(*June 5, 2009*)

Decomposition and estimates for cuspforms

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

The discrete decomposition with finite multiplicities of square-integrable *cuspforms*, and concommitant estimates, are foundational. Lie-group and symmetric-space versions of these results were known [Selberg 1956], [Gelfand-Fomin 1952] in the mid 1950's. The general assertion was made in [Gelfand-Graev 1962] and [Gelfand-PS 1963], the latter observing that an adelic formulation can proceed in the same fashion.

The argument here begins with that of [Godement 1966], similar to Langlands' then-unpublished notes [Langlands 1967] on Eisenstein series, apart from use of Poisson summation. The proofs are readable either classically or adelically, thinking of $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ or $SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A})$ when convenient.

---

1. *Pointwise boundedness (statement)*

The compactly-supported continuous functions $C_c^o(G)$ on a unimodular topological group G act on any reasonable representation space V of G by the Gelfand-Pettis integrals [1]

$$\varphi \cdot v = \int_G \varphi(g)\, g \cdot v\, dg \qquad \text{(for } \varphi \in C_c^o(G) \text{ and } v \in V\text{)}$$

The continuity of $G \times V \to V$ immediately yields the continuity of

$$C_c^o(G) \times V \to V$$

Even when more familiar classical details (such as differential operators) *are* available, integral operators behave better. Further, such integral operators are available in the p-adic case where differentiation has no immediate sense.

The result stated in this section does not itself prove that spaces of cuspforms decompose discretely, nor directly give the corollaries on admissibility. However, here the main technical difficulty *is* overcome.

---

[1] The topological vector space V should be locally convex (of course), and *quasi-complete*, also called *locally complete*, meaning that every *bounded* (in topological vector space sense) Cauchy *net* is convergent. This subsumes the complete metric case. Then the Gelfand-Pettis (also called *weak integral*) compatibility requirement with functionals

$$\lambda(\int_G \varphi(g)\, g \cdot v\, dg) = \int_G \varphi(g)\, \lambda(g \cdot v)\, dg \qquad \text{(for all } \lambda \in V^*\text{)}$$

determines the integral uniquely, if it exists, since (by Hahn-Banach) the functionals separate points on V. There is also the *estimate*

$$\int_G \varphi(g)\, g \cdot v\, dg) \in \text{meas}\,(\text{spt}\,\varphi) \cdot (\text{closure of convex hull of } \varphi(\text{spt}\,\varphi))$$

Thus, continuity for a *finer* topology on some subset of $C_c^o(G)$ follows *a fortiori.*

For G a real Lie group[2] the space $C_c^\infty(G)$ of **test functions** is defined to be compactly-supported smooth[3] complex-valued functions on G. For a totally disconnected group[4] G such as $SL_n(\mathbb{Q}_p)$, the space $C_c^\infty(G)$ of test functions is defined to be compactly-supported *locally constant* functions. [5] An *adele group* can always be expressed as a product

$$G_\mathbb{A} = G_\infty \times G_{\text{fin}}$$

of the corresponding Lie group G_∞ and corresponding totally disconnected group G_{fin} (the product of all the corresponding p-adic groups). The space of test functions is the (algebraic) tensor product of the two genres of test functions, namely

$$C_c^\infty(G_\mathbb{A}) = C_c^\infty(G_\infty) \otimes C_c^\infty(G_{\text{fin}})$$

In what follows, one can imagine at least two cases:

$$\Gamma = SL_n(\mathbb{Z}) \subset G = SL_n(\mathbb{R})$$

or

$$\Gamma = SL_n(\mathbb{Q}) \subset G = SL_n(\mathbb{A})$$

depending upon taste and prior experience. A function f in $L^2(\Gamma\backslash G)$ is a **cuspform** when it meets Gelfand's condition

$$\int_{(N\cap\Gamma)\backslash N} f(ng)\, dg = 0$$

where N runs through a class of subgroups described in examples as follows. [6] For $\Gamma = SL_2(\mathbb{Z})$, one need only consider the subgroup

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

For $\Gamma = SL_n(\mathbb{Z})$, it suffices to consider N running through subgroups of the form

$$N = \begin{pmatrix} 1_i & i\text{-by-}(n-i) \\ 0 & 1_{n-i} \end{pmatrix} \qquad (\text{for } 1 \le i < n)$$

For adelic SL_n, the latter family of N's also suffices. The space of **square-integrable cuspforms** is[7]

$$L^2_{\text{cfm}}(\Gamma\backslash G) = \{f \in L^2(\Gamma\backslash G) : f \text{ meets Gelfand's condition}\}$$

[2] We use no serious properties of Lie groups, and the reader can think of $SL_2(\mathbb{R})$ or $SL_n(\mathbb{R})$ without loss.

[3] *Smooth* in this context means *infinitely differentiable.*

[4] A topological space X is *totally disconnected* if, given two points x, y in X there are open neighborhoods U of x and V of y such that $U \cap V = \phi$ and $U \cup V = X$. Many of the elementary features of p-adic groups depend only upon their total-disconnectedness and not much more.

[5] As the name suggests, a *locally constant* function f on a topological space X has the property that every $x \in X$ has a neighborhood U such that f is constant on U. Of course, unless X is totally disconnected there will not be many such functions.

[6] The *intrinsic* condition is that N runs through unipotent radicals of rational parabolics in G. It suffices to consider representatives from Γ *conjugacy classes* of parabolics. In the adelic case, the conjugacy classes are very easy to understand, by contrast to the classical situation wherein for proper congruence subgroups Γ the number of Γ conjugacy classes of course varies with Γ.

[7] Of course, pointwise equality has no literal sense for L^2 functions, so some *almost everywhere* comment needs to be inserted somewhere here.

[1.0.1] Lemma: Let E be a compact subset of $C_c^\infty(G)$. The image

$$E \cdot B = \{\varphi \cdot f : \ \varphi \in E, \ f \in B\}$$

of the unit ball B in square-integrable cuspforms $L^2_{\text{cfm}}(\Gamma\backslash G)$ consists of *continuous* functions, and these functions are *uniformly bounded.* [8]

The proof will occupy the following sections.

[1.0.2] Remark: This modest lemma is the main technical point!

---

2. *Compactness of a set of test functions*

This section proves compactness of a collection of functions involved in the kernel attached to a test function on G. This results does *not* use the Gelfand property. This general compactness property is used in the next section to give *uniform asymptotic estimates for cuspforms.*

For $G = SL_n$, let P be upper triangular matrices, M diagonal matrices, and N upper triangular with 1's on the diagonal. [9] *Wind up* the integral for $\varphi \cdot f$ along $N \cap \Gamma$, to have

$$(\varphi \cdot f)(y) = \int_G \varphi(x)\, f(yx)\, dx = \int_G \varphi(y^{-1}x)\, f(x)\, dx = \int_{(N\cap\Gamma)\backslash G} \left(\sum_{\gamma \in N\cap\Gamma} \varphi(y^{-1}\gamma x) \right) f(x)\, dx$$

In both the Lie case and adelic, since N is unipotent we have an *exponential map*

$$\exp : \mathfrak{n} \to N \qquad\qquad (\mathfrak{n} \text{ the } \textit{algebraic} \text{ Lie algebra of } N)$$

of the Lie algebra $\mathfrak{n}$ of N to the group N. [10] For fixed x, y the map on $\mathfrak{n}$ by

$$\nu \to \varphi(y^{-1} \cdot \exp(\nu) \cdot x)$$

is in $C_c^\infty(\mathfrak{n})$.

There is a discrete additive subgroup Λ in $\mathfrak{n}$ such that $\exp(\Lambda)$ differs from $N \cap \Gamma$ by finite index. [11] For simplicity assume we have *equality*, as with $SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A})$. Rewrite the partly wound-up integral as

$$(\varphi \cdot f)(y) \;=\; \int_{(N\cap\Gamma)\backslash G} \left(\sum_{\nu \in \Lambda} \varphi(y^{-1} \cdot \exp(\nu) \cdot x) \right) f(x)\, dx$$

---

[8] This sense of *uniformly bounded* is the natural sense that there is an absolute constant C such that for all $F \in E \cdot B$ and for all $x \in \Gamma\backslash G$ we have $|F(x)| \le C$.

[9] Intrinsically, consider a minimal rational parabolic P, N its unipotent radical, M a Levi component of P.

[10] The behavior of the exponential map can be explained easily for the case we need: for N consisting of n-by-n upper triangular matrices with 1's on the diagonal, $\mathfrak{n}$ consists of strictly upper triangular matrices, and the usual algebraic expression for the exponential *terminates* (after n terms). Inversely, the series for logarithm terminates. Thus, this exponential map behaves as we'd like in either the Lie case or the adelic case.

[11] In the real-Lie setting, for $\Gamma = SL_2(\mathbb{Z})$ for example, the strictly upper triangular integer matrices exponentiate exactly to $N \cap \Gamma$. For $SL_n(\mathbb{Z})$ with $n > 2$ this is not quite true, due to the denominators in the series for the exponential function. In the adelic case, for $\Gamma = SL_n(\mathbb{Q})$, for example, the rational strictly upper triangular matrices exponentiate exactly to rational matrices in N, nicely avoiding problems of denominators.

Poisson summation for the lattice Λ in $\mathfrak{n}$ gives

$$\sum_{\nu\in\Lambda} \varphi(y^{-1}\cdot \exp(\nu)\cdot x) = \sum_{\psi\in\Lambda^*} \int_{\mathfrak{n}} \overline{\psi}(\nu)\, \varphi(y^{-1}\cdot \exp(\nu)\cdot x)\, d\nu$$

with suitably normalized measure, where $\mathfrak{n}$ is simply treated as a (real or adelic) vector space, hence abelian topological group. [12]

Use the *reduction theory* available for quotients such as $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ or $SL_n(Q)\backslash SL_n(\mathbb{A})$. Specifically, there is a good type of set *surjecting to* the quotient $\Gamma\backslash G$, described as follows. [13] Let Σ^o be the collection of *positive simple roots*, which for SL_n and upper-triangular N (with 1's on the diagonal) are the characters $\alpha_1,\ldots,\alpha_{n-1}$ on diagonal matrices given by

$$\alpha_j \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} = m_j/m_{j+1} \qquad\qquad (\text{for } 1\leq j\leq n-1)$$

For fixed positive real t_o, fixed compact subset Ω_N of N, and fixed maximal compact subset K of G, the corresponding **Siegel set** is the set

$$\mathfrak{S} = \mathfrak{S}(\Omega_N, t_o) \;=\; \Omega_N \cdot \{m\in M : |\alpha(m)|\geq t_o, \text{ for all } \; \alpha\in\Sigma^o\}\cdot K$$

The main result of reduction theory [14] is that for sufficiently large compact Ω_N in N and for sufficiently small t_o the corresponding Siegel set $\mathfrak{S}$ *surjects* to the quotient $\Gamma\backslash G$.

The *Iwasawa decomposition* is [15]

$$G = N\cdot M\cdot K$$

For $x\in G$, write

$$x = n_x\cdot m_x\cdot k_x \qquad\qquad (\text{with } n_x\in N,\ m_x\in M, \text{ and } k_x\in K)$$

The factors in an Iwasawa decomposition are not unique, but in all cases the only ambiguities involve adjusting n_x or m_x (and, thus, k_x) by an element of the maximal compact. In both archimedean and p-adic cases,

$$N\cap M = \{1\}$$

so there is no ambiguity there.

[2.0.1] Claim: A point in the fundamental domain is well approximated by its M-component in an Iwasawa decomposition. That is, for $z\in G$ in a fixed Siegel set $\mathfrak{S}(\Omega_N, t_o)$, there is a compact subset Ω'_N of N such that

$$z \;\in\; m_z\cdot\Omega'_N\cdot K$$

---

[12] Admittedly, treating $\mathfrak{n}$ as an abelian topological group is strange, but forgetting the Lie bracket structure is allowed, if unexpected. Harish-Chandra used such aggressive parametrization devices to great advantage on many occasions.

[13] In a general *classical* setting, for example with congruence conditions, more than one such set is needed to cover the quotient. This leads to baroque indexing schemes, and is un-necessary, because rewriting the same story adelically restructures everything so that there is a *single* cusp (or flag thereof, for higher-rank groups such as SL_n with $n>2$). In effect, the adele group bundles the classical cusps into a single structured object. I first heard this idea asserted vigorously in 1978 from Piatetski-Shapiro, but did not appreciate its significance until later.

[14] Reduction theory can be posed either for quotients of semi-simple real Lie groups by arithmetic subgroups, as done by Borel and Harish-Chandra, or for adele groups, as Godement did in Sem. Bourbaki.

[15] Such a decomposition holds in reductive real Lie groups, and also in reductive p-adic groups. We use all these *local* decompositions at once.

Proof: *(of claim)* That z is in the Siegel set $\mathfrak{S}(\Omega_N, t_o)$ gives

$$z \;=\; n_z \cdot m_z \cdot k_z \;\in\; \Omega_N \cdot m_z \cdot K \;=\; m_z \cdot m_z^{-1}\Omega_N m_z \cdot K$$

A crucial point is that the lower bound $|\alpha(m)| \ge t_o$ for all $\alpha \in \Sigma^o$ gives a compact set Ω'_N in N depending only upon t_o and Ω_N such that

$$m^{-1}\Omega_N m \subset \Omega'_N \qquad\qquad (\text{for } m \in M \text{ with } |\alpha(m)| \ge t_o \text{ for all } \alpha \in \Sigma^o)$$

In particular, $m_z^{-1}\Omega_N m_z \subset \Omega'_N$. Thus,

$$z \;\in\; m_z \cdot \Omega'_N \cdot K$$

as claimed. ///

[2.0.2] Claim: For x, y both in a Siegel set the $(N \cap \Gamma)$-wound-up kernel vanishes unless the Iwasawa decomposition M-components of the two are close. That is, for fixed compact subset $E \subset C_c^\infty(G)$ and for points x, y in a fixed Siegel set $\mathfrak{S}(\Omega_N, t_o)$, there is a compact subset $\Omega_M \subset M$ such that if there exists $n \in N$ and $\varphi \in E$ with $\varphi(y^{-1} \cdot n \cdot x) \ne 0$, then

$$m_x \;\in\; m_y \cdot \Omega_M$$

Proof: *(of claim)* From the previous claim, there is a compact subset Ω'_N of N such that $m_y^{-1} y \in \Omega'_N \cdot K$. A compact set of test functions has a common compact support Ω_G. Non-vanishing of $\varphi(y^{-1}nx)$ implies $y^{-1}nx \in \Omega_G$, so

$$nx \in y \cdot \Omega_G \subset m_y \cdot \Omega'_N \cdot K \cdot \Omega_G \subset m_y \cdot \Omega'_G \qquad\qquad (\text{with } \Omega'_G = \Omega'_N K \Omega_G = \text{compact})$$

That is,

$$\Omega'_G \ni m_y^{-1} \cdot nx = m_y^{-1} \cdot nn_x \cdot m_x \cdot k_x = (m_y^{-1} nn_x m_y) \cdot m_y^{-1} m_x \cdot k_x$$

That is,

$$(m_y^{-1} nn_x m_y) \cdot m_y^{-1} m_x \in \Omega'_G \cdot K = \text{compact}$$

Since M normalizes N, the element $m_y^{-1} nn_x m_y$ is in N. Since $N \cap M = \{1\}$, the multiplication map $N \times M \to NM$ is a homeomorphism. Thus, for nm to lie in a compact set in G requires that n lies in a compact set in N and m lies in a compact set in M. Thus,

$$m_y^{-1} m_x \in \Omega_M$$

for some compact subset Ω_M of M, as claimed. ///

[2.0.3] Corollary: *(of claims)* For x, y in a fixed Siegel set, and for fixed compact set E of test functions φ, there is a compact Ω_M such that, if $\varphi(y^{-1}nx) \ne 0$ for some $n \in N$ and some $\varphi \in E$, then $m_y^{-1} x$ lies in Ω_M.

Proof: By the first of the two claims, there is a compact Ω'_G in G such that $x \in m_x \cdot \Omega'_G$. By the second, there is a compact Ω_M in M such that $m_x \in m_y \Omega_M$. Thus, $x \in m_y \Omega_M \Omega'_G$, as claimed. ///

For x, y in a fixed Siegel set and $\varphi(y^{-1}nx) \ne 0$ for some $n \in N$ and $\varphi \in E$, we will get a *uniform* estimate on the asymptotic behavior of the Fourier transforms of the functions $\nu \to \varphi(y^{-1} \exp(\nu) x)$. We have[16]

$$\int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi\left(y^{-1} \cdot \exp(\nu) \cdot x\right) \, d\nu = \int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi\left(y^{-1} m_y \cdot \exp(m_y^{-1} \nu m_y) \cdot m_y^{-1} x\right) \, d\nu$$

[16] We just write *conjugation* of ν by m_y rather than writing *adjoint*.

Above we saw that $\omega_y = y^{-1}m_y$ and $\omega_{x,y} = m_y^{-1}x$ are in fixed compacts in G (depending only upon the given Siegel set and upon the test function φ). The left and right translation actions of G on test functions $C_c^\infty(G)$ are continuous maps

$$G \times G \times C_c^\infty(G) \to C_c^\infty(G)$$

so the image

$$\{\omega_y : y\} \times \{\omega_{x,y} : x, y\} \times E \to C_c^\infty(G) \qquad (x, y \text{ in fixed Siegel set, } \varphi(y^{-1}nx) \neq 0)$$

is *compact*. Since N is *closed*[17] in G, the restriction map

$$C_c^\infty(G) \to C_c^\infty(N) \approx C_c^\infty(\mathfrak{n})$$

is continuous. A continuous image of a compact set is compact, so the collection of functions

$$\nu \to \varphi_{x,y}(\nu) = \varphi(\omega_y \cdot \exp(\nu) \cdot \omega_{x,y}) \qquad (x, y \text{ in fixed Siegel set, } \varphi(y^{-1}nx) \neq 0,\ \varphi \in E)$$

is a *compact* subset of $C_c^\infty(\mathfrak{n})$. The inclusion $C_c^\infty(\mathfrak{n}) \subset \mathscr{S}(\mathfrak{n})$ of test functions to Schwartz functions is continuous (in both archimedean and adelic situations).

Thus, making a change of variables, the ψ^{th} summand in the expression for the kernel is

$$\int_{\mathfrak{n}} \overline{\psi}(\nu)\, \varphi_{x,y}(m_y^{-1}\nu m_y)\, d\nu \;=\; \delta(m_y) \int_{\mathfrak{n}} \overline{\psi}(m_y \nu m_y^{-1}) \cdot \varphi_{x,y}(\nu)\, d\nu$$

where δ is the modular function on P. Defining ψ^{m_y} by

$$\psi^{m_y}(\nu) = \psi(m_y \nu m_y^{-1})$$

this is

$$\psi^{th} \text{ summand } \;=\; \delta(m_y) \cdot \widehat{\varphi}_{x,y}(\psi^{m_y})$$

and we have proven that the functions $\varphi_{x,y}$ lie in a compact set in $C_c^\infty(\mathfrak{n}) \subset \mathscr{S}(\mathfrak{n})$.

3. *Asymptotics of images $\varphi \cdot f$ for $SL_2(\mathbb{Z})$*

The compactness conclusion of the previous section and the defining property of cuspforms will lead to *asymptotics* for images $\varphi \cdot f$ where f is a cuspform and φ is in a compact set of test functions on G: we will show that such images are of *rapid decay in Siegel sets*. This is done in this section for the *simplest* classical situation, namely, $\Gamma = SL_2(\mathbb{Z}) \subset G = SL_2(\mathbb{R})$. The following section will treat the general case.

Let

$$\alpha \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \;=\; m_1/m_2$$

[3.0.1] Proposition: Given a compact set E in $C_c^\infty(SL_2(\mathbb{R}))$, a Siegel set $\mathfrak{S}$, and positive ℓ, there is a uniform C such for all x, y in $\mathfrak{S}$, for all $\varphi \in E$, and for all L^2 cuspforms f,

$$|(\varphi \cdot f)(y)| \;\leq\; C \cdot |\alpha(m_y)|^{-\ell} \cdot |f|_{L^2(\Gamma\backslash G)}$$

[17] Both in the real Lie group setting and in the adelic setting, the subgroup N is a closed subgroup. In the adelic case the archimedean part N_∞ is closed in G_∞, and the finite-prime part N_{fin} is closed in G_{fin}. Indeed, N is *collared* in G, or whatever strong condition we might want.

Proof: For SL_2

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \qquad \mathfrak{n} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

and the exponential function is simply

$$\exp \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

In this classical setting, the entries in N or $\mathfrak{n}$ are *real*, and we use the usual norm $|*|$ on $\mathbb{R}$, thus on $\mathfrak{n}$. We index (continuous unitary) homomorphisms

$$\psi : \mathfrak{n} \approx \mathbb{R} \to \mathbb{C}^\times$$

in the usual fashion, by $\xi \in \mathbb{R}$, via

$$\psi_\xi \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = e^{2\pi i \xi x}$$

By this parametrization identify $\mathfrak{n}^*$ with $\mathbb{R}$, and give it the usual norm.

The compactness result of the previous section gives a uniform estimate on Fourier transforms, namely, for every exponent $\ell > 0$, there is a *uniform* constant c_ℓ such that for all x, y (in a fixed Siegel set)

$$|\widehat{\varphi}_{x,y}(\psi)| \le c_\ell \cdot (1 + \|\psi\|)^{-\ell}$$

Thus, with conjugation by m_y,

$$\delta(m_y)\, |\widehat{\varphi}_{x,y}(\psi^{m_y})| \le c_\ell \cdot \delta(m_y) \cdot (1 + |\psi^{m_y}|)^{-\ell}$$

So far, the fact that f is a cuspform has still not been used.

[3.0.2] Claim: For a character $\psi \in \mathfrak{n}^*$ vanishing identically on $\mathfrak{n}$, namely the 0-element in $\mathfrak{n}^*$, the corresponding function $\widehat{\varphi}_{x,y}(\psi)$ is left N-invariant in x.

Proof: Let $n \in N$, and replace x by nx in the original integral defining $\widehat{\varphi}_{x,y}(\psi)$, with $n = \exp(\nu')$, obtaining

$$\int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi(y^{-1} \exp(\nu) \cdot nx)\, d\nu \;=\; \int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi(y^{-1} \exp(\nu + \nu') \cdot x)\, d\nu$$

using the abelian-ness of N. Replacing ν by $\nu - \nu'$ in the integral gives

$$\int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi(y^{-1} \exp(\nu) \cdot nx)\, d\nu \;=\; \psi(\nu') \cdot \widehat{\varphi}_{x,y}(\psi) \;=\; \widehat{\varphi}_{x,y}(\psi)$$

proving the left N-invariance in x. ///

Let

$$K_\varphi(x,y) = \sum_{\nu \in \Lambda} \varphi(y^{-1} \cdot \exp(\nu) \cdot x) = \delta(m_y) \sum_{\psi \in \Lambda^*} \widehat{\varphi}_{x,y}(\psi^{m_y})$$

be the $(N \cap \Gamma)$-wound-up kernel. Then

$$(\varphi \cdot f)(y) = \delta(m_y) \sum_{\psi \in \Lambda^*} \int_{(N\cap\Gamma)\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y}) \cdot f(x)\, dx$$

For ψ left N-invariant, the corresponding integral is

$$\int_{(N\cap\Gamma)\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y}) \cdot f(x)\,dx = \int_{N\backslash G}\int_{(N\cap\Gamma)\backslash N} \widehat{\varphi}_{nx,y}(\psi^{m_y}) \cdot f(nx)\,dn\,dx$$

$$= \int_{N\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y}) \cdot \left(\int_{(N\cap\Gamma)\backslash N} f(nx)\,dn\right)\,dx$$

and the inner integral is 0 exactly because f is a cuspform. Thus, for cuspforms f the image $\varphi\cdot f$ is computed dropping from the summation the character ψ vanishing on $\mathfrak{n}$, namely $\psi = 1$.

Let α be the unique positive root

$$\alpha\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = m_1/m_2$$

The modular function $\delta(m)$ is the norm $|\alpha(m)|$ of $\alpha(m)$, for $m \in M$. Thus, for $0 \neq \psi$, in the expression for $\varphi\cdot f$, with x, y in a fixed Siegel set, there is a constant c_ℓ uniform over $\psi \neq 0$ such that

$$|\widehat{\varphi}_{x,y}(\psi^{m_y})| \;\leq\; c_\ell \cdot \delta(m_y)\cdot(1+|\psi^{m_y}|)^{-\ell} \;\leq\; c_\ell\cdot|\alpha(m_y)|^{1-\ell}\cdot(1+|\psi|)^{-\ell}$$

For $1-\ell < -1$, the sum of norms $|\psi|^{-\ell}$ over non-zero elements of $\Lambda^* \approx \mathbb{Z}$ is absolutely convergent. Thus, for $\ell > 2$ and for cuspforms f there is a constant C independent of x, y, f such that

$$|(\varphi\cdot f)(y)| \;\leq\; C\cdot|\alpha(m_y)|^{1-\ell} \qquad \text{(for } x,y \text{ in a fixed Siegel set)}$$

In the simple case of $SL_2(\mathbb{Z})$ this is the desired asymptotic assertion for the kernel.

Recall that, for y in the fixed Siegel set $\mathfrak{S} = \mathfrak{S}(\Omega_N, t_o)$, there is a compact Ω_M in M such that, for $\varphi(y^{-1}nx)$ to be nonzero, the M-component m_x of x must lie in $m_y\cdot\Omega_M$. Thus,

$$\{x \in \mathfrak{S} : \varphi(y^{-1}nx) \neq 0 \text{ for some } n \in N\} \;\subset\; \Omega_N\cdot m_y\Omega_M\cdot K$$

Combining this with the estimate just obtained for the kernel,

$$|(\varphi\cdot f)(y)| \;\leq\; \text{const}\cdot\delta(m_y)\cdot\sup_{\alpha\in\Sigma^o}|\alpha(m_y)|^{-\ell/\sigma}\cdot\int_{\Gamma\backslash\Gamma(\Omega_N\cdot m_y\Omega_M\cdot K)}|f(x)|\,dx$$

By Cauchy-Schwarz-Bunyakowsky

$$\int_{\Gamma\backslash\Gamma(\Omega_N\cdot m_y\Omega_M\cdot K)}|f(x)|\,dx \leq \left(\int_{\Gamma\backslash\Gamma(\Omega_N\cdot m_y\Omega_M\cdot K)} 1\,dx\right)^{1/2}\cdot\left(\int_{\Gamma\backslash\Gamma(\Omega_N\cdot m_y\Omega_M\cdot K)}|f(x)|^2\,dx\right)^{1/2}$$

$$\leq\; \text{meas}\,(\Gamma\backslash G)\cdot|f|_{L^2}$$

That is, for y in a fixed Siegel set,

$$|(\varphi\cdot f)(y)| \leq \text{const}\cdot\sup_{\alpha\in\Sigma^o}|\alpha(m_y)|^{-\ell}\cdot|f|_{L^2}$$

This is the desired decay for $SL(2,\mathbb{Z})\backslash SL_2(\mathbb{R})$. ///

---

4. *Asymptotics of images $\varphi\cdot f$ in general*

Again, the compactness of a certain set of test functions proven above leads to proof that $\varphi\cdot f$ is *of rapid decay on Siegel sets*, with f a cuspform and φ a test function on G:

[4.0.1] Proposition: For y in a fixed Siegel set, and for φ in compact set E in $C_c^\infty(G)$, for every $\ell > 0$ there is a constant such that, for every L^2 cuspform f,

$$|(\varphi \cdot f)(y)| \;\le\; \text{const} \cdot \sup_{\alpha \in \Sigma^o} |\alpha(m_y)|^{-\ell} \cdot |f|_{L^2(\Gamma\backslash G)}$$

[4.0.2] Remark: This result incidentally yields the assertion of the first lemma, that such images are *pointwise bounded*, which will lead to proof of the compactness of the operator $f \to \varphi \cdot f$ on cuspforms, and corollaries concerning various classes of cuspforms.

Proof: Return to the Fourier transform

$$\psi^{th} \text{ summand} = \delta(m_y)\, \widehat{\varphi}_{x,y}(\psi^{m_y})$$

For $G = SL_n$, with α the i^{th} standard simple root $\alpha(m) = m_i/m_{i+1}$, let N_α be matrices of the shape[18]

$$N_\alpha = \begin{pmatrix} 1_i & i\text{-by-}(n-i) \\ 0 & 1_{n-i} \end{pmatrix}$$

and let $\mathfrak{n}_\alpha$ be its Lie algebra

$$N_\alpha = \begin{pmatrix} 0_i & i\text{-by-}(n-i) \\ 0 & 0_{n-i} \end{pmatrix}$$

[4.0.3] Claim: If a character $\psi \in \mathfrak{n}^*$ vanishes identically on $\mathfrak{n}_\alpha$, then $\widehat{\varphi}_{x,y}(\psi)$ is left N_α-invariant in x, and (therefore) integrates to 0 (over $(N_\alpha \cap \Gamma)\backslash N_\alpha$ so also integrates to 0 over $(N \cap \Gamma)\backslash G$) against a cuspform.

Proof: Let $n \in N_\alpha$, and replace x by nx in the original integral defining $\widehat{\varphi}_{x,y}(\psi)$, obtaining

$$\int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi\left(y^{-1} \exp(\nu) \cdot nx\right)\, d\nu = \int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi\left(y^{-1} \exp(\nu + \nu') \cdot x\right)\, d\nu$$

where $\nu' \in \mathfrak{n}_\alpha$ is a (continuous) function of ν determined by

$$\nu + \nu' = \log\left(\exp(\nu) \cdot n\right)$$

using the fact that the series for log terminates after finitely-many steps. The fact that ν' is in the subalgebra $\mathfrak{n}_\alpha$ rather than merely $\mathfrak{n}$ follows from the Baker-Campbell-Hausdorff-Dynkin formula. Replacing ν by $\nu - \nu'$ in the integral gives

$$\int_{\mathfrak{n}} \overline{\psi}(\nu) \cdot \varphi(y^{-1} \exp(\nu) \cdot nx)\, d\nu = \psi(\nu') \cdot \widehat{\varphi}_{x,y}(\psi) = \widehat{\varphi}_{x,y}(\psi)$$

proving the left N_α-invariance in x. With

$$K_\varphi(x,y) = \sum_{\nu \in \Lambda} \varphi(y^{-1} \cdot \exp(\nu) \cdot x) = \delta(m_y) \sum_{\psi \in \Lambda^*} \widehat{\varphi}_{x,y}(\psi^{m_y})$$

we have

$$(\varphi \cdot f)(y) = \delta(m^y) \sum_{\psi \in \Lambda^*} \int_{(N\cap\Gamma)\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y}) \cdot f(x)\, dx$$

[18] Intrinsically, for a simple positive root α, let $\mathfrak{n}_\alpha = \sum_{\beta \ge \alpha} \mathfrak{g}_\beta$ be the sum of all the rootspaces $\mathfrak{g}_\beta$ with positive roots $\beta \ge \alpha$. This $\mathfrak{n}_\alpha$ is the unipotent radical of a maximal proper parabolic determined by the root α.

For ψ left N_α-invariant, the corresponding integral is

$$\int_{(N\cap\Gamma)\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y})\cdot f(x)\,dx = \int_{(N\cap\Gamma)N_\alpha\backslash G}\int_{(N_\alpha\cap\Gamma)\backslash N_\alpha} \widehat{\varphi}_{nx,y}(\psi^{m_y})\cdot f(nx)\,dn\,dx$$

$$= \int_{(N\cap\Gamma)N_\alpha\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y})\cdot \left(\int_{(N_\alpha\cap\Gamma)\backslash N_\alpha} f(nx)\,dn\right)dx$$

and the inner integral is 0 exactly because f is a cuspform. ///

Thus, for *cuspforms* f

$$(\varphi\cdot f)(y) = \delta(m_y)\sum_\psi{}' \int_{(N\cap\Gamma)\backslash G} \widehat{\varphi}_{x,y}(\psi^{m_y})\,f(x)\,dx \qquad (\text{over } \psi\in\Lambda^* \text{ with } \mathrm{pr}_\alpha\psi\neq 0 \text{ for all simple } \alpha)$$

Give the dual $\mathfrak{g}_\beta^*$ of each rootspace $\mathfrak{g}_\beta$ any (positive linear homogeneous) norm,[19] and give $\mathfrak{n}_\alpha^* = \sum_{\beta\geq\alpha}\mathfrak{g}_\beta^*$ the sup-of-norms-of-summands norm

$$\left|\sum_{\beta\geq\alpha} v_\beta\right|_\alpha = \sup_{\beta\geq\alpha} |v_\beta|_\beta$$

Via the natural maps

$$\mathrm{pr}_\alpha : \mathfrak{n}^*\to\mathfrak{n}_\alpha^* \qquad (\text{by } (\mathrm{pr}_\alpha\psi)(\nu) = \psi(\nu), \text{ for } \nu\in\mathfrak{n}_\alpha)$$

put a *norm* on the *archimedean* part $\mathfrak{n}_\infty^*$ of the dual $\mathfrak{n}^*$ (of $\mathfrak{n}$), by[20]

$$|\psi_\infty| = \sup_\alpha |\mathrm{pr}_\alpha\psi_\infty|_\alpha \qquad (\text{sup over positive simple } \alpha)$$

In these terms, the uniform estimate on Fourier transforms is that, for every exponent $\ell>0$, there is a *uniform* constant c_ℓ and a compact open subgroup $C_{\mathrm{fin}}\subset\mathfrak{n}_{\mathrm{fin}}^*$ such that for all x,y (in the fixed Siegel set)

$$|\widehat{\varphi}_{x,y}(\psi)| \leq \begin{cases} c_\ell\cdot|\psi_\infty|^{-\ell} & (\text{for } \psi\in\mathfrak{n}^* \text{ with } \psi_{\mathrm{fin}}\in C_{\mathrm{fin}})\\ 0 & (\text{for } \psi\in\mathfrak{n}^* \text{ with } \psi_{\mathrm{fin}}\notin C_{\mathrm{fin}})\end{cases}$$

However, the argument to $\widehat{\varphi}_{x,y}$ in the expression for $(\varphi\cdot f)(y)$ is ψ^{m_y}, not just ψ. Note that the estimate does apply to all $\psi\in\mathfrak{n}^*$, not only to ψ in the dual lattice Λ^*.

The last thing needed from reduction theory is that $M_\mathbb{Q}\backslash M_\mathbb{A}$ has representatives of the form $m_\infty\cdot m_{\mathrm{fin}}$ with m_{fin} in a compact subset $\Omega_{M,\mathrm{fin}}$ of M_{fin}.[21] Replace the compact open subgroup C_{fin} of $\mathfrak{n}_{\mathrm{fin}}^*$ by a larger

[19] As usual, a function $v\to|v|$ on an $\mathbb{R}$-vectorspace is a (positive-linear homogeneous) *norm* when: $|v|\geq 0$ with strict inequality unless $v=0$, $|v+v'|\leq|v|+|v'|$, and for $t\in\mathbb{R}$ there is the homogeneity $|t\cdot v| = |t|\cdot|v|$ (with $|t|$ the usual absolute value on $\mathbb{R}$).

[20] Of course, the sup over positive simple roots α of norms of projections is the same as the sup over all positive roots β, in light of the definition of the norm on $\mathfrak{n}_\alpha$, but we will use this grouping.

[21] In cases such as SL_n, this reduces to the fact that the quotient of the ideles of norm 1 by principal ideles is *compact*. This assertion for arbitrary number fields is often called *Fujisaki's lemma*, dating from the mid or early 1950's. It implies finiteness of generalized class numbers as well as the generalized units theorem, and replaces these classical assertions in the context of harmonic analysis on adele groups. More generally, the Levi component M of a minimal parabolic may have a significant anisotropic factor, but, happily, the arithmetic quotient of an anisotropic group is compact.

compact open subgroup $C'_{\rm fin}$ with the property that if $\psi_{\rm fin}^m \in C_{\rm fin}$ for *some* $m \in \Omega_{M,{\rm fin}}$ then $\psi_{\rm fin} \in C'_{\rm fin}$. Then, the uniform asymptotic description becomes

$$|\widehat{\varphi}_{x,y}(\psi^{m_y})| \le \begin{cases} c_\ell \cdot |\psi_\infty^{m_y}|^{-\ell} & (\text{for } \psi_{\rm fin} \in C'_{\rm fin}) \\ 0 & (\text{for } \psi_{\rm fin} \not\in C'_{\rm fin}) \end{cases}$$

Now we come to the slightly delicate part. Each projection

$$\mathrm{pr}_\alpha(\Lambda^* \cap (\mathfrak{n}_\infty^* \times C'_{\rm fin}))$$

is a *lattice* in $(\mathfrak{n}_\alpha^*)_\infty$. Thus, the infimum μ_α of norms of non-zero elements is *strictly positive*, that is,

$$\mu_\alpha = \inf |\mathrm{pr}_\alpha \psi_\infty|_\alpha > 0 \qquad\qquad (\text{over } \psi \in \Lambda^* \text{ with } \mathrm{pr}_\alpha\psi \ne 0 \text{ and } \psi_{\rm fin} \in C'_{\rm fin})$$

For convenience in the estimates below, we can adjust these (positive) lower bounds μ_α so that $\mu_\alpha \le 1$. Let σ be the cardinality of the set Σ^o of positive simple roots. Without loss of generality suppose that the (positive) constant t_o in the definition of the Siegel set satisfies $t_o \le 1$. Let L be the maximum of the sum $\sum_\alpha n_\alpha$ of coefficients needed to express any positive root β as $\beta = \sum_\alpha n_\alpha \alpha$ as a non-negative integer linear combination of positive simple roots α. (For $G = SL_n$, this L is $L = n-1$, but the specific value is an inessential detail.) Then the action of m_y on a vector v in $\mathfrak{n}_\alpha^*$ can be analyzed in detail. Express v as a sum $v = \sum_{\beta \ge \alpha} v_\beta$ of vectors v_β in root spaces $\mathfrak{g}_\beta$. Then

$$v^{m_y} = \sum_\beta v_\beta^{m_y} = \sum_\beta \beta(m_y) \cdot v_\beta$$

Since $|\alpha(m_y) \ge t_o$ for all α,

$$|\beta(m_y)| \ge t_o^{L-1} |\alpha(m_y)| \qquad\qquad (\text{for all } \beta \ge \alpha, \text{ with } y \in \mathfrak{S})$$

Then

$$|v^{m_y}|_\alpha = \sup_\beta |v_\beta^{m_y}| = \sup_\beta |\beta(m_y) \cdot v_\beta| \ge t_o^{L-1} \cdot |\alpha(m_y)| \cdot \sup_\beta |v_\beta| = t_o^{L-1} \cdot |\alpha(m_y)| \cdot |v|_\beta$$

Therefore,

$$|\psi_\infty^{m_y}|^\sigma \ge \Pi_{\alpha\in\Sigma^o} |\mathrm{pr}_\alpha \psi_\infty^{m_y}|_\alpha \ge \Pi_{\alpha\in\Sigma^o} \left(t_o^{L-1}\, |\alpha(m_y)|\, |\mathrm{pr}_\alpha \psi_\infty|_\alpha \right)$$

$$\ge t_o^{(L-1)\sigma}\, (\Pi_{\alpha\in\Sigma^o} |\alpha(m_y)|)\, |\psi_\infty| \ge \left(t_o^{(L-1)\sigma}\, t_o^{\sigma-1}\, \Pi_\alpha \mu_\alpha \right) \sup_{\alpha\in\Sigma^o} |\alpha(m_y)| \cdot |\psi_\infty|$$

using at the end the simplifying normalization that $\mu_\alpha \le 1$. That is, for some constant C,

$$|\psi_\infty^{m_y}| \le C \cdot \sup_{\alpha\in\Sigma^o} |\alpha(m_y)|^{1/\sigma} \cdot |\psi_\infty|^{1/\sigma}$$

Then

$$|\widehat{\varphi}_{x,y}(\psi^{m_y}| \le \begin{cases} c_\ell \cdot \delta(m_y) \cdot C^{-\ell} \cdot \sup_{\alpha\in\Sigma^o} |\alpha(m_y)|^{-\ell/\sigma} \cdot |\psi_\infty|^{-\ell/\sigma} & (\text{for } \psi_{\rm fin} \in C'_{\rm fin}) \\ 0 & (\text{for } \psi_{\rm fin} \not\in C'_{\rm fin}) \end{cases}$$

For ℓ sufficiently large, the sum

$$\sum_\psi{}' |\psi_\infty|^{-\ell/\sigma}$$

is convergent, and then for some constant

$$\sum_\psi{}' \, |\widehat{\varphi}_{x,y}(\psi^{m_y})| \le \mathrm{const} \cdot \delta(m_y) \cdot \sup_{\alpha\in\Sigma^o} |\alpha(m_y)|^{-\ell/\sigma}$$

This is our desired estimate on the kernel.

Recall that, for x, y in the fixed Siegel set $\mathfrak{S}(\Omega_N, t_o)$, there is a compact Ω_M in M such that, for $\varphi(y^{-1}nx)$ to be nonzero for any $\varphi \in E$, the M-component m_x of x must lie in $m_y \cdot \Omega_M$. Thus,

$$\{x \in \mathfrak{S} : \varphi(y^{-1}nx) \neq 0 \text{ for some } n \in N\} \subset \Omega_N \cdot m_y\Omega_M \cdot K$$

Combining this with the estimate just obtained for the kernel,

$$|(\varphi \cdot f)(y)| \leq \text{const} \cdot \delta(m_y) \cdot \sup_{\alpha \in \Sigma^o} |\alpha(m_y)|^{-\ell/\sigma} \cdot \int_{\Gamma\backslash\Gamma(\Omega_N \cdot m_y\Omega_M \cdot K)} |f(x)|\, dx$$

By Cauchy-Schwarz-Bunyakowsky

$$\int_{\Gamma\backslash\Gamma(\Omega_N \cdot m_y\Omega_M \cdot K)} |f(x)|\, dx \leq \left(\int_{\Gamma\backslash\Gamma(\Omega_N \cdot m_y\Omega_M \cdot K)} 1\, dx\right)^{1/2} \cdot \left(\int_{\Gamma\backslash\Gamma(\Omega_N \cdot m_y\Omega_M \cdot K)} |f(x)|^2\, dx\right)^{1/2}$$

$$\leq \text{meas}\,(\Gamma\backslash G) \cdot |f|_{L^2}$$

That is, at last, we have the desired estimate (for y in a fixed Siegel set)

$$|(\varphi \cdot f)(y)| \leq \text{const} \cdot \delta(m_y) \cdot \sup_{\alpha \in \Sigma^o} |\alpha(m_y)|^{-\ell/\sigma} \cdot |f|_{L^2}$$

This is the desired decay property. ///

[4.0.4] Corollary: There is a uniform pointwise bound for the images $\varphi \cdot f$ for f a cuspform with $|f|_{L^2} \leq 1$ and for φ in a compact E inside $C_c^\infty(G)$.

Proof: The proposition shows that $\varphi \cdot f$ is pointwise bounded on a Siegel set by a bound that does not depend upon f. A sufficiently large Siegel set surjects to the quotient. ///

---

5. *Compactness arguments*

Some corollaries bring us closer to proving the compactness of the operators $f \to \varphi \cdot f$ on cuspforms f.

Recall that a collection E of continuous functions on G or $\Gamma\backslash G$ is *uniformly equicontinuous* when, given $\varepsilon > 0$, there is a neighborhood U of 1 in G such that

$$|f(x) - f(y| < \varepsilon \qquad \text{(for all } f \in E\text{, for all } x \in G\text{, for all } y \in x \cdot U)$$

We have the unsurprising

[5.0.1] Lemma: The left-derivative map

$$\mathfrak{g} \times C_c^\infty(G_\infty) \to C_c^\infty(G_\infty)$$

by

$$X \times \varphi \to \left(g \to \frac{d}{dt}\bigg|_{t=0} \varphi(e^{-tX}\, g)\right)$$

is a continuous map $\mathfrak{g} \times C_c^\infty(G_\infty) \to C_c^\infty(G_\infty)$.

Proof: The topology on $C_c^\infty(G_\infty)$ is that of a colimit of Fréchet spaces, the limit being taken over spaces $\mathcal{D}_K$ of smooth functions on G_∞ supported on compacts K. The topology on each $\mathcal{D}_K$ is given by seminorms taking sups of *derivatives* of all orders, but the notion of *derivative* is *a priori* ambiguous, since there are at least two different choices of global vectorfields, *left* derivatives by $\mathfrak{g}$, and *right* derivatives by $\mathfrak{g}$. But a simple general assertion is true, for more elementary reasons. On a smooth manifold, let X^i be a tuple of (smooth) vector fields on an open U containing a given compact set K such that, for every x in U, the values X^i_x at x are a basis for the tangent space at x. Another such tuple Y^j can be expressed (smoothly, pointwise) as linear combinations of the X^i, and vice-versa. The determinant of the matrix of coefficients does not vanish on the compact K, the coefficients are smooth functions, and the inverse is also smooth on K. Thus, the two sets of seminorms

$$\sup_{x\in K}\sup_i (X_i\varphi)(x) \qquad \sup_{x\in K}\sup_j (Y_j\varphi)(x)$$

are topologically equivalent. With this ambiguity removed, the assertion of the lemma is essentially the definition. ///

[5.0.2] Corollary: For a compact set E of test functions on G_∞, for a compact C in $\mathfrak{g}$, and for f ranging over cuspforms in the unit ball in $L^2(\Gamma\backslash G)$, there is a uniform constant C such that

$$\left| \frac{d}{dt}\Big|_{t=0} (\varphi\cdot f)(g\,e^{tX}) \right| \le C \qquad \text{(for all } \varphi\in E\text{, for all } X\in C)$$

Proof: The differentiation of $\varphi\cdot f$ can be rewritten as a differentiation of (the archimedean part of) φ, as follows.

$$(X\cdot\varphi\cdot f)(x) = \frac{d}{dt}\Big|_{t=0}\int_G \varphi(y)\,f(x\,e^{tX}\,y)\,dy$$

$$= \frac{d}{dt}\Big|_{t=0}\int_G \varphi(e^{-tX}y)\,f(x\,y)\,dy = \int_G \left(\frac{d}{dt}\Big|_{t=0}\varphi(e^{-tX}y)\right) f(x\,y)\,dy$$

by replacing y by $e^{-tX}y$. [22] That is,

$$X\cdot\varphi\cdot f = (X^{\text{left}}\varphi)\cdot f$$

with X^{left} denoting the *left* action. The previous lemma shows that the map $X\to X^{\text{left}}\varphi$ is continuous. By the boundedness result of the previous section, there is a constant depending only upon E and U such that for all $\varphi\in X$

$$|X\cdot\varphi\cdot f|_{L^2} = |(X^{\text{left}}\varphi)\cdot f|_{L^2} \le \text{const}\cdot|f|_{L^2}$$

which proves the uniform boundedness for $|f|_{L^2}\le 1$. ///

[5.0.3] Remark: Each function $\varphi\cdot f$ is smooth as a function on G_∞, since φ is.

It is not surprising that a uniform bound on derivatives implies uniform continuity:

[5.0.4] Lemma: Let F be a smooth function on G_∞, with a uniform pointwise bound for all $X\cdot F$ with X in a *compact* neighborhood C of 0 in $\mathfrak{g}$

$$|(X\cdot F)(x)| \le B \qquad \text{(for all } x\in G\text{, all } X\in C)$$

Then F is *uniformly continuous*, in the sense that for every $\varepsilon>0$ there is a neighborhood U of 1 in G_∞ such that for all $x\in G_\infty$ and $y\in xU$

$$|F(x)-F(y)| < \varepsilon$$

Proof: Let V be a small enough open containing 1 such that V is contained in $\exp C$, and that the exponential map on $\exp^{-1}(C\cap V)$ is injective to V. Let $y = x\cdot e^{sX}$ for $X\in C$ and $0\le s\le 1$. By hypothesis, the function

$$h(t) = F(x\cdot e^{sX})$$

[22] And with the interchange of differentiation and integration justified by the fact that differentiation is a continuous linear map on test functions φ, so commutes with the Gelfand-Pettis integral.

has

$$h'(s) = \frac{d}{dt}\Big|_{t=0} h(s+t) = F(x \cdot e^{sX} \cdot e^{tX})$$

bounded by B. From the mean value theorem,

$$|F(x \cdot e^{tX}) - F(x)| \le t \cdot B$$

Thus, for all $|t| < B \cdot \varepsilon$ we have the desired inequality. ///

[5.0.5] Corollary: For a compact set E of test functions on G_∞, for a compact C in $\mathfrak{g}$, and for f ranging over cuspforms in the unit ball in $L^2(\Gamma\backslash G)$, the family of images $\varphi \cdot f$ is *equicontinuous* on G_∞. ///

The non-archimedean analogue of an inference of uniform continuity from smoothness is simpler, due to the simpler nature of smoothness in the non-archimedean situation. Still, there is a minor issue of a left-right ambiguity somewhat akin to that in the archimedean case.

[5.0.6] Lemma: Let E be a compact subset of $C_c^\infty(G_{\rm fin})$. Then there is a compact open subgroup $K_{\rm fin}$ of $G_{\rm fin}$ such that, for any $\mathbb{C}$-valued function F on $G_{\rm fin}$, all functions $\varphi \cdot F$ are right $K_{\rm fin}$-invariant.

Proof: The topology on $C_c^\infty(G_{\rm fin})$ for any totally disconnected group $G_{\rm fin}$ is as colimit (ascending union) of finite-dimensional subspaces

$$V_{K,C} = \{f \in C_c^\infty(G_{\rm fin}) : f \text{ is right } K\text{-invariant, supported on } C\}$$

as K varies over compact open subgroups and C varies over compact subsets of $G_{\rm fin}$. The topology on the colimit [23] is such that any compact subset lies in one of the limitands. Thus, there is a compact open subgroup K such that every φ in E is *right* K-invariant. However, the condition that will arise concerns *left* invariance, as is visible from writing out the definition: for $g \in G$ and $k \in K$

$$(\varphi \cdot f)(gk) = \int_{G_{\rm fin}} \varphi(h)\, f(gkh)\, dh = \int_{G_{\rm fin}} \varphi(k^{-1}h)\, f(gh)\, dh$$

We also use the fact that there is a uniform compact subset C containing the support of φ (φ in a compact set E). Using the compactness, there is a *finite* cover

$$C \subset \bigcup_i x_i \cdot K$$

We claim that right K-invariant φ supported within C is *left* invariant by the (compact open) subgroup

$$K' = \bigcap_i x_i^{-1} K x_i$$

Indeed, for $h \in K'$ and $x = x_j k_o$ in C with $k_o \in K$,

$$\varphi(hx) = \varphi(h \cdot x_j k_o) = \varphi(h \cdot x_j) = \varphi(x_j \cdot x_j^{-1} h x_j) = \varphi(x_j) = \varphi(x_j k_o)$$

This proves the uniform left invariance. Thus, from the integral expressions for $\varphi \cdot f$, we conclude that $\varphi \cdot f$ is right invariant by this K'. ///

[23] A colimit of finite-dimensional vector spaces is one of the least subtle topological vector spaces, but not completely trivial. In particular, to think that there is *no* topology on it needlessly loses the sense of topological features such as compactness.

[5.0.7] Corollary: For a compact set E of test functions on G_{fin}, for f ranging over cuspforms in the unit ball in $L^2(\Gamma\backslash G)$, the family of images $\varphi \cdot f$ is *equicontinuous* on G_{fin}. ///

In summary,

[5.0.8] Corollary: For a compact set E of test functions on G, for f ranging over cuspforms in the unit ball in $L^2(\Gamma\backslash G)$, the family of images $\varphi \cdot f$ is uniformly bounded and *equicontinuous* on G. ///

We are almost done.

---

6. *A compactness lemma*

Let G be a unimodular topological group, Γ a discrete subgroup, with the natural measure on $\Gamma\backslash G$ *finite*. Let $C^o_{\text{bdd}}(\Gamma\backslash G)$ be the Banach space of *bounded* continuous functions, with sup norm. Suppose that G has a countable dense subset $\{x_i\}$.

[6.0.1] Proposition: Let E be a subset of the unit ball in $C^o_{\text{bdd}}(\Gamma\backslash G)$, and suppose that E is *(uniformly) equicontinuous.* Then E has *compact closure* in $L^2(\Gamma\backslash G)$.

[6.0.2] Remark: Superficially, this is reminiscent of the Arzela-Ascoli theorem [24] and it is common to end a sketch of this discrete decomposition by an allusion to Arzela-Ascoli. However, in fact, we need *less* than Arzela-Ascoli, and this is fortunate since adaptation of the literal Arzela-Ascoli result to the present circumstance seems awkward.

Proof: The proof is a maneuver to invoke the fact that a totally bounded subset of a complete metric space has compact closure. [25] Without loss of generality, all the functions in E are bounded (in absolute value) by 1, and the total measure of $\Gamma\backslash G$ is 1.

Take $\varepsilon > 0$. Using the equicontinuity, let U be a small enough neighborhood (with compact closure) of 1 in G such that for any $x \in G$ and $y \in xU$ we have $|F(x) - F(y)| < \varepsilon$ for all $F \in E$.

Let $U_i = x_i U$. Let $q : G \to \Gamma\backslash G$ be the quotient map. Let V_1 be the image of U_1 in $\Gamma\backslash G$, and recursively

$$V_{n+1} = \{x \in \Gamma\backslash G : \; x \in qU_i \;\text{ but }\; x \notin q(U_1 \cup \ldots \cup U_n)\}$$

Since the V_i's are disjoint and their union is $\Gamma\backslash G$, which has finite measure,

$$\sum_i \text{meas}\,(V_i) < +\infty$$

In particular, the measures $\text{meas}\, V_i$ go to 0 as $i \to \infty$. Take m large enough such that

$$\sum_{i>m} \text{meas}\,(V_i) < \varepsilon$$

Let X be a *finite* set of complex numbers such that any complex number of absolute value at most 1 is within $\varepsilon/2$ of an element of X.

For each m-tuple $\xi = (\xi_1, \ldots, \xi_m)$ of elements of X, define a function $F_{i,\xi}$ on $\Gamma\backslash G$ by

$$F_{i,\xi}(x) = \begin{cases} \xi_i & (\text{for } x \in V_i,\; i \leq m) \\ 0 & (\text{for } x \in V_i,\; i > m) \end{cases}$$

[24] The standard Arzela-Ascoli theorem asserts that an equicontinuous and uniformly bounded subset of $C^o(K)$ (with sup norm) for a compact topological space K has compact closure in $C^o(K)$.

[25] If the quotient $\Gamma\backslash G$ were *compact* then we could simply invoke Arzela-Ascoli.

Given $F \in E$, for each $i \le m$ choose ξ_i such that

$$|F(x_i) - \xi_i| < \varepsilon$$

Then by the choice of U

$$|F(x) - \xi_i| \le |F(x) - F(x_i)| + |F(x_i) - \xi_i| < 2\varepsilon \qquad \text{(for } x \in V_i\text{)}$$

Then

$$\int_{\Gamma\backslash G} |F - F_\xi|^2 < \int_{V_1 \cup \ldots \cup V_m} (2\varepsilon)^2 + \int_{V_{m+1}\cup\ldots} 1 \le 4\varepsilon^2 \cdot \text{meas}\,\Gamma\backslash G + \text{meas}\,(V_{m+1} \cup \ldots) \le 4\varepsilon^2 + \varepsilon$$

Going back and tweaking the estimates, for given $\varepsilon > 0$ we can cover E by a finite number of balls of radius $\varepsilon > 0$ in $L^2(\Gamma\backslash G)$.

This sets up a typical argument that the closure is compact. [26] ///

[6.0.3] Remark: This is a weaker conclusion than an assertion of compactness in a space of continuous functions, which would *imply* compactness in $L^2(\Gamma\backslash G)$ for $\Gamma\backslash G$.

---

7. *The main compactness conclusion*

[7.0.1] Theorem: Let $\varphi \in C_c^\infty(G)$. The operator $f \to \varphi \cdot f$ gives a *compact* operator

$$L^2_{\text{cfm}}(\Gamma\backslash G) \to L^2_{\text{cfm}}(\Gamma\backslash G)$$

Proof: We review the argument. The asymptotics of the kernels prove pointwise boundedness of the image of the unit ball B of $L^2_{\text{cfm}}(\Gamma\backslash G)$. Discussion of derivatives and non-archimedean smoothness prove equicontinuity of the image of B. The missing compactness lemma (faux-Arzela-Ascoli) proves compactness of the closure of the image $\varphi \cdot B$. And these operators, being integrated versions of *right* translations, stabilize the subspace of cuspforms, which is defined by a *left* integral condition. Thus, the operator given by φ maps the unit ball to a set with compact closure, so is a compact operator. ///

---

8. *Decomposition by compact operators*

Let A be a (not necessarily commutative)$\mathbb{C}$-algebra. In the present context, a *representation* of A is a Hilbert space V on which A acts by continuous linear operators. Recall V is said to be *(topologically) irreducible* (with respect to A) when it has no proper *closed* subspace stable under the action of A. An A-homomorphism $T : V \to W$ of A-representation spaces V, W is a continuous linear map T commuting with A in the sense that $T(av) = aT(v)$ for $a \in A$ and $v \in V$.

The *multiplicity* of an A-irreducible V in a larger A-representation H is

$$\text{multiplicity of } V \text{ in } H \;=\; \dim_{\mathbb{C}}\, \text{Hom}_A(V, H)$$

[26] For example, we recall an argument that shows that every sequence $\{x_n\}$ in E has a convergent subsequence. In a finite cover by balls of radius 1, there is at least one such ball, B_0, with infinitely-many x_i in it. Inductively, cover $E \cap B_n$ with finitely-many balls of radius $2^{-(n+1)}$. At least one, say B_{n+1}, has infinitely-many elements of E in it. Then choose a subsequence with $x_{i_j} \in B_j$. By construction, this subsequence is Cauchy, and since the metric space is complete it converges.

[8.0.1] Theorem: Let A be an *adjoint-stable* algebra of *compact* operators on a Hilbert space V, *non-degenerate* in the sense that for every non-zero $v \in V$ there is $a \in A$ with $a \cdot v \neq 0$. Then V is a direct sum of *closed* A*-irreducible* subspaces, and each isomorphism class of irreducible A-space occurs with finite multiplicity.

Proof: To prove decomposability in the first place, we immediately reduce to the case that V allegedly has no proper irreducible A-subspaces, by replacing V by the orthogonal complement to the sum of all irreducible A-subspaces. There exists a *non-zero* self-adjoint operator T in A, since for a non-zero operator S in A, either $S + S^*$ or $S - S^*$ is non-zero (and $S + S^*$ and $(S - S^*)/i$ are self-adjoint).

From the spectral theorem for self-adjoint compact operators (on Hilbert spaces) there is a non-zero eigenvalue λ of the (non-zero) self-adjoint compact operator T on V. Among all A-stable closed subspaces choose one, W, such that the λ-eigenspace W_λ is of *minimal positive* dimension. Let w be a non-zero vector in W_λ. Then the closure of $A \cdot w$ is a closed subspace of W, and we claim that it is irreducible. Suppose that

$$\text{closure}(A \cdot w) = X \oplus Y$$

is a decomposition into orthogonal closed A-stable subspaces. With $w = w_X + w_Y$ the corresponding decomposition,

$$\lambda w_X + \lambda w_Y = \lambda w = Tw = T(w_X + w_Y)$$

By the orthogonality and stability,

$$\lambda w_X = T w_X \qquad \lambda w_Y = T w_Y$$

By the minimality of the λ-eigenspace in W, either $w_X = 0$ or $w_Y = 0$. That is, $\lambda w = Tw \subset X$ or $\lambda w = Tw \subset Y$. That is, since $\lambda \neq 0$, either $w \in X$ or $w \in Y$. Thus, either $A \cdot w \subset X$ or $A \cdot w \subset Y$, and likewise for the *closures*. But this implies that one or the other of X, Y is 0. This proves the irreducibility of the closure of $A \cdot w$. ///

[8.0.2] Corollary: The space $L^2_{\text{cfm}}(\Gamma\backslash G)$ of square-integrable cuspforms decomposes discretely with finite multiplicities into irreducibles for $C_c^\infty(G)$.

Proof: We have shown that $C_c^\infty(G)$ acts by compact operators. Adjoints are easily computed: letting $\langle , \rangle$ be the inner product on $L^2(\Gamma\backslash G)$,

$$\langle \varphi \cdot f, F\rangle = \int_{\Gamma\backslash G}\int_G \varphi(x)\, f(yx)\, \overline{F}(y)\, dx\, dy = \int_{\Gamma\backslash G}\int_G \varphi(x)\, f(y)\, \overline{F}(yx^{-1})\, dx\, dy$$

$$= \int_{\Gamma\backslash G}\int_G f(y)\, \varphi(x^{-1})\, \overline{F}(yx)\, dx\, dy = \int_{\Gamma\backslash G} f(y)\, \overline{\varphi^\vee \cdot F}(y)\, dy$$

where $\varphi^\vee(x) = \overline{\varphi}(x^{-1})$ as suggested by the computation. Certainly the space of test functions is stable under the operation $\varphi \to \varphi^\vee$.

What remains is to prove the *non-degeneracy*, which is not surprising. Let φ_i be an *approximate identity* in $C_c^\infty(G)$, meaning non-negative real-valued test functions whose supports shrink to $\{1\}$ and each of whose integrals is 1. [27] For *any* (continuous) representation of G on a locally convex quasi-complete topological vector space V, the continuity of the map

$$G \times V \to V$$

and the basic estimate for the Gelfand-Pettis integral together give an estimate on $\varphi_i \cdot v$. By the continuity, given a neighborhood $N + v$ of v in V (with neighborhood N of 0 in V) there is a small enough neighborhood

[27] Existence of approximate identities in $C_c^\infty(G)$ is not difficult to verify.

U of 1 in G such that for all $g \in U$ we have $g \cdot v \in N + v$. Take the index i large enough such that the support of φ_i is inside U. Consider, then, the V-valued function

$$F(x) = \begin{cases} x \cdot v - v & (\text{for } x \in \text{spt}\varphi_i) \\ 0 & (\text{for } x \notin \text{spt}\varphi_i) \end{cases}$$

on the measure space sptφ_i with measure $\varphi_i(x)\, dx$. The measure space has total mass 1, and F takes values in N. Thus, the basic estimate on Gelfand-Pettis integrals gives

$$ph_i \cdot v - v = \int_G (\varphi_i(x)\, x \cdot v - \varphi_i(x)\, v)\; dx = \int F(x)\, dx \in N$$

proving that $\varphi_i \cdot v \to v$ in V, for every $v \in V$. In particular, for $v \neq 0$ eventually $\varphi_i \cdot v$ is not 0, proving the non-degeneracy. ///

[8.0.3] Remark: From the definition of the action of $C_c^\infty(G)$, when a Hilbert space with a continuous action of G decomposes as a sum of closed G-stable subspaces it certainly decomposes similarly into $C_c^\infty(G)$-subspaces. The converse is not obviously true. Thus, as it stands, the assertion that the space of cuspforms decomposes discretely with finite multiplicities for $C_c^\infty(G)$ is *stronger* than the analogous assertion for G. In particular, we have *not* inadvertently lost either discreteness of finite multiplicities by using $C_c^\infty(G)$ rather than the action of G itself.

[Gelfand Fomin 1952] I.M. Gelfand, S.V. Fomin, *Geodesic flows on manifolds of constant negative curvature*, Uspekh. Mat. Nauk. **7** (1952), no. 1, 118-137. English translation, A.M.S. ser. 2, **1** (1965), 49-65.

[Gelfand-Graev 1959] I.M. Gelfand, M.I. Graev, *Geometry of homogeneous spaces, representations of groups in homogeneous spaces, and related problems of integral geometry*, Trudy Moskov. Obshch. **8** (1962), 321-390.

Gelfand-PS 1963] I.M. Gelfand, I.I. Piatetski-Shapiro, *Automorphic functions and representation theory*, Trudy Moskov. Obshch. **8** (1963), 389-412 [Trans.: Trans. Moscow Math. Soc. **12** (1963), 438-464.]

[Godement 1966] R. Godement, *Spectral decomposition of cuspforms*, in *Proc. Symp. AMS* IX, 1969 (Boulder Conference, 1966).

[Langlands 1967] R.P.Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, Berlin and New York, 1976.

[Selberg 1956] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with application to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-87.
