Sketch of simple Siegel-Weil formula

Paul Garrett

in T.H. Chen's *Geometric methods in Langlands Program* Seminar, March 28, 2022

Will also be posted at the top of
http://www.math.umn.edu/˜garrett/m/v/
with filename Siegel-Weil.pdf

C.L. Siegel, *Über die analytische Theorie der quadratische Formen, I, II, III*, Ann. of Math. **36** (1935), 527-606; **37** (1936), 230-263; **38** (1937), 212-291.

C.L. Siegel, *Indefinite quadratische Formen und Funktionentheorie, I, II*, Math. Ann. **124** (1952), 17-54; (1952), 364-387.

A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1-87.

Vague (classic) Siegel-Weil: *Certain* linear combinations of holomorphic *theta series* are (exactly) holomorphic *Eisenstein series*.

Example, and arithmetic content: As modular forms for the congruence subgroup Γ_θ of $SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$\sum_{v \in \mathbb{Z}^8} e^{-\pi i |v|^2 z} \;=\; E_4^{(i\infty)}(z)$$

where $E_4^{(i\infty)}$ is the weight-four Eisenstein series taking value 1 at $i\infty$ and 0 at the other cusp.

$\theta_8(z) = \sum_{v \in \mathbb{Z}^8} e^{-\pi i |v|^2 z}$ is a *theta series*.

The Fourier expansion of the Eisenstein series is

$$1 + \frac{(2\pi)^4}{3!\zeta(4)(2^4-1)} \sum_{N \geq 1} \Big(\sum_{0 < c | N} c^3 \cdot (-1)^{N+c} \Big) e^{\pi i N z}$$

Theta series and Eisenstein series are *opposites*, in construction and in Fourier expansions.

The N^{th} Fourier coefficient of the theta series is the *representation number* $\nu_8(N)$, the number of ways to express N as a sum of 8 squares of integers. In particular, an integer.

The N^{th} Fourier coefficient of the Eisenstein series involves $\zeta(4)/\pi^4$ and sums-of-divisors.

For example, with $N = 1$,

$$16 \;=\; \nu_8(1) \;=\; \frac{(2\pi)^4}{3!\zeta(4)(2^4-1)}$$

Thus,

$$\zeta(4) \;=\; \frac{(2\pi)^4}{3!(2^4-1)\cdot 16} \;=\; \frac{\pi^4}{90}$$

Oppositely, for p an odd prime,

$$\nu_8(p) \;=\; 16 \cdot \sum_{0<c|p} c^3 \cdot (-1)^{p+c} \;=\; 16 \cdot (1+p^3)$$

And, for another odd prime $q \neq p$,

$$\frac{\nu_8(pq)}{16} \;=\; \sum_{0<c|pq} c^3 \cdot (-1)^{pq+c}$$

$$=\; 1 + p^3 + q^3 + (pq)^3 \;=\; (1+p^3)(1+q^3)$$

$$=\; \frac{\nu_8(p)}{16} \cdot \frac{\nu_8(q)}{16}$$

Similarly, for relatively prime, odd m, n,

$$\frac{\nu_8(mn)}{16} \;=\; \frac{\nu_8(m)}{16} \cdot \frac{\nu_8(n)}{16}$$

None of these facts is obvious.

Another example: There do exist 8-by-8 symmetric integer matrices with determinant 1 and *even* diagonal entries:

$$Q = \begin{pmatrix} 8 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The associated theta series

$$\theta_Q(z) = \sum_{v \in \mathbb{Z}^8} e^{-\pi i (v^t Q v) z}$$

is a weight-four level-one holomorphic modular form. Thus,

$$\theta_Q(z) = E_4(z) = 1 + \frac{(2\pi)^4}{3!\zeta(4)} \sum_{n \geq 1} \sigma_3(n) e^{2\pi i n z}$$

As with θ_8, the Fourier coefficients of θ_Q are *representation numbers*

$$\nu_Q(n) \;=\; \text{card}\{v \in \mathbb{Z}^n : v^t Q v = n\}$$

Cor From the coefficient of $e^{2\pi i z}$,

$$\nu_Q(1) \;=\; \frac{(2\pi)^4}{3!\zeta(4)} \;=\; \frac{2^4}{3!} \cdot 90 \;=\; 240$$

Thus,

$$\nu_Q(n) \;=\; \frac{(2\pi)^4}{3!\zeta(4)}\,\sigma_3(n) \;=\; 240\sigma_3(n)$$

And, again, $\nu_Q(n)/240$ is *weakly multiplicative*: for relatively prime $0 < m, n \in \mathbb{Z}$,

$$\frac{\nu_Q(mn)}{240} \;=\; \frac{\nu_Q(m)}{240} \cdot \frac{\nu_Q(n)}{240}$$

Likewise, because the only weight-eight level-one holomorphic elliptic modular form is (the Eisenstein series) E_8,

$$\begin{aligned}\sum_{n\geq 0} \nu_{Q\oplus Q}(n)\, e^{\pi i n z} \;&=\; \theta_{Q\oplus Q}(z) \\ &=\; \theta_Q(z)\cdot\theta_Q(z) \\ =\; E_8(z) \;=\; 1 + \frac{(2\pi)^8}{7!\zeta(8)} &\sum_{n\geq 1} \sigma_7(n) e^{2\pi i n z}\end{aligned}$$

entailing more non-obvious identities, for example,

$$\zeta(8) \;=\; \frac{2^8\pi^8}{7!\cdot 480} \;=\; \frac{\pi^8}{9450}$$

Patterns of easy equality of theta series and Eisenstein series cannot continue simply, because there *are* holomorphic cuspforms of higher weights.

The futility of a naive hope that *all* theta series are Eisenstein series reflects the non-triviality of the precise Siegel-Weil relation.

Why are theta series modular forms?

The classical argument mirrors proofs that *local* Weil representations are representations, and that for *k-rational* quadratic forms, the *global* Weil representation has properties reflecting *global* arithmetic.

Gunning 1962 echoes the most classical argument. My *Holomorphic Hilbert Modular Forms* 1990 modernizes that argument to a degree (and might suggest revising the whole approach to *overtly* use the Weil representation).

Of course, this is an anachronistic and causality-reversing description.

To be clear, for holomorphic Siegel-Weil, the first substantive issue is that such a theta series *is* a modular form. *And* this is essentially equivalent to construction (and details) of the local and global Weil representation. The subtler, second issue is about arranging linear combinations to obtain *exactly* Eisenstein series.

A more general set-up

To simplify, consider quadratic spaces of the form $Q = Q_1 \oplus Q_1$, so that the Weil representation descends from a two-fold cover to the symplectic groups $Sp_{2n}(\mathbb{A})$.

Suppose Q is *positive-definite* at archimedean places (which then must be *real*). This entails that all the theta series and Eisenstein series correspond to *holomorphic* modular forms, for *local* Weil-representation reasons.

The essential issues already arise for $SL_2 \times O(Q)$ over $\mathbb{Q}$. The same ideas apply to $Sp_{2n} \times O(Q)$, over totally real number fields k.

The global Weil representation restricted to $Sp_{2n} \times O(Q)$ acts on the Schwartz functions φ on $Q_{\mathbb{A}} \times \mathbb{A}^n$. View the latter as $\dim Q \times n$ rectangular matrices.

The action of $h \in O(Q)_{\mathbb{A}}$ is elementary, induced from the natural linear action on $Q_{\mathbb{A}}$, on functions by $(h \cdot \varphi)(v) = \varphi(h^{-1} \cdot v)$.

$g \in Sp_{2n}(\mathbb{A})$ acts via the Weil representation, defined in pieces: using the simplifying assumptions on Q, with standard additive character ψ on $\mathbb{A}/k$,

$$\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \varphi(v)$$

$$= \ \chi_Q(\det a) \cdot |\det a|^{\frac{1}{2}\dim Q} \cdot \varphi(v \cdot a)$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \varphi(v) \ = \ \psi(\tfrac{1}{2}\mathrm{tr}\big(Q(v) \cdot x)\big) \cdot \varphi(v)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \varphi(v)$$

$$= \ \chi_Q(-1) \cdot \widehat{\varphi}(v) \quad \text{(locally everywhere)}$$

For present purposes, a *theta kernel* Φ_φ is a function of $g \in Sp_{2n}(\mathbb{A})$:

$$\Phi_\varphi(g) \ = \ \sum_{v \in Q_k \otimes k^n} (g \cdot \varphi)(v)$$

The *theta series* in the Siegel-Weil formula is

$$\theta_\varphi(g) \ = \ \int_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} \Big(\sum_{v \in Q_k \otimes k^n} (g \cdot \varphi)(h^{-1} v) \Big) \, dh$$

In fact, especially because $O(Q)_k \backslash O(Q)_{\mathbb{A}}$ is *compact*, the integral easily passes inside the sum:

$$\int_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} \Big(\sum_{v \in Q_k \otimes k^n} (g \cdot \varphi)(h^{-1}v) \Big) \, hd$$

$$= \sum_{v \in Q_k \otimes k^n} \int_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} (g \cdot \varphi)(h^{-1}v) \, dh$$

$$= \sum_{v \in Q_k \otimes k^n} (g \cdot \widetilde{\varphi})(h^{-1}v) \; = \; \Phi_{\widetilde{\varphi}}(g)$$

where

$$\widetilde{\varphi}(g) \; = \; \int_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} h \cdot \varphi \, dh$$

That is, with compact $O(Q)_k \backslash O(Q)_{\mathbb{A}}$,

$$\theta_\varphi \; = \; \Phi_{\widetilde{\varphi}}$$

Claim: Every Φ_φ is left $Sp_{2n}(k)$-invariant.

Proof: We prove that it is left-invariant by

$$N_k \;=\; \{n_x = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : S^t = S,\ S \text{ over}\, k\}$$

by

$$M_k \;=\; \{m_a = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in GL_n(k)\}$$

and by the Weyl element

$$w \;=\; \{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A Bruhat decomposition shows that these generate $Sp_{2n}(k)$.

First, in fact, each summand $(g \cdot \varphi)(v)$ for v in $Q_k \otimes k^n$ is left N_k-invariant:

$$((n_x g) \cdot \varphi)(v) \;=\; (n_x \cdot (g \cdot \varphi))(v)$$

$$=\; \psi(\tfrac{1}{2}\mathrm{tr}\big(Q(v) \cdot x\big)) \cdot (g \cdot \varphi)(v) \;=\; 1 \cdot (g \cdot \varphi)(v)$$

because ψ is trivial on k.

Second,

$$\sum_v ((m_a g)\cdot\varphi)(v) \;=\; \sum_v (m_a\cdot(g\cdot\varphi))(v)$$

$$=\; \chi_Q(\det a)\cdot|\det a|^{\frac{1}{2}\dim Q}\sum_v (g\cdot\varphi)(vm)$$

$$=\; 1\cdot\sum_v (g\cdot\varphi)(v)$$

by the product formula, by the fact that $GL_n(k)$ stabilizes $Q_k\otimes k^n$, and by the fact that χ_Q is a Hecke character.

Last,

$$\sum_v ((wg)\cdot\varphi)(v) \;=\; \sum_v (w\cdot(g\cdot\varphi))(v)$$

$$=\; \chi_Q(-1)\cdot\sum_v (\widehat{g\cdot\varphi})(v) \;=\; \sum_v (g\cdot\varphi)(v)$$

by Poisson summation, since χ_Q is a Hecke character. ///

On the other hand, Siegel-parabolic Eisenstein series E_f on Sp_{2n}, holomorphic or not, are attached to functions f on Sp_{2n} left-invariant by

$$N_{\mathbb{A}} \;=\; \{\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : S^t = S,\; S \text{ adelic}\}$$

and by

$$M_k \;=\; \{\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in GL_n(k)\}$$

Under various hypotheses assuring convergence,

$$E_f(g) \;=\; \sum_{\gamma \in N_k M_k \backslash Sp_{2n}(k)} f(\gamma \cdot g)$$

The relevant $f = f_\varphi$ for Siegel-Weil is

$$f_\varphi(g) \;=\; (g \cdot \varphi)(0)$$

Siegel-Weil (classical, holomorphic)

Given n, for $\dim Q$ sufficiently large,

$$\theta_\varphi \;=\; E_{f_\varphi} \qquad\qquad (\text{with } f_\varphi(g) = (g\cdot\varphi)(0))$$

Expanded:

$$\int\limits_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} \sum_v (g\cdot\varphi)(h^{-1}v)\; dh$$

$$= \sum_{\gamma \in N_k M_k \backslash Sp_{2n}(k)} \big(\gamma g \cdot \varphi\big)(0)$$

Cor Equality of 0^{th} Fourier coefficients is the **Siegel Mass Formula**.

Note By the positive-definiteness of Q at archimedean places, the integral over $O(Q)_k \backslash O(Q)_{\mathbb{A}}$ is actually a *finite sum*, weighted by various volumes:

Proof: Generally, when H_k is globally anisotropic, $H_k \backslash H_{\mathbb{A}}$ is *compact.* For classical groups, this is Mahler's criterion: Godement's Sem. Bourbaki talk on Reduction Theory, 1963.

For orthogonal groups $O(Q)$, *global* anisotropy is non-solvability of $Q(x) = 0$ for non-zero $x \in Q_k$. Certainly this is implied by *local* anisotropy at any completion k_v, meaning $Q(x) = 0$ has no non-zero solutions $x \in Q_{k_v}$.

Hasse-Minkowski is the converse!

For $\dim Q > 4$, there are no p-adic anisotropic quadratic forms, so global anisotropy occurs exactly for anisotropy at some archimedean place. This does not happen at complex places, so there must be a *real* place where the form Q is positive-definite or negative-definite.

When the archimedean factors of $H_{\mathbb{A}}$ are all compact,

$$H_k \backslash H_{\mathbb{A}} / K_{\mathbb{A}} \;\approx\; (H_k)_{\text{fin}} \backslash H_{\text{fin}} / K_{\text{fin}}$$

with the projection of H_k to non-archimedean factors, and with finite-prime adele groups. The quotient $(H_k)_{\text{fin}} \backslash H_{\text{fin}}$ inherits compactness.

Since K_{fin} is (compact and) *open*, that further quotient

$$(H_k)_{\text{fin}} \backslash H_{\text{fin}} / K_{\text{fin}} \;=\; \text{compact/open}$$

is *finite*. ///

Sketch of Proof of Siegel-Weil:

Positive-definiteness at archimedean places *greatly* simplifies the argument, but is inessential.

Local computation shows that θ_φ generates *holomorphic discrete series* at archimedean places (classical avatar a *holomorphic* modular form). This is worth some attention:

Up to $\mathbb{R}$-isomorphism, the positive-definite form is $Q(v_1, \ldots, v_{2\ell}) = \sum_j v_j^2 = |v|^2$. We use the Lie algebra version of the representation, best referenced as Segal-Shale-Weil. In $\mathfrak{sl}_2$, as usual, let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Historically backward) differentiating the Weil representation, these Lie algebra elements act on functions on $\mathbb{R}^{2\ell}$ by

(up to normalizing constants)

$$X \;\to\; \text{multiplication by } \frac{|v|^2}{2}$$

$$Y \;\to\; \frac{\Delta}{2} \qquad\qquad H \;\to\; \ell + \sum_{j=1}^{2\ell} v_j \frac{\partial}{\partial v_j}$$

To determine possible *principal series* representations I_χ to which this has non-trivial maps, we'd compute its Jacquet module, namely, X-cofixed vectors, due to the Frobenious adjunction

$$\mathrm{Hom}_{\mathfrak{sl}_2}(V, I_s) \;\approx\; \mathrm{Hom}_{\mathfrak{m}}(V_{\mathfrak{n}}, |\cdot|^s)$$

with $\mathfrak{n}$ the Lie algebra of N, $\mathfrak{m}$ that of M.

To simplify, dualize to consider *fixed* vectors rather than *co-fixed*: consider the *tempered distributions* $\mathscr{S}^*$. Those fixed/annihilated by multiplication by $|v|^2$ are supported at 0. Thus, δ and its derivatives $\partial^\alpha \delta$ such that $\widehat{\partial^\alpha \delta}$ is a harmonic polynomial.

By Euler's identity,

$$\Big(\ell + \sum_{j=1}^{2\ell} v_\ell \frac{\partial}{\partial v_j}\Big)\partial^\alpha \delta \;=\; \big(\ell - (2\ell + |\alpha|)\big) \cdot \partial^\alpha \delta$$

$$= \; -(\ell + |\alpha|) \cdot \partial^\alpha \delta$$

Un-dualizing (at the level of characters, not principal series), vectors in the archimedean Weil representation map *at most* to principal series induced from

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \to a^{\ell + |\alpha|} = a^{\frac{1}{2}\dim Q + |\alpha|} \qquad (\text{for } a > 0)$$

These non-unitarizable principal series are the ones that contain holomorphic discrete series as subrepresentations. The $(\ell + |\alpha|)^{th}$ holomorphic discrete series has lowest K_∞ type $\ell + |\alpha|$.

The intertwining $I_s \to I_{1-s}$ by (analytic continuation of)

$$f \longrightarrow \int_N f(wn_x g)\, dx$$

is generically an isomorphism. For $s = \ell+|\alpha|$, its kernel is the holomorphic discrete series of that weight. Since the Weil representation has no non-zero map to $I_{1-(\ell+|\alpha|)}$, its image in $I_{\ell+|\alpha|}$ must be inside the holomorphic discrete series.

In fact, the Weil representation has images

$$H_d \otimes \pi_{d+\frac{1}{2}\dim Q}$$

as $O(2\ell) \times SL_2(\mathbb{R})$ representation, for homogeneous harmonic polynomials H_d of degree d, and holomorphic discrete series $\pi_{d+\frac{1}{2}\dim Q}$ with lowest weight $d + \frac{1}{2}\dim Q$, for all d.

This entails considerable simplification for the sequel.

A *global* idea is that $f_\varphi(g) = (g \cdot \varphi)(0)$ is the *constant term*

$$c_P\theta_\varphi(g) \;=\; \int_{N_k\backslash N_{\mathbb{A}}} \theta_\varphi(ug)\,du \;=\; f_\varphi(g)$$

of θ_φ along the Siegel parabolic $P = NM$. Since the summands in θ_φ and Φ_φ are N_k-invariant, we can compute this constant term summandwise:

$$\int\limits_{N_k\backslash N_{\mathbb{A}}} (n_x g\cdot\varphi)(v)\,dx \;=\; \int\limits_{N_k\backslash N_{\mathbb{A}}} (n_x\cdot(g\cdot\varphi))(v)\,dx$$

$$=\; \int\limits_{N_k\backslash N_{\mathbb{A}}} \psi(\tfrac{1}{2}\mathrm{tr}\big(Q(v)\cdot x\big))\;(g\cdot\varphi)(v)\,dx$$

$$=\;(g\cdot\varphi)(v)\cdot \int\limits_{N_k\backslash N_{\mathbb{A}}} \psi(\tfrac{1}{2}\mathrm{tr}\big(Q(v)\cdot x\big))\,dx$$

The integral is 0 unless the character on $N_k\backslash N_{\mathbb{A}}$ is trivial, which is exactly for $Q(v) = 0$.

Since Q is globally anisotropic (positive-definite at least one real place), this is exactly for $v = 0$. Thus, the constant-term contribution of the v^{th} summand is 0 unless $v = 0$, in which case it is $(g \cdot \varphi)(0) = f_\varphi(g)$.

$O(Q)_\mathbb{A}$ stabilizes 0, so integration over $O(Q)_k \backslash O(Q)_\mathbb{A}$ does not change the constant term. But this integration *does* assure, via local representation theory, that the resulting $\theta_\varphi = \Phi_{\widetilde{\varphi}}$ is the lowest-K-type vector in a holomorphic discrete series.

In general, for an SL_2 automorphic form F with constant term F_0, the difference $F - E_{F_0}$ is *not* a cuspform. But in the holomorphic situation (for lowest-K-type) this *does* hold. So $\theta_\varphi - E_{f_\varphi}$ is a cuspform. We first sketch the rest of the proof for SL_2, then look at the complications for Sp_{2n}, still in the holomorphic case.

The second idea is that θ_φ and E_{f_φ} generate the same *principal series* representation *locally* at almost all finite primes. Thus, their difference locally generates that representation.

For large $\dim Q$, the non-archimedean principal series are in a range where the principal series is *irreducible*, and *outside* the unitarizability range.

Holomorphic cuspforms are square-integrable, so local representations generated by them are unitary/unitarizable. Thus, $\theta_\varphi - E_{f_\varphi} \neq 0$ would generate a principal series at almost all finite places, outside the unitarizable range, but required to be unitary by cuspidality. Contradiction. So the difference is 0. This completes the sketch for SL_2. ///

Note: At archimedean places, the holomorphic discrete series (which are unitarizable) *do* occur as subrepresentations of principal series outside the unitarizable range. So high-weight holomorphic Eisenstein series *locally* generate unitarizable representations at archimedean places, but *not* at finite primes.

For *holomorphic* Siegel modular forms, *Klingen 1967* says: for large-enough weight, every holomorphic Siegel modular form F is

$$F \;=\; c \cdot E_{2k}^{Sp_{2n}} + E_{f_1}^{Sp_{2n}} + E_{f_2}^{Sp_{2n}} + \ldots + E_{f_{n-1}}^{Sp_{2n}} + f_n$$

with f_m's holomorphic cuspforms on Sp_{2m}, and Eisenstein series induced from cuspforms f (*Klingen-type* Eisenstein series).

In the more general case, $\theta_\varphi - E_{f_\varphi}$ has leading term 0 in the Klingen expansion. If non-zero, it would generate a principal series at almost all finite places, outside the unitarizable range.

Klingen-type holomorphic Eisenstein series are made from holomorphic cuspforms, which are L^2, and therefore generate unitarizable representations locally. The Eisenstein series of large weight formed from cuspforms locally generate representations at finite places that are not only not-unitarizable, but also in a different parameter range from the Siegel-type Eisenstein series.

Thus, $\theta_\varphi - E_{f_\varphi} = 0$. ///

Fancier examples of arithmetic content

It has been known for a long time (since 1979 at least) that Eisenstein series decompose under restriction as

$$E_{2k}^{Sp_4}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$$

$$= E_{2k}(z) \cdot E_{2k}(w) + \sum_{\text{cfm } f} c_f \cdot f(z)\, f(w)$$

summed over an orthogonal basis for cuspforms, where c_f is a ratio of an L-function value and a Petersson norm.

Expression of $E_{2k}^{Sp_4}$ as a linear combination of theta series implies *at least* that the Fourier coefficients of $E_{2k}^{Sp_4}$ are rational with bounded denominators.

Some linear algebra implies that holomorphic cuspforms f have Fourier coefficients and Petersson norms and L-function values with very nice rationality properties.

The extension of this restriction formula to $Sp_{2m} \times Sp_{2n} \to Sp_{2m+2n}$, with $m \leq n$ is (for $z \in \mathfrak{H}_m$, $w \in \mathfrak{H}_n$), is

$$E^{Sp_{2m+2n}} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} = E_{2k}^{Sp_{2m}}(z) \cdot E_{2k}^{Sp_{2m}}(w)$$

$$+ \sum_{\text{cfm } f \text{ on } SL_2} c_f \cdot E_f^{Sp_{2m}}(z) E_f^{Sp_{2n}}(w)$$

$$+ \sum_{\text{cfm } f \text{ on } Sp_4} c_f \cdot E_f^{Sp_{2m}}(z) E_f^{Sp_{2n}}(w)$$

$$\ldots + \sum_{\text{cfm } f \text{ on } Sp_{2m}} c_f \cdot f(z)\, E_f^{Sp_{2n}}(w)$$

Combining this with rationality properties of $E_{2k}^{Sp_{2m+2n}}$ directly implies that Siegel modular varieties are defined over number fields, etc.

Shimura's 1970 proofs of field-of-definition required substantial algebraic geometry of moduli spaces (*canonical models*) of abelian varieties.

The combination of the restriction/pullback formula with holomorphic Siegel-Weil also implies that Klingen-type Eisenstein series have Fourier coefficients with good rationality and Galois properties. (Harris 1981 proved this in a different way, akin to part of the argument here for Siegel-Weil.)

Similarly, algebraicity/Galois properties of normalized values of certain L-functions on Sp_{2n}'s and related classical groups originally required substantial algebraic geometry.

Further, *integrality properties* of both definition of the Siegel modular varieties and special values of L-functions follow with the decomposition formula and Siegel-Weil, not just from the canonical models viewpoint.

The restriction formula and Siegel-Weil also resolve *the basis problem*, expressing holomorphic Siegel modular forms as theta series. (Böcherer, PiatetskiShapiro-Rallis, *et al*)