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Characters of principal series

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

The characters of principal series representations of reductive groups G over local fields, whether p-adic or archimedean, admit a simple common computation. The archimedean case was known to Gelfand-Graev, HarishChandra, and others in the early 1950's. The book of Gelfand-Graev-PiatetskiShapiro GL_2 emphasizes the commonalities of $GL_2(\mathbb{R})$ and $GL_2(\mathbb{Q}_p)$. Jacquet's work also made clear that the commonality of p-adic and archimedean cases extends to larger reductive groups.

Let P be a minimal parabolic subgroup of G, and χ a one-dimensional complex representation of P factoring through the quotient P/N of P by its unipotent radical N. Let $M \approx P/N$ be a Levi component of P. Let I^{nf}_χ be the naively normalized induced representation constructed as

$$I^{\mathrm{nf}}_\chi \;=\; \mathrm{Ind}_P^G \chi \;=\; \{\mathbb{C}\text{-valued functions } f \text{ on } G \;:\; f(pg) = \chi(p)\cdot f(g)\}$$

with G acting by right translations. Note that, by an Iwasawa decomposition $G = P \cdot K$, the values of a function $f \in I^{\mathrm{nf}}_\chi$ are determined by its values on K.

Functions $\eta \in C^o_c(G)$ act as usual on any representation π of G by superpositions

$$\eta \cdot v \;=\; \pi(\eta)(v) \;=\; \int_G \eta(g)\; g\cdot v\; dg \;=\; \int_G \eta(g)\; \pi(g)(v)\; dg$$

On $\pi = I_\chi$, with $x \in K$,

$$(\eta\cdot f)(x) \;=\; \int_G \eta(g)\; f(xg)\; dg \;=\; \int_G \eta(x^{-1}g)\; f(g)\; dg$$

$$= \int_K \int_P \eta(x^{-1}py)\; f(py)\; \frac{dp}{\delta(p)}\, dy \;=\; \int_K \int_P \eta(x^{-1}py)\; \chi(p)\; f(y)\; \frac{dp}{\delta(p)}\, dy$$

where δ is the modular function of P, and dp is a right Haar measure, so $dp/\delta(p)$ is a left Haar measure. This operator can be described by a kernel: extract the inner integral over P by setting

$$Q_\eta(x,y) \;=\; \int_P \eta(x^{-1}py)\; \chi(p)\; \frac{dp}{\delta(p)}$$

so then

$$(\eta\cdot f)(x) \;=\; \int_K Q_\eta(x,y)\, f(y)\, dy$$

The integral for $Q_\eta(x,y)$ has support inside $K\cdot \mathrm{spt}(\eta)\cdot K$, which is compact because the support of η is compact. By Gelfand-Pettis $Q_\eta(x,y)$ is continuous on $K\times K$. Thus, the operator $f \to \eta\cdot f$ is Hilbert-Schmidt. *If* it were of trace class, it would have trace computed by

$$\mathrm{tr}(f \to \eta\cdot f) \;=\; \int_K Q_\eta(x,x)\, dx \;=\; \int_K \int_P \eta(x^{-1}px)\; \chi(p)\; \frac{dp}{\delta(p)}\, dx$$

To make the symmetries of the latter integral more apparent, reparametrize the copy of P via the map

$$N\times M \longrightarrow P \qquad\qquad n\times m \to n^{-1}mn$$

The image includes all semi-simple elements of P, and is injective on the *regular* elements $M^{\rm reg}$ of M, essentially by definition. This map is N-equivariant, so the change-of-measure can be determined near $1 \times m$. From

$$n^{-1}mn \;=\; \Big(n^{-1} \cdot mnm^{-1}\Big) \cdot m \;\in\; N \cdot M$$

the change-of-measure is

$$\Delta(m) \;=\; \prod_{\alpha>0} \big|\alpha(m) - 1\big| \qquad\qquad (\text{for } m \in M^{\rm reg})$$

where the product is over positive roots α, with multiplicities. Thus,

$$\mathrm{tr}(f \to \eta \cdot f) \;=\; \int_K \int_N \int_{M^{\rm reg}} \eta(x^{-1}\,(n^{-1}mn)\,x)\;\chi(n^{-1}mn)\;\Delta(m)\,\frac{dn\;dm}{\delta(n^{-1}mn)}\,dx$$

$$=\; \int_K \int_N \int_{M^{\rm reg}} \eta((nx)^{-1}\,m\,(nx))\;\chi(m)\;\Delta(m)\,dn\,\frac{dm}{\delta(m)}\,dx \;=\; \int_{M\backslash G} \int_{M^{\rm reg}} \eta(\dot{g}^{-1}\,m\,\dot{g})\;\chi(m)\;\frac{\Delta(m)}{\delta(m)}\,dm\,d\dot{g}$$

Observe that Δ/δ is *almost* invariant under conjugation of M by the Weyl group W, but we are off by a factor of $\delta^{\frac{1}{2}}$. Thus, replacing the naive χ by the better-normalized $\delta^{\frac{1}{2}} \cdot \chi$,

$$\delta^{\frac{1}{2}} \cdot \chi \cdot \delta^{-1} \cdot \prod_{\alpha>0} |\alpha - 1| \;=\; \chi \cdot \prod_{\alpha>0} |\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}|$$

if we ignore the issue of ambiguity of individual square roots of characters. Since the totality of roots consists of positive roots and negatives of positive roots, and since the Weyl group preserves the negation of roots, the latter product is W-stable. Thus, we should use the less naive normalization

$$I_\chi \;=\; I^{\rm nf}_{\delta^{1/2}\cdot\chi}$$

Thus, summing over conjugates of the integrand by W, on the less-naively normalized I_χ,

$$\mathrm{tr}(f \to \eta \cdot f) \;=\; \frac{1}{\#W} \int_{M\backslash G} \int_{M^{\rm reg}} \eta(\dot{g}^{-1}\,m\,\dot{g})\;\frac{\Delta(m)}{\delta^{\frac{1}{2}}(m)} \sum_{w\in W} \chi^w(m)\;dm\,d\dot{g}$$

$$=\; \int_{M^{\rm reg}} \frac{\Delta(m)}{\delta^{\frac{1}{2}}(m)} \sum_{w\in W} \chi^w(m)\;\Big(\int_{M\backslash G} \eta(\dot{g}^{-1}\,m\,\dot{g})\;d\dot{g}\Big)dm$$

Without worrying further about normalizations, this expresses the trace in terms of an *orbital integral*

$$\mathcal{O}^M_\eta(m) \;=\; \int_{M\backslash G} \eta(\dot{g}^{-1}\,m\,\dot{g})\;d\dot{g}$$

of η, namely

$$\mathrm{tr}(f \to \eta \cdot f) \;=\; \int_{M^{\rm reg}} \Big(\frac{\Delta(m)}{\delta^{\frac{1}{2}}(m)} \sum_{w\in W} \chi^w(m)\Big) \;\cdot\; \mathcal{O}^M_\eta(m)\;\;dm$$