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# Gauges on groups, convergence arguments

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This discussion is sufficient for many convergence arguments. There are similar, more complicated arguments possible in *relative* situations, where a function is already invariant under a subgroup of the discrete group $\Gamma$. It may be clearest to give the basic arguments first, without complications.

Let $G$ be a countably-based, locally compact, Hausdorff, unimodular topological group. Fix a compact subgroup $K$ of $G$.

**[1.1] Definition** A *gauge* $g \to \|g\|$ on $G$ is a positive real-valued *continuous* function on $G$ with properties

- $\|e\| = 1$, $\|g\| \geq 1$ and $\|g^{-1}\| = \|g\|$
- *Submultiplicativity*: $\|gh\| \leq \|g\| \cdot \|h\|$
- $K$-*invariance*: for $g \in G$, $k \in K$, $\|k \cdot g\| = \|g \cdot k\| = \|g\|$
- *Integrability*: for some $\sigma_o \geq 0$

$$\int_G \|g\|^{-\sigma}\, dt \;<\; +\infty \qquad\qquad (\text{for } \sigma > \sigma_o)$$

Associated to a gauge is a *metric* $d(,)$ on $G$,

$$d(x,y) \;=\; \log \|x^{-1}y\|$$

**[1.2] Summability and convergence** For a *discrete* subgroup $\Gamma$ of $G$, we claim the corresponding *summability*:

$$\sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^{\sigma}} \;<\; +\infty \qquad\qquad (\text{for all } \sigma > \sigma_o)$$

The proof is as follows. From

$$\|\gamma \cdot g\| \;\leq\; \|\gamma\| \cdot \|g\|$$

for $\sigma > 0$

$$\frac{1}{\|\gamma\|^{\sigma} \cdot \|g\|^{\sigma}} \;\leq\; \frac{1}{\|\gamma\cdot g\|^{\sigma}}$$

Invoking the discreteness of $\Gamma$ in $G$, let $C$ be a small open neighborhood of $1 \in G$ such that

$$C \cap \Gamma \;=\; \{1\}$$

Then,

$$\int_C \frac{dg}{\|g\|^{\sigma}} \cdot \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^{\sigma}} \;\leq\; \int_C \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\cdot g\|^{\sigma}}\, dg \;=\; \sum_{\gamma\in\Gamma} \int_{\gamma^{-1}C} \frac{1}{\|g\|^{\sigma}}\, dg \;\leq\; \int_G \frac{dg}{\|g\|^{\sigma}} \;<\; +\infty$$

This gives the indicated summability.

**[1.3] Moderate growth, sufficient decay** A complex-valued function $f$ on $G$ is of *moderate growth* when

$$|f(g)| \;\ll\; \|g\|^{\sigma} \qquad\qquad (\text{for some } \sigma > 0)$$

If $f$ is of moderate growth with exponent $\sigma$, it is of moderate growth of exponent for all $\sigma' \leq \sigma$, as well. The function $f$ is *rapidly decreasing* when

$$|f(g)| \;\ll\; \|g\|^{-\sigma} \qquad\qquad (\text{for } \textit{all } \sigma > 0)$$

The function $f$ is *sufficiently rapidly decreasing* (for a given purpose) when

$$|f(g)| \ll \|g\|^{-\sigma} \qquad \text{(for } \textit{some} \text{ sufficiently large } \sigma > 0)$$

**[1.4] Pointwise convergence, moderate growth of Poincaré series** We claim that, for $f$ sufficiently rapidly decreasing, the *Poincaré series*

$$P_f(g) \;= \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

*converges absolutely and uniformly* on compacts, and is of moderate growth. Indeed, for $\sigma > 0$, from

$$\|\gamma\| \;=\; \|\gamma\cdot g\cdot g^{-1}\| \;\le\; \|\gamma\cdot g\|\cdot\|g^{-1}\| \;=\; \|\gamma g\|\cdot\|g\|$$

we have

$$\frac{1}{\|\gamma\cdot g\|^\sigma} \;\le\; \frac{\|g\|^\sigma}{\|\gamma\|^\sigma} \qquad \text{(for } \sigma > 0)$$

Then we have the moderate growth estimate:

$$P_f(g) \;= \sum_{\gamma\in\Gamma} f(\gamma\cdot g) \;\ll \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\cdot g\|^\sigma} \;\le\; \|g\|^\sigma \cdot \sum_{\gamma\in\Gamma} \frac{1}{\|\gamma\|^\sigma} \;=\; \|g\|^\sigma \qquad \text{(for all } \sigma > \sigma_o)$$

**[1.5] Square integrability of Poincaré series** Next, we claim that for $f$ sufficiently rapidly decreasing $P_f$ is *square-integrable* on $\Gamma\backslash G$. Specifically, assume that

$$|f(g)| \;\ll\; \|g\|^{-2\sigma} \qquad \text{(for some } \sigma > \sigma_o)$$

Unwind, and use the assumed estimate on $f$ along with the above-proven moderate growth of the Poincaré series:

$$\int_{\Gamma\backslash G} |P_f|^2 \;=\; \int_G |f|\cdot|P_f| \;\ll\; \int_G \|g\|^{-2\sigma}\cdot\|g\|^\sigma\, dg \;=\; \int_G \|g\|^{-\sigma}\, dg \;<\; +\infty$$

since $\sigma > \sigma_o$. This proves the square integrability of the Poincaré series.

**[1.6] Existence of gauges on reductive groups** First, we construct a gauge on $GL(n, k_v)$ for every local field $k_v$ of characteristic 0. Second, a general reductive group $G$ over a local field $k_v$ acts on its own rational Lie algebra $\mathfrak{g}$, giving a mapping of $G$ modulo its center to the general linear group of $\mathfrak{g}$. Restriction of the gauge on $GL_{k_v}(\mathfrak{g})$ to the image of $G_v$ gives a gauge on $Z_v\backslash G_v$.

On a $k_v$-vectorspace $V = k_v^n$ with a basis, the usual norms are

$$|(x_1,\ldots,x_n)| \;=\; \begin{cases} (|x_1|^2+\ldots+|x_n|^2)^{1/2} & \text{(for } v \text{ archimedean)} \\ \\ \sup_i |x_i|_v & \text{(for } k_v \text{ non-archimedean)} \end{cases}$$

Define an operator norm $|\cdot|$ on $GL(n, k_v)$ by

$$|g| \;=\; \sup_{|x|\le 1} |g\cdot x|$$

Then $\|g\|$ is

$$\|g\| \;=\; \max\big(|g|,\, |g^{-1}|\big)$$

It is clear that $\|e\| = 1$ and $\|g\| = \|g^{-1}$. The sub-multiplicativity follows from the submultiplicativity of the operator norm, namely,

$$\|g \cdot h\| \;=\; \|g\| \;=\; \max\big(|gh|,\, |(gh)^{-1}|\big) \;\leq\; \max\big(|gh|, |gh^{-1}|, |(gh)^{-1}|, |(gh^{-1})^{-1}|\big)$$

$$\leq\; \max\big(|g||h|,\, |g||h^{-1}|,\, |h^{-1}||g^{-1}|,\, |h||g^{-1}|\big) \;\leq\; \max(|g|, |g^{-1}|) \cdot \max(|h|, |h^{-1}|) \;=\; \|g\| \cdot \|h\|$$

Clearly the gauge on $GL_n(k_v)$ is left and right invariant under the standard compact groups $K_v = O(n, \mathbb{R})$ or $K_v = U(n)$ in the real and complex cases, and invariant under $K_v = GL_n(\mathfrak{o}_v)$ in the non-archimedean case.

Given a reductive group $G$ over a local field $k_v$ of characteristic 0, the Adjoint action of $G$ on the $k_v$-rational Lie algebra $\mathfrak{g}$, together with a choice of $k_v$-basis for $\mathfrak{g}$, gives a $k_v$-homomorphism $\rho : G \to GL_n$. For quasi-split $G$, there is a choice of basis for $\mathfrak{g}$ so that a maximal $k_v$-split torus maps to diagonal elements of $GL_n(k_v)$, and so that

$$\rho^{-1}\big(\rho(G) \cap K_v\big) \;=\; Z_v \cdot \big((\text{good}) \text{ maximal compact in } G\big)$$

Generally, in the non-archimedean case, given a gauge only bi-invariant by a finite-index subgroup of the desired compact $K$ of $G$, the integral

$$\|g\|_{\text{new}} \;=\; \int_{K \times K} \|k_1 \cdot g \cdot k_2\| \, dk_1 \, dk_2$$

produces a new gauge with the desired bi-invariance.

The *integrability condition* is verified for such gauges by integrating in Cartan coordinates $K \cdot A \cdot K$ modulo the center $Z$, where $A$ is a maximal $k_v$-split torus in $G$, and choosing an imbedding $\rho$ by choosing a basis of $\mathfrak{g}$ consisting of weight vectors for $A$. For Lie groups, such computations appear in Cartan and Weyl. For p-adic groups, the argument is potentially more elementary.

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