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Self-adjoint operators on automorphic forms

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Informal report [1] *on joint work with E. Bombieri. Details will appear elsewhere.*

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We give a rigorous setting allowing precise statements and proofs illustrated by the following simple examples:

• Extrapolating and refining comments at the end of [CdV 1983]: there is a Hilbert space $\mathfrak{E}^0$ of automorphic forms such that, given a complex quadratic field $k = \mathbb{Q}(\sqrt{d})$, there is a natural unbounded operator $\widetilde{\Delta}_d$ on $\mathfrak{E}^0$, an extension of a restriction of the invariant Laplacian Δ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, whose discrete spectrum, *if any*, is of the form $\lambda_s = s(s-1)$ for $\zeta_k(s) = 0$.

• *No low-hanging fruit:* at most 94% of the zeros s of $\zeta(s)$ appear as eigenvalues $s(s-1)$ in the spectrum of $\widetilde{\Delta}_d$. The proof uses *exotic eigenfunction expansions* in the Lax-Phillips space of pseudo-cuspforms (below), the relatively regular behavior of $\arg\zeta(s)$ on $\Re(s) = 1$, and Montgomery's pair correlation conjecture.

• *Spacing of zeros.* For complex quadratic fields $k = \mathbb{Q}(\sqrt{d})$ with $d < 0$, *half* the on-line zeros w_o of

$$J(w) \;=\; \frac{h_d^2}{-\lambda_w \cdot \langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \Big|\frac{\zeta_k(s)}{\zeta(2s)}\Big|^2 - \Big|\frac{\zeta_k(w)}{\zeta(2w)}\Big|^2 \;\; \frac{ds}{\lambda_s - \lambda_w}$$

(depending on sign of derivative) *repel upward* the on-line zeros s_o of $\zeta_k(s)$, in the sense that, given $\varepsilon > 0$, there is sufficiently large T such that above a zero w_o of $J(w)$ with $\Im(w_o) \geq T$ the *next* on-line zero s_o of $\zeta_k(s)$ must satisfy

$$\Im(s_o) - \Im(w_o) \;\geq\; (\tfrac{1}{2} - \varepsilon) \cdot \text{average spacing} \;\geq\; (\tfrac{1}{2} - \varepsilon) \cdot \frac{\pi}{\log T}$$

[1] This is an informal, more detailed version of a talk given in Bristol, UK on June 4, 2018, in the conference *Perspectives on the Riemann Hypothesis*, hosted by the Heilbronn Institute, organized by B. Conrey, J. Keating, P. Sarnak, and A. Wiles. My talk was the first half of a two-part talk together with E. Bombieri concerning our on-going joint work. My talk and these notes primarily address analytical details. Bombieri's talk and notes give more of the number theory details. A version of this document is also at http://www.math.umn.edu/~garrett/m/v/bristol_2018.pdf

1. *Perspective and context*

As a Diplomarbeit, [Haas 1977] attempted a numerical determination of the eigenvalues of the invariant Laplacian $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on $\Gamma\backslash\mathfrak{H}$ with $\Gamma = SL_2(\mathbb{Z})$, parametrizing the eigenvalues as $\lambda_s = s(s-1)$. A copy of the list of spectral parameters s was sent to A. Terras in San Diego in early 1979. She showed the list to H. Stark and D. Hejhal. [2] Terras noted to Hejhal that Stark had noticed the lowest zero of $\zeta(s)$ in the list, and that they (Terras and Stark) had requested a copy of the Diplomarbeit, but only an empty envelope arrived. Hejhal compared the list to a list of zeros of L-functions in the Scripps Oceanographic Library, and noticed coincidences with zeros of $L(s, \chi_{-3})$, with the quadratic character of conductor 3.

Of course, if $s(s-1)$ is real and non-positive, either $\Re(s) = \frac{1}{2}$ or $s \in [0,1]$. To show that $s(s-1)$ is real an non-positive, it suffices to show that it is an eigenvalue of a non-positive, self-adjoint operator on a Hilbert space (clarifications below). Thus, to prove the Riemann Hypothesis it would suffice (for example) so show that there is a non-positive self-adjoint operator T on some Hilbert space so that for every zero s of $\zeta(s)$, $\lambda_s = s(s-1)$ is an eigenvalue of T. [3] Even though a numerical coincidence would not explain causality, exposure of an apparent fact would be provocative.

Naturally, Hejhal attempted to reproduce Haas' results, approximately confirmed most of the spectral parameter values, but found that the zeros of $\zeta(s)$ and $L(s, \chi_{-3})$ were exactly the discrepancy between his list and Haas'. In May 1979, Hejhal did acquire a copy of the Diplomarbeit, and realized that, in terms of numerical procedures, Haas had misapplied the Henrici collocation method [Fox-Henrici-Moler 1967]. In effect, Haas had solved equations $(\Delta - \lambda_s)u = A \cdot \delta_\omega^{\text{afc}}$ with varying constants, where $\omega = e^{2\pi i/3}$, and $\delta_{z_o}^{\text{afc}}$ is the $SL_2(\mathbb{Z})$-periodic automorphic Dirac δ at z_o. In particular, for $A \neq 0$, this is not a *homogeneous* equation, so it does not follow that $\lambda_s \in \mathbb{R}$.

Several things were known about solutions u_s to such equations, being instances of automorphic Green's functions, then recently investigated in [Elstrodt 1973], [Neunhöffer 1973], [Fay 1979], and others. For example, it was known that the constant term of u_s is eventually (for large y),

$$c_P u_s(iy) \;=\; \int_0^1 u_s(x+iy)\,dx \;=\; \frac{y^{1-s}\,E_s(\omega)}{2s-1}$$

[2] In addition to information about this episode from [Hejhal 1981], my Minnesota colleague D. Hejhal gave me further details in [Hejhal 2015].

[3] Such ideas are sometimes refered to as the *Polya-Hilbert conjecture*. See [Odlyzko 1981/2].

Also, it had been long known that

$$E_s(\omega) \;=\; \Big(\frac{\sqrt{3}}{2}\Big)^{s/2} \cdot \frac{\zeta(s)\,L(s,\chi_{-3})}{\zeta(2s)}$$

While these connections were and are provocative, there is no obvious, intuitive bridge to eigenvalue equations for self-adjoint operators.

Yet there was precedent for a seemingly magical conversion of certain inhomogeneous equations $(\Delta-\lambda_s)u=\theta$ to homogeneous equations $(\widetilde{\Delta}_\theta-\lambda_s)u=0$ for self-adjoint operators $\widetilde{\Delta}_\theta$, with the same λ_s and the same u, in [Lax-Phillips 1976] and [CdV 1981]. Namely, for $a>1$, let

$$L^2_a(\Gamma\backslash\mathfrak{H}) \;=\; \{f\in L^2(\Gamma\backslash\mathfrak{H}) : c_Pf(iy)=0 \text{ for } y\ge a\}$$

be the space of *pseudo-cuspforms* with cut-off height a,

$$\Delta_a \;=\; \Delta\Big|_{C^\infty_c(\Gamma\backslash\mathfrak{H})\cap L^2_a(\Gamma\backslash\mathfrak{H})}$$

and η_a the *distribution* defined by $\eta_af=c_Pf(ia)$. Then the *Friedrichs self-adjoint extension* $\widetilde{\Delta}_a$ of Δ_a (described in the sequel) is partly characterized by

$$\widetilde{\Delta}_au \;=\; f \qquad\Longleftrightarrow\qquad \Delta u=f+b\cdot\eta_a \;\text{ and }\; \eta_au=0 \;\text{ with some constant } b$$

Implicit in this is that u is in a *global automorphic Sobolev space*

$$H^1(\Gamma\backslash\mathfrak{H}) \;=\; \text{completion of } C^\infty_c(\Gamma\backslash\mathfrak{H}) \text{ with respect to } |f|^2_{H^1}=\langle(1-\Delta)f,f\rangle_{L^2}$$

Thus, [4]

$$(\widetilde{\Delta}_a-\lambda_w)u \;=\; 0 \qquad\Longleftrightarrow\qquad (\Delta-\lambda_w)u=b\cdot\eta_a \;\text{ and }\; \eta_au=0 \;\text{ with some constant } b$$

That is, the inhomogeneous equation $(\Delta-\lambda_w)u=\eta_a$ with $u\in L^2_a(\Gamma\backslash\mathfrak{H})$ is converted into a homogeneous equation $(\widetilde{\Delta}_a-\lambda_w)u=0$, with an additional *boundary condition* $\eta_au=0$. If the differential equation $(\Delta-\lambda_w)u=\eta_a$ holds, then one finds (see below) that $u\in H^1(\Gamma\backslash\mathfrak{H})$.

Among other things, [Lax-Phillips 1976] essentially showed that the space of pseudo-cuspforms $L^2_a(\Gamma\backslash\mathfrak{H})$ decomposes *discretely* for $\widetilde{\Delta}_a$, by proving the Rellich-type lemma that $H^1_a\to L^2_a$ is *compact*, where the source is given the finer H^1 topology. (The inverse $(1-\widetilde{\Delta}_a)^{-1}:L^2\to H^1$ is continuous when H^1 has its finer topology.) Naturally, in the perception of many number theorists, as opposed to that of expert analysts, there are many non-trivial details omitted. On the face of it, this discreteness would seem to

[4] The space L^2_a can be characterizaed as the orthogonal complement to the space Θ of pseudo-Eisenstein series $\Psi_\varphi(z)=\sum_{\Gamma_\infty\backslash\Gamma}\varphi(\Im\gamma z)$ with $\varphi\in C^\infty_c(a,\infty)$. One can show that the closure $\overline{\Theta}$ of this space in $H^{-1}=(H^1)^*$ includes the functional η_a, and that on $H^1\cap L^2_a$ the only non-trivial functional induced by $\overline{\Theta}$ is η_a, up to constants.

generalize earlier arguments that the space of (genuine) cuspforms decomposes discretely for Δ. However, some work must be done to show that genuine cuspforms are in the H^1-closure of $C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$. Since there are apparently no test-function cuspforms except 0, that this decomposition extends that of genuine cuspforms is not trivial.

The new proof of the meromorphic continuation of Eisenstein series for $SL_2(\mathbb{Z})$ in [CdV 1981] made essential use of the Lax-Phillips discretization, as well as some details concerning the distributional equation $(\Delta - \lambda_w)u = \eta_a$ as above. Since this meromorphic continuation was already known by many more prosaic means, the conclusion was not doubted. But the proof mechanism itself was difficult to understand for many number theorists.

In particular, even granting the non-trivial point that genuine cuspforms are indeed eigenfunctions for the pseudo-Laplacian $\widetilde{\Delta}_a$, it is extremely unclear how any part of the *continuous spectrum* in $L^2(\Gamma\backslash\mathfrak{H})$, expressible as wave packets

$$f \;=\; \frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle f, E_s\rangle \cdot E_s\, ds \qquad \text{(in an } L^2 \text{ sense)}$$

in terms of Eisenstein series

$$E_s(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im\,\gamma z)^s$$

can be discretized. And there are the incidental issues about rigorous interpretation of the pairings $\langle f, E_s\rangle$ since E_s is not in L^2, analogous to technical complications with Fourier transform on $L^2(\mathbb{R})$.

Lax-Phillips had already made the explicit point that the genuine eigenfunctions for $\widetilde{\Delta}_a$ not among cuspforms, with eigenvalues $\lambda_w < -1/4$, are exactly the truncated Eisenstein series $\wedge^a E_{s_j}$ with $a^{s_j} + c_{s_j} a^{1-s_j} = 0$, where $a^s + c_s a^{1-s}$ is the constant term of E_s, with $c_s = \xi(2s-1)/\xi(2s)$, where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$ is the completed zeta function. Generations of number theorists have wondered how these truncations, obviously not smooth since their constant terms are not smooth, could be eigenfunctions for an elliptic operator such as Δ, and, presumably, $\widetilde{\Delta}_a$. [5] For that matter, if a function with discontinuous derivative could be an eigenfunction for such an operator, why not a discontinuous function? Thus, why not *every* truncated Eisenstein series $\wedge^a E_s$? And this is obviously ridiculous. But perhaps this seeming paradox was not on anyone's front burner, both because sufficiently expert analysts did not see a problem, and because the number-theoretic consequences were known for other reasons. Unfortunately, the perceived improbability of these details led many people to rationalize that the whole argument was only a heuristic, and could not conceivably be made into a genuine proof.

[5] This author recalls intense and apparently inconclusive discussions at Stanford c. 1980 between A. Selberg and P. Cohen about this issue. P. Sarnak was also in the room. The non-smoothness of eigenfunctions issue plagued me terribly until about 2011, when by good fortune reflection on (global) automorphic Sobolev spaces clarified aspects of Friedrichs' self-adjoint extensions.

In fact, $\widetilde{\Delta}_a$ is *not* an elliptic operator. It is a subtly different thing, in that requiring $\Delta u = f + c\cdot\eta_a$ for some constant c, and $\eta_a u = 0$, is a *differential equation with boundary condition*. This is precisely analogous to Sturm-Liouville problems $(\Delta-\lambda)u = 0$ on $[a,b]$ with boundary conditions, wherein the differential operator condition holds only in the *interior*. At the endpoints, boundary conditions such as Dirichlet's $u(a) = 0 = u(b)$ make differentiability of eigenfunctions impossible a significant fraction of the time: on $[0, 2\pi]$, the eigenfunctions for the boundary-value problem are $\sin(nx/2)$ with integer n. For odd n, although these functions have one-sided derivatives at the endpoints, they cannot be smoothly extrapolated as either periodic functions or as 0 outside $[0, 2\pi]$. This is not a paradox, because the Friedrichs extension $\widetilde{\Delta}$ can be distributionally characterized on $[0, 2\pi]$ by

$$\widetilde{\Delta}u = f \quad\Longleftrightarrow\quad \Delta u = f + a\cdot\delta_0 + b\cdot\delta_{2\pi} \text{ and } \delta_0 u = 0 = \delta_{2\pi}u$$

That is, the non-smoothness of these *exotic eigenfunctions*, producing extra distributional terms, is evidently *ignored* by the Friedrichs extension of the restriction of Δ to the simultaneous kernel of δ_0 and $\delta_{2\pi}$ on the Sobolev space $H^1(\mathbb{R}//2\pi\mathbb{Z})$.

In the case of truncated Eisenstein series $\wedge^a E_{s_j}$ with $\eta_z(\wedge^a E_{s_j}) = a^{s_j} + c_{s_j}a^{1-s_j} = 0$, in fact $\wedge^s E_{s_j} \in H^1$, and

$$(\Delta - \lambda_{s_j})(\wedge^a E_{s_j}) = \text{constant}\cdot\eta_a$$

with constant depending on s_j. This fits into the distributional characterization of a Friedrichs extension. The non-smoothness is not a problem: being in H^1 is sufficient.

There is a technical obstacle to application of the previous ideas to homogenize the inhomogeneous equation $(\Delta - \lambda_w)u = \delta_\omega^{\text{afc}}$. To explain the issue, we need the spectral characterization of the (global automorphic) Sobolev spaces $H^r(\Gamma\backslash\mathfrak{H})$: this Hilbert space is the completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$ with respect to the r^{th} Sobolev norm (squared)

$$|f|^2_{H^r} \;=\; \sum_F |\langle f, F\rangle|^2\cdot(1+|\lambda_{s_F}|)^r + \frac{|\langle f,1\rangle|^2}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle f, E_s\rangle|^2\cdot(1+|\lambda_s|)^r\,ds$$

where F runs over an orthonormal basis of cuspforms. Let

$$H^\infty \;=\; \bigcap_r H^r \;=\; \lim_r H^r \qquad\text{and}\qquad H^{-\infty} \;=\; \bigcup_r H^r \;=\; \text{colim}_r H^r$$

An extension of Plancherel shows that H^r and H^{-r} are paired as mutual duals by [6]

$$\langle f, u\rangle \;=\; \sum_F \langle f, F\rangle\cdot\overline{\langle u, F\rangle} + \frac{\langle f,1\rangle\cdot\overline{\langle u,1\rangle}}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})}\langle f, E_s\rangle\cdot\overline{\langle u, E_s\rangle}\,ds$$

[6] This complex-hermitian pairing can also be adjusted to be complex bilinear, if desired, by using pointwise complex conjugation.

This equality holds literally at first for $f, u \in C_c^\infty(\Gamma\backslash\mathfrak{H})$, and then extending by continuity. The Friedrichs extension of restriction to the kernel of a distribution θ requires that $\theta \in H^{-1}$, but a pre-trace formula computation shows that $\delta_\omega^{\text{afc}}$ is *not* in the (global automorphic) Sobolev space $H^{-1} = (H^1)^*$, but only in $H^{-1-\varepsilon}$ for every $\varepsilon > 0$. The same is true in Euclidean spaces, already for local reasons: on $\mathbb{R}^n$, a Dirac δ is in $H^{-\frac{n}{2}-\varepsilon}$ for all $\varepsilon > 0$, but is not in $H^{-\frac{n}{2}}$.

To overcome this technical obstacle, at the very end of [CdV 1982/3] it is suggested to consider the restriction θ of $\delta_\omega^{\text{afc}}$ to a smaller Hilbert space of automorphic forms, *excluding cuspforms*. Modifying slightly the literal assertion there, as an eigenfunction expansion converging at least in $H^{-\infty}$,

$$\theta \;=\; \frac{\langle\theta, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle\theta, E_s\rangle \cdot E_s \; ds$$

and

$$|\theta|^2_{H^r} \;=\; \frac{|\langle\theta, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle\theta, E_s\rangle|^2 \cdot (1 + |\lambda_s|)^r \; ds$$

The restricted version θ of $\delta_\omega^{\text{afc}}$ has the same value on E_s. Since $\langle\theta, E_s\rangle = \zeta(1-s)L(1-s, \chi_{-3})/\zeta(2-2s)$, the second-moment bound for $\zeta(s)$ from [Hardy-Littlewood 1918], and the convexity bound for $L(s, \chi_{-3})$, show that the H^r-norm of θ is certainly finite for $r \le -1$.

That is, let $\mathfrak{E}_c^\infty$ be the collection of pseudo-Eisenstein series

$$\Psi_\varphi(z) \;=\; \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \varphi(\Im\, \gamma z)$$

with $\varphi \in C_c^\infty(0, \infty)$, and $\mathfrak{E}^r$ be the completion of $\mathfrak{E}_c^\infty$ with respect to the H^r norm. Let Δ_θ be the restriction of Δ to the kernel of θ on $\mathfrak{E}_c^\infty$, and $\widetilde{\Delta}_\theta$ its Friedrichs extension. Then, for $u \in \mathfrak{E}^1$,

$$(\Delta - \lambda_w)u = c \cdot \theta \quad \text{for some constant } c, \text{ and } \theta(u) = 0 \qquad \Longleftrightarrow \qquad (\widetilde{\Delta}_\theta - \lambda_w)u = 0$$

This successfully replaces the inhomogeneous equation with a homogeneous one, at the cost of adding the boundary condition $\theta(u) = 0$.

Using the spectral expansion of θ, at least in $\Re(w) > \frac{1}{2}$ we can solve the equation $(\Delta - \lambda_w)u = \theta$ *by division*:

$$u_w \;=\; \frac{\langle\theta, 1\rangle \cdot 1}{(\lambda_1 - \lambda_w)\cdot\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle\theta, E_s\rangle \cdot E_s \, \frac{ds}{\lambda_s - \lambda_w} \qquad \text{(converging in } H^1\text{)}$$

In that region, u_w is an H^1-valued function of w, and

$$\theta(u_w) \;=\; \frac{\langle\theta, 1\rangle^2}{(\lambda_1 - \lambda_w)\cdot\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle\theta, E_s\rangle|^2 \, \frac{ds}{\lambda_s - \lambda_w}$$

Of course, in $\Im(w) > \frac{1}{2}$ and $\Re(w) \neq 0$, necessarily $\theta(u_w) \neq 0$, because if it were to vanish then $\lambda_w = w(w-1)$ would be an eigenvalue of the self-adjoint $\widetilde{\Delta}_\theta$. If we are deceived by the apparent symmetry under $w \to 1-w$ of the integral in the expression for $\theta(u_w)$, we would think that the only possible zeros of the holomorphic function $w \to \theta(u_w)$ can be on $\Re(w) = \frac{1}{2}$ or on $[0,1]$. Thus, we would anticipate that $\widetilde{\Delta}_\theta$ has *many* eigenvalues, possibly suggesting a way to show that many zeros of $\zeta(s)$ are on-line, asymptotically.

However, the apparent symmetry of the integral under $w \to 1-w$ is illusory (see below). This possibly counter-intuitive fact is akin to the more elementary fact that the Cauchy integral

$$F(w) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dw$$

over a circle γ *does* give a holomorphic function F both inside and outside the circle, *but* that the function outside is 0, and this is not generally the analytic continuation of the function inside the circle, namely $f(w)$. Although $w \to u_w$ does meromorphically continue to $w \in \mathbb{C}$, the meromorphic continuation lies only in a larger topological vector space of functions on $\Gamma\backslash\mathfrak{H}$, not lying inside $H^{-\infty}$. In particular, $\theta(u_w) = 0$ in $\Re(w) < \frac{1}{2}$ does *not* give an eigenvalue λ_w for $\widetilde{\Delta}_\theta$ unless also $u_w \in H^1$, which occurs only at isolated points (see below).

In addition to [Hejhal 1981], [Cartier 1980/81] is a then-contemporary account of the provocative nature of the situation, explaining something of both why the Riemann Hypothesis *might* have been proven, but *was not*. Well-grounded treatment of the analytic prerequisites for the above is a significant part of the point of [Garrett 2018].

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2. *Eisenstein-Sobolev spaces, spectral expansions of distributions*

The standard $L^2(\Gamma\backslash\mathfrak{H})$ spectral expansion is

$$f \;=\; \sum_F \langle f, F\rangle \cdot F + \langle f, u_o\rangle \cdot u_o + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s)\cdot E_s\,ds \qquad\qquad (L^2\text{-sense equality})$$

where F runs over an orthonormal basis of cuspforms, u_o is a suitable constant, and $\mathcal{E}(f)$ is extended by L^2 isometry from $\mathcal{E}f(s) = \int_{\Gamma\backslash\mathfrak{H}} f \cdot E_{1-s}$ for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H}) = C_c^\infty(\Gamma\backslash G)^K$ with $G = SL_2(\mathbb{R})$ and $K = SO(2,\mathbb{R})$. There is a Plancherel theorem:

$$|f|^2_{L^2} \;=\; \sum_F |\langle f, F\rangle|^2 + |\langle f, u_o\rangle|^2 + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\mathcal{E}(f)|^2\,ds$$

Let $\mathfrak{E}_c^\infty$ be the space of pseudo-Eisenstein series

$$\Psi_\varphi(z) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\Im(\gamma z)) \;\in\; C_c^\infty(\Gamma\backslash\mathfrak{H})$$

with test-function data $\varphi \in C^\infty(0,\infty)$.

$$\{L^2 \text{ cuspforms}\}^\perp \;=\; L^2 \text{ closure } \mathfrak{E}^0 \text{ of } \mathfrak{E}^\infty_c$$

For $r \in \mathbb{R}$, the (global) Eisenstein-Sobolev space $\mathfrak{E}^r$ is the completion of $\mathfrak{E}^\infty_c$ with respect to the $\mathfrak{E}^r$-norm

$$|f|^2_{\mathfrak{E}^r} \;=\; (1+|\lambda_1|)^r \cdot |\langle f, u_o\rangle|^2 + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (1+|\lambda_s|)^r \cdot |\mathcal{E}f(s)|^2\, ds \qquad (\text{for } f \in \mathfrak{E}^\infty_c)$$

The L^2 pairing against u_o extends by continuity to $\mathfrak{E}^r$, as does $f \to \mathcal{E}f$. We have

$$\lim \mathfrak{E}^r = \mathfrak{E}^\infty \subset \ldots \subset \mathfrak{E}^1 \subset \mathfrak{E}^0 \subset \mathfrak{E}^{-1} \subset \ldots \subset \mathfrak{E}^{-\infty} = \operatorname{colim} \mathfrak{E}^r$$

Distributions $\theta \in \mathfrak{E}^{-\infty}$ have spectral expansions converging in $\mathfrak{E}^{-\infty}$:

$$\theta \;=\; \langle \theta, u_o\rangle \cdot u_o + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}\theta(s)\cdot E_s\, ds \qquad (\mathfrak{E}^{-\infty} \text{ convergence})$$

Plancherel gives the duality of $\mathfrak{E}^r$ and $\mathfrak{E}^{-r}$:

$$\langle \varphi, \psi\rangle_{\mathfrak{E}^r \times \mathfrak{E}^{-r}} \;=\; \langle \varphi, u_o\rangle \cdot \overline{\langle \psi, u_o\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}\varphi(s)\cdot \overline{\mathcal{E}\psi(s)}\, ds$$

For θ a distribution which is (the restriction of) a *compactly-supported measure*, $\mathcal{E}\theta(s) = \theta(E_{1-s})$, since Eisenstein series are continuous, and $s \to \mathcal{E}\theta(s)$ is meromorphic on $\mathbb{C}$. For example, with $\theta(f) = f(i)$,

$$\mathcal{E}\theta(1-s) \;=\; E_s(i) \;=\; \frac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta(2s)}$$

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3. *Example: solving differential equations in automorphic forms*

Since $\Delta : \mathfrak{E}^\infty_c \to \mathfrak{E}^\infty_c$ is continuous when the source is given the $\mathfrak{E}^r$ topology and the target is given the $\mathfrak{E}^{r-2}$ topology, extending by continuity gives continuous $\Delta : \mathfrak{E}^r \to \mathfrak{E}^{r-2}$ consistent with distributional differentiation, and then continuous maps $\Delta : \mathfrak{E}^\infty \to \mathfrak{E}^\infty$ and $\mathfrak{E}^{-\infty} \to \mathfrak{E}^{-\infty}$.

Extending by continuity, since Δ can be applied termwise to spectral expansions of pseudo-Eisenstein series in $\mathfrak{E}^\infty_c$, it can *always* be applied *termwise* to spectral expansions in $\mathfrak{E}^{-\infty}$:

$$\Delta\Big(\langle f, u_o\rangle \cdot u_o + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s)\cdot E_s\, ds\Big) \;=\; \langle f, u_o\rangle \cdot \Delta u_o + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s)\cdot \Delta E_s\, ds$$

$$=\; \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s)\cdot \lambda_s \cdot E_s\, ds$$

The equation $(\Delta - \lambda_w)u = \theta$ with $\theta \in \mathfrak{E}^{-\infty}$ and $\lambda_w \neq 0$ can be uniquely solved *by division* for $u \in \mathfrak{E}^{-\infty}$ by equating spectral coefficients of $(\Delta - \lambda_w)u$ and θ:

$$(\lambda_1 - \lambda_w) \cdot \langle u, u_o \rangle \;=\; \langle \theta, u_o \rangle \qquad \text{and} \qquad (\lambda_s - \lambda_w) \cdot \mathcal{E}u(s) \;=\; \mathcal{E}\theta(s)$$

This proves

Claim: There exists a solution in $\mathfrak{E}^{-\infty}$ to $(\Delta - \lambda_w)u = \theta$ *only if* $\mathcal{E}\theta(s)$ vanishes (in a strong sense) at $s = w$. For $\Re(w) > \frac{1}{2}$ and $w \neq 1$, this unique solution is given by

$$u \;=\; u_w \;=\; \frac{\langle \theta, u_o \rangle}{\lambda_1 - \lambda_w} \cdot u_o + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{\mathcal{E}\theta(s)}{\lambda_s - \lambda_w} \cdot E_s \, ds$$

The function $w \to u_w$ is a holomorphic $\mathfrak{E}^{-\infty}$-valued function in that region. ///

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4. *Meromorphic continuations of solutions of differential equations*

Let θ be the restriction of a compactly-supported real-valued measure, and suppose that $\theta \in \mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$. Thus, in $\Re(w) > \frac{1}{2}$, $u_w \in \mathfrak{E}^{1+\varepsilon}$, and

$$u_w \;=\; \frac{\langle \theta, u_o \rangle}{\lambda_1 - \lambda_w} \cdot u_o + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \mathcal{E}\theta(s)\, E_s - \mathcal{E}\theta(w)\, E_w \, \frac{ds}{\lambda_s - \lambda_w} + \mathcal{E}\theta(w)\, E_w \frac{1}{4\pi i} \int_{(\frac{1}{2})} \frac{ds}{\lambda_s - \lambda_w}$$

$$= \frac{\langle \theta, u_o \rangle}{\lambda_1 - \lambda_w} \cdot u_o + \frac{1}{4\pi i} \int_{(\frac{1}{2})} \mathcal{E}\theta(s)\, E_s - \mathcal{E}\theta(w)\, E_w \, \frac{ds}{\lambda_s - \lambda_w} \;-\; \mathcal{E}\theta(w)\, E_w \frac{1}{2(2w-1)}$$

By a Sobolev-type theorem, $\mathfrak{E}^{1+\varepsilon} \subset C^o(\Gamma\backslash\mathfrak{H})$. The integral has a canonical holomorphic extension to a neighborhood of $\Re(w) = \frac{1}{2}$, as a $C^o(\Gamma\backslash\mathfrak{H})$-valued function, and then as a meromorphic $C^o(\Gamma\backslash\mathfrak{H})$-valued function beyond.

By the functional equation of E_w, the leading term and the integral are invariant under the apparent symmetry $w \to 1 - w$. The denominator in the last term causes it to be *skew-symmetric*.

In particular, in $\Re(w) < \frac{1}{2}$, the meromorphically continued u_w is *not* in $\mathfrak{E}^{1+\varepsilon}$, but only in a larger space such as $C^o(\Gamma\backslash\mathfrak{H})$, unless $\mathcal{E}\theta(w) = 0$.

Corollary: For $\Re(w_o) = \frac{1}{2}$, if $(\Delta - \lambda_{w_o})u = \theta$ has a solution $u \in \mathfrak{E}^{-\infty}$, then $\mathcal{E}\theta(w_o) = 0$, and $u = u_{w_o}$. ///

Corollary: Despite the above meromorphic continuation, the resolvent $(T - \lambda_w)^{-1} : \mathfrak{E}^0 \to \mathfrak{E}^0$ does *not* have a meromorphic continuation to $\mathrm{Re}(w) = \frac{1}{2}$ and beyond. It *does* have a meromorphic continuation as a map $(T - \lambda_w)^{-1} : \mathfrak{E}^0 \to C^o(\Gamma\backslash\mathfrak{H})$. ///

5. *Friedrichs' self-adjoint extensions of restrictions of Laplacians*

Fix $\varepsilon > 0$. Let $\Theta \subset \mathfrak{E}^{-1+\varepsilon}$ consist of restrictions of compactly-supported real measures, and assume that

$$\mathfrak{E}^0 \;\cap\; \mathfrak{E}^{-1}\text{-closure of } \Theta \;=\; \{0\}$$

Let $\ker\Theta$ denote the simultaneous kernel of Θ as continuous linear functionals on $\mathfrak{E}^1$.

Lemma: $\ker\Theta$ is dense in $\mathfrak{E}^0$. ///

Thus, the restriction T of Δ to $\mathfrak{E}^\infty_c \cap \ker\Theta$ is symmetric and densely defined on the Hilbert space $\mathfrak{E}^0$. Friedrichs' self-adjoint extension $\widetilde{T}_\Theta$ is characterized by

$$\langle (1-\widetilde{T}_\Theta)^{-1} v, w\rangle_{\mathfrak{E}^1} \;=\; \langle v, w\rangle_{\mathfrak{E}^0} \qquad\qquad (\text{for } v\in\mathfrak{E}^0,\, w\in\mathfrak{E}^\infty_c\cap\ker\Theta)$$

With non-trivial Θ, the domain of $\widetilde{T}_\Theta$ may be strictly larger than $\mathfrak{E}^2$, that is, $\widetilde{T}_\Theta$ may fail to be *essentially self-adjoint*. Specifically,

Theorem: The domain of $\widetilde{T}_\Theta$ is $\{u\in\ker\Theta : \Delta u \in \mathfrak{E}^0 + \overline{\Theta}\}$ where $\overline{\Theta}$ is the $\mathfrak{E}^{-1}$ closure. $\widetilde{T}_\Theta u = f$ with $u\in\mathfrak{E}^1$ and $f\in\mathfrak{E}^0$ if and only if $u\in\ker\Theta$ and $\Delta u = f+\theta$ for some $\theta\in\Theta$. ///

Corollary: With $\Theta=\{\theta\}$ and $\Re(w_o)=\frac{1}{2}$, λ_{w_o} is an eigenvalue of $\widetilde{T}_\theta$ if and only if both $\mathcal{E}\theta(w_o)=0$ and $\theta(u_{w_o})=0$.

Proof: From above, if there is any eigenfunction in $\mathfrak{E}^{-\infty}$ for λ_{w_o}, then $\mathcal{E}\theta(w_o)=0$. Then u_{w_o} is in $\mathfrak{E}^{-\infty}$, in fact in $\mathfrak{E}^{1+\varepsilon}$, and $(\Delta-\lambda_{w_o})u_{w_o}=\theta$. If also $\theta(u_{w_o})=0$, then $u_{w_o}\in\ker\Theta$, so is in the domain of $\widetilde{T}_\theta$. ///

Via the pairing $\mathfrak{E}^1\times\mathfrak{E}^{-1}$,

$$\theta(u_w) \;=\; \frac{|\theta(u_o)|^2}{\lambda_1-\lambda_w} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\mathcal{E}\theta(s)|^2\,\frac{ds}{\lambda_s-\lambda_w}$$

and holomorphically continues as scalar-valued function to a neighborhood of $\Re(w)=\frac{1}{2}$, by

$$\theta(u_w) \;=\; \frac{|\theta(u_o)|^2}{\lambda_1-\lambda_w} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\mathcal{E}\theta(s)|^2 - \mathcal{E}\theta(w)\mathcal{E}\theta(1-w)\frac{ds}{\lambda_s-\lambda_w} \;-\; \mathcal{E}\theta(1-w)\mathcal{E}\theta(w)\frac{1}{2(2w-1)}$$

On $\Re(w)=\frac{1}{2}$, the leading term and the main term are *real*, and the third term is purely *imaginary*. Thus, both must vanish for $\theta(u_w)=0$. ///

6. *Constant-term and Heegner distributions*

Let $\eta_a(f) = \int_0^1 f(x+ia)\,dx$, and let θ be (the restriction of) a compactly-supported real measure, lying in $\mathfrak{E}^{-1+\varepsilon}$ for some $\varepsilon > 0$, and assume that no non-trivial linear combination of η_a and θ is in $\mathfrak{E}^0$. Let $\widetilde{T}_{a,\theta}$ be the Friedrichs extension of the restriction of Δ to $\ker\theta\cap\ker\eta_a$.

Let u_w be the meromorphically continued solution of $(\Delta - \lambda_w)u = \theta$ as above, and v_w that of $(\Delta - \lambda_w)v = \eta_a$.

The condition that a linear combination $xu_w + yv_w$ is in $\ker\theta \cap \ker\eta_a$ is

$$\begin{cases} 0 = \theta(xu_w + yv_w) = \theta(u_w)x + \theta(v_w)y \\ 0 = \eta_a(xu_w + yv_w) = \eta_a(u_w)x + \eta_a(v_w)y \end{cases}$$

which has a non-trivial solution if and only if

$$\theta(u_w)\cdot\eta_a(v_w) \;-\; \eta_a(u_w)\cdot\theta(v_w) \;=\; 0$$

From the spectral expansions and pairings, by residues,

$$\eta_a(v_w) \;=\; \frac{a^w + c_w a^{1-w}}{1-2w}\cdot a^{1-w}$$

For θ (the restriction of) the sum of Dirac deltas at the Heegner points attached to a complex quadratic field $k = \mathbb{Q}(\sqrt{d})$, and for $a \gg_d 1$,

$$\eta_a(u_w) \;=\; \theta(v_w) \;=\; \frac{\theta(E_w)a^{1-w}}{1-2w} \;=\; \frac{\zeta_k(w)}{\zeta(2w)}\frac{a^{1-w}}{1-2w}$$

Corollary: $\lambda_w < -1/4$ is in the discrete spectrum of $\widetilde{T}_{a,\theta}$ if and only if

$$\theta(u_w)\cdot\frac{a^w + c_w a^{1-w}}{1-2w}\cdot a^{1-w} \;-\; \Big(\frac{\zeta_k(w)}{\zeta(2w)}\frac{a^{1-w}}{1-2w}\Big)^2 \;=\; 0$$

///

Using the regularization and meromorphic continuation of $\theta(u_w)$,

Claim: This vanishing condition is symmetric under $w \to 1-w$. Thus, all zeros of this expression are on $\Re(s) = \frac{1}{2} \cup [0,1]$.

Proof: The symmetry is by rearranging, using the regularization, and using the functional equation of the Eisenstein series. The vanishing condition is equivalent to λ_w being an eigenvalue of $\widetilde{T}$. The Friedrichs extension $\widetilde{T}$ is non-positive self-adjoint, so any eigenvalues are non-positive. Thus, either $\Re(w) = \frac{1}{2}$ or $w \in [0,1]$. ///

7. *Exotic eigenfunction expansions*

[Lax-Phillips 1976] essentially showed that the Friedrichs extension of the restriction of Δ to the space $L^2_a(\Gamma\backslash\mathfrak{H})$ of automorphic functions with constant term vanishing above height $y = a > 1$ has *purely discrete spectrum.* The same holds for the Friedrichs extension $\widetilde{T}_{\ge a}$ of the restriction $T_{\ge a}$ of Δ to the space $L^2_a \cap \mathfrak{E}^\infty_c$ of test-function-data pseudo-Eisenstein series with constant term vanishing at heights $y \ge a$.

Let $\{f_n : n = 1, 2, \ldots\}$ be an $\mathfrak{E}^0$-orthogonal basis of eigenfunctions for $\widetilde{T}_{\ge a}$, with eigenvalues λ_{s_n}, in $\mathfrak{E}^{\frac{3}{2}+\varepsilon}$ for every $\varepsilon > 0$. Without loss of generality, take all f_n real-valued.

Let $j : \mathfrak{E}^1 \cap L^2_a \to \mathfrak{E}^1$ be the inclusion. The adjoint $j^* : \mathfrak{E}^{-1} \to (\mathfrak{E}^{-1} \cap L^2_a)^*$ is a *quotient* map:

$$\begin{array}{ccccc} \mathfrak{E}^1 & \xrightarrow{\text{inc}} & \mathfrak{E}^0 & \xrightarrow{\text{inc}} & \mathfrak{E}^{-1} \\ {\scriptstyle j}\uparrow & & & & \downarrow{\scriptstyle j^*} \\ \mathfrak{E}^1 \cap L^2_a & \xrightarrow{\text{inc}} & \mathfrak{E}^0 \cap L^2_a & \xrightarrow{\text{inc}} & (\mathfrak{E}^{-1} \cap L^2_a)^* \end{array}$$

Extend the self-adjoint operator $\widetilde{T}_{\ge a}$ to a map $\widetilde{T}^\#_{\ge a} : \mathfrak{E}^1 \cap L^2_a \to (\mathfrak{E}^1 \cap L^2_a)^*$ by

$$\widetilde{T}^\#_{\ge a}(u)(v) \;=\; \langle u, v\rangle_{\mathfrak{E}^1}$$

Take θ (the restriction of) a compactly supported real measure, with $\theta \in \mathfrak{E}^{-1}$. For $a \gg_\theta 1$, there is a spectral expansion convergent in $(\mathfrak{E}^1 \cap L^2_a)^* = j^*\mathfrak{E}^{-1}$, a *quotient* of $\mathfrak{E}^{-1}$:

$$j^*\theta \;=\; \sum_n (j^*\theta)(f_n)\cdot f_n \;=\; \sum_n \theta(jf_n)\cdot f_n \;=\; \sum_n \theta(f_n)\cdot f_n \qquad (\text{convergent in } j^*\mathfrak{E}^{-1})$$

If $(\Delta - \lambda_w)u = \theta$ and $u \in \mathfrak{E}^1 \cap L^2_a$, then certainly $(\widetilde{T}^\#_{\ge a} - \lambda_w)u = j^*\theta$, so, noting that the inclusion j identifies u with its image,

$$j^*\theta \;=\; \Big(j^* \circ (\Delta - \lambda_w) \circ j\Big)u \;=\; (\widetilde{T}^\#_{\ge a} - \lambda_w)u$$

The equation $(\widetilde{T}^\#_{\ge a} - \lambda_w)u = j^*\theta$ can be solved by division: producing a spectral expansion convergent in $\mathfrak{E}^1 \cap L^2_a \subset \mathfrak{E}^1$:

$$u \;=\; u_w \;=\; \sum_n \frac{\theta(f_n)}{\lambda_{s_n} - \lambda_w}\cdot f_n \qquad (\text{convergent in } \mathfrak{E}^1)$$

Via the $\mathfrak{E}^1 \times \mathfrak{E}^{-1}$ pairing, the condition $\theta(u_w) = 0$ is

$$0 \;=\; \theta(u_w) \;=\; \sum_n \frac{\theta(f_n)^2}{\lambda_{s_n} - \lambda_w}$$

By the intermediate value theorem, there is exactly one solution to the latter equation between successive spectral parameters s_n with $\theta(f_n) \ne 0$, giving

Claim: The equations $(\Delta - \lambda_w)u = \theta$ and $\theta u = 0$ have a solution u for at most one w in each interval $\Im(s_n) \le \Im(w) \le \Im(s_{n+1})$. ///

8. *Example: the 94% limitation*

With the simplest types of self-adjoint operators whose discrete spectrum $\lambda_s = s(s-1)$, *if any*, can only appear for $\zeta_k(s) = 0$, invoking exotic eigenfunction expansions, the regularity of $\zeta(s)$ on $\Re(s) = 1$, and Montgomery's pair correlation (for simplicity presented in the conjecturally strongest form), we find a conflict, as follows.

Take θ to be the sum of (restrictions of) automorphic Dirac deltas at the Heegner points attached to $k = \mathbb{Q}(\sqrt{d})$, so that

$$\theta E_w \;=\; \Big|\frac{\sqrt{|d|}}{2}\Big|^w \frac{\zeta_k(w)}{\zeta(2w)}$$

Corollary: At most 94% of the zeros of zeta appear among the spectral parameters for $\widetilde{T}_\theta$.

Proof: All the eigenfunctions f_n with eigenvalues $< -1/4$ are constant multiples of truncated Eisenstein series $\wedge^a E_s$ such that $a^s + c_s a^{1-s} = 0$.

The spacing of the spectral parameters s_j such that $a^{s_j} + c_{s_j} a^{1-s_j} = 0$ becomes essentially regular when $\log\log t$ is large. Namely, from [Titchmarsh 1986] 5.17.4 page 112 (in an earlier edition, page 98), $\psi(t) = \arg \xi(1+2it)$ satisfies

$$\psi(t) \;=\; t\log t + O\Big(\frac{t\log t}{\log\log t}\Big) \qquad \text{and} \qquad \psi'(t) \;=\; \log t + O\Big(\frac{\log t}{\log\log t}\Big)$$

Thus, given $\epsilon > 0$, there is t_o sufficiently large such that, for w_1, w_2 on $\Re(w) = \frac{1}{2}$ with $\Im(w_j) \ge t_o$, such that λ_{w_1} and λ_{w_2} are eigenvalues for $\widetilde{T}_\theta$, $|\Im(w_1) - \Im(w_2)| \ge (1-\varepsilon)\cdot \frac{\pi}{\log t}$. Otherwise, adjust the cut-off height a slightly to put both w_1, w_2 in the same interval between some s_n, s_{n+1}.

On the other hand, Montgomery's pair correlation conjecture is that, for imaginary parts of zeros $\ldots \le \gamma_{-1} < 0 < \gamma_1 \le \gamma_2 \le \ldots$, for $0 \le \alpha < \beta$,

$$\#\Big\{m < n \;:\; 0\gamma_m, \gamma_n \le T \text{ with } \frac{2\pi\alpha}{\log T} \le \gamma_n - \gamma_m \le \frac{2\pi\beta}{\log T}\Big\} \;\sim\; \int_\alpha^\beta \Big(1 - (\frac{\sin \pi u}{\pi u})^2\Big)\, du$$

For example,

$$\#\Big\{m < n \;:\; 0\gamma_m, \gamma_n \le T \text{ with } \gamma_n - \gamma_m \le \frac{\pi}{\log T}\Big\} \;\sim\; \int_0^{\frac{1}{2}} \Big(1 - (\frac{\sin \pi u}{\pi u})^2\Big)\, du \;\approx\; 0.11315$$

From the asymptotic lower bound for separation of zeros appearing as spectral parameters, for at least one of every such pair m, n the zero cannot appear. ///

9. *Example: spacing of zeros*

For example, *without* assuming anything about existence or non-existence of discrete spectra:

Let

$$J(w) \;=\; \frac{h_d^2}{-\lambda_w \cdot \langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \Big|\frac{\zeta_k(s)}{\zeta(2s)}\Big|^2 - \Big|\frac{\zeta_k(w)}{\zeta(2w)}\Big|^2 \;\; \frac{ds}{\lambda_s - \lambda_w}$$

and $w = \frac{1}{2} + i\tau$. By rearranging, the condition that λ_w be an eigenvalue of $\widetilde{T}_{a,\theta}$ and/or of $\widetilde{T}_{\geq a,\theta}$ is

$$\cos(\tau \log a + \psi(\tau)) \cdot J(w) \;=\; \sin(\tau \log a + \psi(\tau)) \cdot \frac{\theta E_{1-w} \cdot \theta E_w}{2\tau}$$

As above, between any two consecutive zeros of $\cos(\tau \log a + \psi(\tau))$ there is a unique τ such that $\lambda_{\frac{1}{2}+i\tau}$ is an eigenvalue of $\widetilde{T}_{\geq a,\theta}$.

[9.0.1] Corollary: Let $\tau < \tau'$ be large such that $\frac{1}{2} + i\tau$ and $\frac{1}{2} + i\tau'$ are adjacent zeros of θE_w, and neither a zero of $J(w)$. If there is a unique on-the-line zero of $J(w)$ between the two, with $\frac{\partial}{\partial\tau} J(\frac{1}{2} + i\tau) > 0$, then

$$|\tau - \tau'| \;\geq\; \frac{\pi}{\log t} \cdot (1 + O(\frac{1}{\log\log t}))$$

That is, *in this configuration*, the distance between consecutive zeros must be at least essentially the average. ///

10. *Technical basics about unbounded self-adjoint operators*

A self-contained, reasonably efficient account of the relevant ideas, aimed at number-theoretic applications, is one of the main goals of [Garrett 2018].

Unsurprisingly, outside of the finite-dimensional situation, and outside of the context of *bounded*[7] operators, a suitable notion of *self-adjointness* is subtler.

There is a suitable, precise notion of *symmetric, unbounded operator* T on a Hilbert space V, and its *adjoint* T^* from [Stone 1929/30] and [vonNeumann 1929]. First, as is often only implicit, we assume that the domain D_T of T is *dense* in V, although is necessarily (due to the unboundedness) *not* the whole Hilbert space V. The *symmetry* is that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for

[7] Recall that, on Hilbert spaces $T : V \to V$, a linear map $T : V \to V$ is *continuous* if and only if it is *continuous at* 0, if and only if it is *bounded*, in the sense that there is $0 \leq B < \infty$ such that $|Tv| \leq B$ for all $|v| \leq 1$. In this context, such a linear map is *implicitly* assumed to be everywhere defined on V.

$v, w \in D_T$. The *unboundedness* allows for a lack of continuity in the Hilbert space topology on the whole space V, such as for the archetype $T = \Delta|_{C_c^\infty(\mathbb{R}^n)}$ on $V = L^2(\mathbb{R}^n)$. There, the symmetry property follows because in integration by parts there are no boundary terms. The *adjoint* T^* of a symmetric operator T is the unique maximal extension of T satisfying $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for $v \in D_T$ and $w \in D_{T^*}$. Specification of $D_{T^*} \supset D_T$ is an essential part of the characterization. Letting $U : V \oplus V \to V \oplus V$ by $U(v \oplus w) = -w \oplus v$, the graph Γ_{T^*} of the adjoint T^* is characterized as the orthogonal complement to the image $U\Gamma_T$ of the graph Γ_T under U. There are details to check.

An unbounded, symmetric, densely-defined T is *self-adjoint* if $T = T^*$. This includes the assertion that $D_{T^*} = D_T$, that is, that T admits no *proper* extension T' satisying $\langle Tv, w\rangle = \langle v, T'w\rangle$ for $v \in D_T$ and $w \in D_{T'} \supset D_T$. Every self-adjoint operator is symmetric, but not vice-versa. For unbounded operators, it is important to appreciate the distinction between T and T^*, if they are not equal, because in general T^* is *not* symmetric, and has many non-real eigenvalues. This is not a pathology: already in the simplest Sturm-Liouville problems, with $\Delta = \frac{d^2}{dx^2}$ on $L^2[a, b]$, the natural symmetric operator to begin with is $T = \Delta|_{C_c^\infty(a,b)}$, that is, the restriction of Δ to test functions supported properly in the interior of the interval. The domain of the adjoint T^* includes *at least* all smooth functions on the closed interval $[a, b]$, but without a suitable choice of boundary conditions T^* is *not symmetric*, because integration by parts produces non-trivial boundary terms obstructing the symmetry.

Why did intuitive manipulation of such operators in the hands of physicists in the 1920s and 1930s reach mathematically correct conclusions? For one, many of the operators employed are *essentially self-adjoint*, in the sense that they have a unique self-adjoint extension, *and* it is the *graph closure*[8] of the given operator. Proof of the essential self-adjointness, when it does hold, is non-trivial. In effect, the essential self-adjointness assures that naive, intuitive manipulation of the operator leads to correct conclusions. Typically, essential self-adjointness of Laplace-Beltrami operators, restricted to test functions, holds in absence of boundary conditions. This is the case for $T = \Delta|_{C_c^\infty(\mathbb{R}^n)}$, a densely-defined, unbounded, symmetric operator on $L^2(\mathbb{R}^n)$. Thus, being unaware of the *possibility* that an operator might have several distinct self-adjoint extensions does not lead to disaster when, as fortunately often happens, the operator has a unique self-adjoint extension.

[Stone 1929] and [vonNeumann 1929] gave a necessary and sufficient criterion for a densely-defined symmetric unbounded operator T to *have* a self-adjoint extension, and also classified all possible self-adjoint extensions, as follows. For $\lambda \in \mathbb{C}$ but $\lambda \notin \mathbb{R}$, the λ^{th} *deficiency subspace* of T is the kernel of $T^* - \lambda$. The λ^{th} *deficiency index* is the dimension of the λ^{th}

[8] The graph closure, often called simply the closure, of an unbounded operator $T : V \to V$ is the extension whose graph is the topological closure of the graph of T in $V \oplus V$. In general, this closure will not be the graph of an operator. Operators whose graph-closure *is* the graph of an operator are called *closeable*. All symmetric operators admitting self-adjoint extensions are closeable.

deficiency subspace. The first main result is that there *exists* a self-adjoint extension of T if and only if the λ^{th} deficiency index is equal to the $\overline{\lambda}^{th}$ deficiency index. (On separable Hilbert spaces, this is correct whether those indices are finite or infinite.) When the two indices are equal, the collection of all possible self-adjoint extensions of T is indexed by isometric isomorphisms $\ker(T^* - \lambda) \to \ker(T^* - \overline{\lambda})$.

[Friedrichs 1934/35] description of a canonical self-adjoint extension for *semi-bounded* operators requires no checking of hypotheses. A symmetric, densely-defined operator T is *semi-bounded* when either there is a constant c such that $\langle Tv, v\rangle \geq c \cdot \langle v, v\rangle$ for all v in the domain of T, or there is a constant c such that $\langle Tv, v\rangle \leq c \cdot \langle v, v\rangle$ for all v in the domain of T. K. Klinger-Logan has pointed out to me that this particular self-adjoint extension was already noted (page 103) in [vonNeumann 1929]. For the construction, take the case $\langle Tv, v\rangle \geq \langle v, v\rangle$ without loss of generality, so that T is something like a restriction of $1 - \Delta$. Give the domain D_T of T a Sobolev-like norm via $\langle v, w\rangle_1 = \langle Tv, v\rangle$, and let V^1 be the completion of D_T in that norm. We have a natural continuous imbedding $V^1 \to V$. Define $B : V \to V^1$ by $\langle Bv, w\rangle_1 = \langle v, w\rangle$ for $v \in V$ and $w \in V^1$, via Riesz-Fréchet. Then $\widetilde{T}^{-1} = B$ for $\widetilde{T}$ the Friedrichs extension of T.

In contrast to the case of semi-bounded operators, where Friedrichs' construction immediately gives existence of at least one self-adjoint extension, in general a symmetric, densely-defined, unbounded operator need not have *any* self-adjoint extensions. Yes, seemingly *natural* symmetric operators should have at least one self-adjoint extension, for either physical or mathematical reasons, but it is not hard to find simply-expressed symmetric operators with *no* self-adjoint extensions. For example, from [MSE-2363904 2017] and [MSE-2364766 2017], consider the unbounded operator $T = x^3 \circ i\frac{d}{dx} + i\frac{d}{dx} \circ x^3$ on $L^2(\mathbb{R})$, where x^3 is the multiplication-by-x^3 operator. A seeming paradox is that the $L^2(\mathbb{R})$ function

$$u(x) \;=\; \begin{cases} \dfrac{e^{-1/4x^2}}{|x|^{3/2}} & (\text{at } x \neq 0) \\ 0 & (\text{at } x = 0) \end{cases}$$

is apparently an eigenfunction with eigenvalue $-i$, which would be impossible for an eigenvector of a symmetric operator (and the operator L is indeed symmetric on the dense subspace $\mathscr{S}(\mathbb{R})$ of $L^2(\mathbb{R})$). That is, one can solve the differential equation

$$(x^3 \circ i\partial_x + i\partial_x \circ x^3)u \;=\; \lambda \cdot u$$

by elementary means:

$$u \;=\; \frac{e^{i\lambda/4x^2}}{|x|^{3/2}}$$

For $\mathrm{Im}(\lambda) > 0$, the function is in $L^2(\mathbb{R})$, being square-integrable both at 0 and at ∞. For $\mathrm{Im}(\lambda) \leq 0$, the function is not square-integrable at 0. For $\mathrm{Im}(\lambda) > 0$, the solution vanishes to infinite order at 0, so we can splice together a solution in $x < 0$ and another solution in

$x > 0$: the L^2 solutions are

$$u = \begin{cases} a\dfrac{e^{i\lambda/4x^2}}{|x|^{3/2}} & (\text{for } x > 0) \\ b\dfrac{e^{i\lambda/4x^2}}{|x|^{3/2}} & (\text{for } x < 0) \end{cases} \qquad (\text{for arbitrary constants } a, b)$$

For $\mathrm{Im}(\lambda) > 0$, certainly these functions are λ-eigenvectors for the adjoint T^*, so the *deficiency index* for such λ is 2. But for $\mathrm{Im}(\lambda) < 0$, there are no L^2 λ-eigenvectors for T^*. Thus, by von Neumann's criterion, there is *no* self-adjoint extension of S. In particular, there is no paradox about non-real eigenvalues of self-adjoint operators.

While it is not surprising that most basic classical analysis does behave consistently with physical intuition, given the history and historical purposes of calculus, it is more surprising that the fairly extravagant mathematics initiated in [Dirac 1928a,b] and [Dirac 1930], for example, was so successful in producing correct, testable physical conclusions. Genuine mathematical justification, as opposed to experimental, had to wait for [Stone 1929/30], [vonNeumann 1929], [Friedrichs 1934/5], [Gelfand 1936], [Pettis 1938] [Sobolev 1937], [Sobolev 1938], [Sobolev 1950], [Schwartz 1950/1], [Schwartz 1950], [Schwartz 1953/4], [Grothendieck 1953a,b], and others.

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