$$\widehat{\Phi}(0) = |\mathfrak{d}|^{r/2} \qquad\qquad \xi(r, \Phi(0, *)) = \xi(r)$$

where ξ is the usual zeta function with standard gamma factors, but without any epsilon factor or accounting for conductors. The residue at $s = 1$ is
$$\text{Res}_{s=1} E = \frac{\kappa \cdot |\mathfrak{d}|^{r/2}}{n \cdot \xi(r)}$$

At $s = 0$, the relevant residue is

$$\text{Res}_{s=0} \xi(rs, \Phi(0, *)) \;\cdot\; E_s = -\frac{\kappa \, \Phi(0)}{r}$$

6. *Appendix: degenerate Eisenstein series for* $P^{r-2,1,1}$

Let $P = P^{r-2,1,1}$, and

$$\varphi_{s_1,s_2,s_3}\begin{pmatrix} A & * & * \\ & a & * \\ & & d \end{pmatrix} = |\det A|^{s_1} \cdot |a|^{s_2} \cdot |d|^{s_3} \qquad \text{(with } A \in GL_{r-2},\ a,d \in GL_1\text{)}$$

and extend φ to a function on $G(\mathbb{A}) = GL_r(\mathbb{A})$ by requiring right $K_{\mathbb{A}}$-equivariance. Define an Eisenstein series on $G = GL_r$ by

$$E_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-2,1,1}\backslash G_k} \varphi_{s_1,s_2,s_3}(\gamma g)$$

Symmetry in s_2 and s_3: This can be rewritten as an interated sum

$$E_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-2,2}\backslash G_k} \varphi_{s_1 \otimes E^{1,1}_{s_2,s_3}}(\gamma g)$$

where

$$\varphi_{s_1 \otimes E^{1,1}_{s_2,s_3}}\begin{pmatrix} A & * \\ & D \end{pmatrix} = |\det A|^{s_1} \cdot E^{1,1}_{s_2,s_3}(D) \qquad \text{(with } A \in GL_{r-2} \text{ and } D \in GL_2\text{)}$$

and $E^{1,1}_{s_2,s_3}$ is the GL_2 Eisenstein series

$$E^{1,1}_{s_2,s_3}(g) = \sum_{\gamma \in P_k^{1,1}\backslash GL_2(k)} \varphi_{s_2,s_3}(\gamma g) \qquad \left(\text{with } \varphi_{s_2,s_3}\begin{pmatrix} a & * \\ & d \end{pmatrix} = |a|^{s_2}|d|^{s_3}\right)$$

Since

$$|a|^{s_2}|d|^{s_3} = |ad|^{\frac{s_2+s_3}{2}} \cdot |a/d|^{\frac{s_2-s_3}{2}}$$

this GL_2 Eisenstein series can be expressed in terms of an Eisenstein series with trivial central character, namely

$$E^{1,1}_{s_2,s_3}(g) = |\det g|^{\frac{s_2+s_3}{2}} \cdot E_{\frac{s_2-s_3}{2}}(g)$$

where

$$E_s(g) = \sum_{\gamma \in P_k^{1,1}\backslash GL_2(k)} \varphi_s(\gamma g) \qquad \text{with} \qquad \varphi_s\begin{pmatrix} a & * \\ & d \end{pmatrix} = |a/d|^s$$

From the functional equation

$$\xi(2s) \cdot E_s = \xi(2s-1) \cdot E_{1-s}$$

the $P^{1,1}$ Eisenstein series $E^{1,1}_{s_2,s_3}$ has a functional equation under

$$(s_2,s_3) = (\tfrac{1}{2},-\tfrac{1}{2}) + (s_2-\tfrac{1}{2}, s_3+\tfrac{1}{2}) \to (\tfrac{1}{2},-\tfrac{1}{2}) + (s_3+\tfrac{1}{2}, s_2-\tfrac{1}{2}) = (s_3+1, s_2-1)$$

given by

$$E^{1,1}_{s_2,s_3} = \frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)} \cdot E^{1,1}_{s_3+1,s_2-1}$$

Thus,

$$E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)} \cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1}$$

Pole at $s_1 - s_2 = r - 1$: There is another iterated sum expression

$$E^{r-2,1,1}_{s_1,s_2,s_3}(g) = \sum_{\gamma \in P_k^{r-1,1} \backslash G_k} \varphi_{E^{r-2,1}_{s_1,s_2} \otimes s_3}(\gamma g)$$

where

$$\varphi_{E^{r-2,1}_{s_1,s_2} \otimes s_3} \begin{pmatrix} A & * \\ & d \end{pmatrix} = E^{r-2,1}_{s_1,s_2}(A) \qquad \text{(with } A \in GL_{r-1} \text{ and } d \in GL_1)$$

and $E^{r-2,1}_{s_1,s_2}$ is the GL_{r-1} Eisenstein series

$$E^{r-2,1}_{s_1,s_2} \begin{pmatrix} A & * \\ & d \end{pmatrix} = \sum_{\gamma \in P_k^{r-2,1} \backslash GL_{r-1}(k)} \varphi_{s_1,s_2}(\gamma g) \qquad \left(\text{with } \varphi_{s_1,s_2} \begin{pmatrix} A & * \\ & d \end{pmatrix} = |A|^{s_1} |d|^{s_2}\right)$$

Since

$$|A|^{s_1}|d|^{s_2} = \left| \frac{\det A}{d^{r-2}} \right|^{\frac{s_1 - s_2}{r-1}} \cdot |\det A \cdot d|^{\frac{(r-2)s_1 + s_2}{r-1}}$$

we can express $E^{r-2,1}_{s_1,s_2}$ in terms of an Eisenstein series with trivial central character, as

$$E^{r-2,1}_{s_1,s_2}(h) = |\det h|^{\frac{(r-2)s_1+s_2}{r-1}} \cdot E_{\frac{s_1-s_2}{r-1}}(g)$$

where

$$E_s(h) = \sum_{P_k^{r-2,1} \backslash GL_{r-1}(k)} \varphi_s(\gamma h)$$

with

$$\varphi_s \begin{pmatrix} A & * \\ & d \end{pmatrix}(g) = \left| \frac{\det A}{d^{r-2}} \right|^s \qquad \text{(for } A \in GL_{r-2} \text{ and } d \in GL_1)$$

From the previous appendix, the Eisenstein series E_s for $P^{r-2,1}$ has a pole at $s = 1$ with constant residue

$$\mathrm{Res}_{s=1} E_s = \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1) \cdot \xi(r-1)}$$

Thus, at $\frac{s_1-s_2}{r-1} = 1$, that is, at $s_1 - s_2 = r-1$, $E^{r-2,1}_{s_1,s_2}(h)$ has residue

$$\mathrm{Res}_{s_1-s_2=r-1} E^{r-2,1}_{s_1,s_2}(h) = |\det h|^{\frac{(r-2)s_1 + (s_1-(r-1))}{r-1}} \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1)\cdot \xi(r-1)} = |\det h|^{s_1-1} \cdot \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1)\cdot \xi(r-1)}$$

Thus, since

$$|\det A|^{s_1} \cdot |a|^{s_1-(r-1)} = \left|\det \begin{pmatrix} A & \\ & a \end{pmatrix}\right|^{s_1-1} \cdot \left| \frac{\det A}{a^{r-2}} \right|^1 \qquad \text{(for } A \in GL_{r-2} \text{ and } a \in GL_1)$$

the residue is

$$\mathrm{Res}_{s_1-s_2=r-1} E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\kappa \cdot |\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1)\cdot \xi(r-1)} \cdot E^{r-1,1}_{s_1-1,s_3}$$

Residue at $s_1 - s_3 = r$: The functional equation in s_2 and s_3 (from GL_2) is

$$E^{r-2,1,1}_{s_1,s_2,s_3} = \frac{\xi(s_2 - s_3 - 1)}{\xi(s_2 - s_3)} \cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1}$$

This functional equation and the pole of $E^{r-2,1,1}_{s_1,s_2,s_3}$ at $s-1-s_2=r-1$ give a pole of $E^{r-2,1,1}_{s_1,s_2,s_3}$ at $s_1-s_3=r$ with residue

$$\mathrm{Res}_{s_1-s_3=r}E^{r-2,1,1}_{s_1,s_2,s_3} = \mathrm{Res}_{s_1-s_3=r}\left(\frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)}\cdot E^{r-2,1,1}_{s_1,s_3+1,s_2-1}\right)$$

$$= \frac{\kappa\cdot|\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1)\cdot\xi(r-1)}\cdot\frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)}\cdot E^{r-1,1}_{s_1-1,s_2-1}$$

$$= \frac{\kappa\cdot|\mathfrak{d}|^{\frac{r-1}{2}}}{(r-1)\cdot\xi(r-1)}\cdot\frac{\xi(s_2-(s_1-r)-1)}{\xi(s_2-(s_1-r))}\cdot E^{r-1,1}_{s_1-1,s_2-1}$$

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