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The Classical Groups and Domains

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The complex unit disk

$$\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$$

has four families of generalizations to bounded open subsets in $\mathbb{C}^n$ with groups acting transitively upon them. Such domains, defined more precisely below, are **bounded symmetric domains**. First, we recall some standard facts about the unit disk, the upper half-plane, the ambient complex projective line, and corresponding groups acting by linear fractional (Möbius) transformations. Happily, many of the higher-dimensional bounded symmetric domains behave in a manner that is a simple extension of this simplest case.

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1. *The disk, upper half-plane, $SL_2(\mathbb{R})$, and $U(1,1)$*

The group

$$GL_2(\mathbb{C}) \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C},\ ad-bc \neq 0\}$$

acts on the extended complex plane $\mathbb{C} \cup \infty$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

with the traditional natural convention about arithmetic with ∞. But we can be more precise, in a form helpful for higher-dimensional cases: introduce *homogeneous coordinates* for the *complex projective line* $\mathbb{P}^1$, by defining $\mathbb{P}^1$ to be a set of cosets

$$\mathbb{P}^1 \;=\; \{\begin{pmatrix} u \\ v \end{pmatrix} :\ \text{not both } u,v \text{ are } 0\}/\mathbb{C}^\times \;=\; \left(\mathbb{C}^2 - \{0\}\right)/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts by scalar multiplication. That is, $\mathbb{P}^1$ is the set of equivalence classes of non-zero 2-by-1 vectors under the equivalence relation

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} u' \\ v' \end{pmatrix} \quad \text{if and only if} \quad \begin{pmatrix} u \\ v \end{pmatrix} = c \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} \quad \text{for some non-zero } c \in \mathbb{C}^\times$$

In other words, $\mathbb{P}^1$ is the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $\mathbb{C}^\times$. The complex line $\mathbb{C}^1$ is imbedded in this model of $\mathbb{P}^1$ by

$$z \to \begin{pmatrix} z \\ 1 \end{pmatrix}$$

The point at infinity ∞ is now easily and precisely described as

$$\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Certainly the matrix multiplication action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$ stabilizes non-zero vectors, and commutes with scalar multiplication, so $GL_2(\mathbb{C})$ has a natural action on $\mathbb{P}^1$. This action, linear in the homogeneous coordinates, becomes the usual linear *fractional* action in the usual complex coordinates:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \;=\; \begin{pmatrix} (az+b)/(cz+d) \\ 1 \end{pmatrix}$$

when $cz+d \neq 0$, since we can multiply through by $(cz+d)^{-1}$ to normalize the lower entry back to 1, unless it is 0, in which case we've mapped to ∞.

The action of $GL_2(\mathbb{C})$ is *transitive* on $\mathbb{P}^1$, since it is transitive on $\mathbb{C}^2 - \{0\}$, and $\mathbb{P}^1$ is the image of $\mathbb{C}^2 - \{0\}$.

The condition defining the open unit disk

$$\mathfrak{D} \;=\; \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$$

can be rewritten as

$$\mathfrak{D} \;=\; \{z \in \mathbb{C} : \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} > 0\}$$

where T^* is conjugate transpose of a matrix T. The standard **unitary group** $U(1,1)$ of signature $(1,1)$ is

$$U(1,1) \;=\; \{g \in GL_2(\mathbb{C}) : g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

We can easily prove that $U(1,1)$ stabilizes the disk $\mathfrak{D}$, as follows. For $z \in \mathfrak{D}$ and

$$g \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$$

compute

$$1 - |g(z)|^2 \;=\; \begin{pmatrix} g(z) \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g(z) \\ 1 \end{pmatrix} \;=\; \frac{1}{(cz+d)^*} \cdot \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \cdot \frac{1}{(cz+d)}$$

$$= \frac{1}{|cz+d|^2} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} z \\ 1 \end{pmatrix} \;=\; \frac{1}{|cz+d|^2} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

since g is in the unitary group. Then

$$1 - |g(z)|^2 \;=\; \frac{1}{|cz+d|^2}(1 - |z|^2) \;>\; 0$$

In fact, to be sure that $cz+d \neq 0$ in this situation, we should have really computed (by the same method) that

$$|cz+d|^2 \cdot (1 - |g(z)|^2) \;=\; 1 - |z|^2$$

Since $1 - |z|^2 > 0$ and $|cz+d| \geq 0$, necessarily $|cz+d| > 0$ and $1 - |g(z)|^2 > 0$.

The complex upper half-plane is

$$\mathfrak{H} \;=\; \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

In the spirit of giving matrix expressions for defining inequalities,

$$\operatorname{Im} z \;=\; \frac{1}{2i} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

This suggests taking

$$G \;=\; \{g \in GL_2(\mathbb{C}) : g^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$$

Just as in the discussion of the disk, for $z \in \mathfrak{H}$ and

$$g \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

compute

$$2i \cdot \mathrm{Im}\, g(z) \;=\; \begin{pmatrix} g(z) \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g(z) \\ 1 \end{pmatrix} \;=\; \frac{1}{(cz+d)^*} \cdot \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \cdot \frac{1}{(cz+d)}$$

$$= \frac{1}{|cz+d|^2} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} z \\ 1 \end{pmatrix} \;=\; \frac{1}{|cz+d|^2} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

since g is in G. Then

$$\mathrm{Im}\, g(z) \;=\; \frac{1}{|cz+d|^2} \mathrm{Im}\, z > 0$$

To be sure that $cz + d \neq 0$, compute in the same way that

$$|cz+d|^2 \cdot \mathrm{Im}\, g(z) \;=\; \mathrm{Im}\, z$$

Since $\mathrm{Im}\, z > 0$ and $|cz + d| \geq 0$, necessarily $|cz + d| > 0$ and $\mathrm{Im}\, g(z) > 0$.

Since the center

$$Z = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t \in \mathbb{C}\}$$

of $GL_2(\mathbb{C})$ acts trivially on $\mathbb{C} \subset \mathbb{P}^1$, one might expect some simplification of matters by restricting our attention to the action of

$$SL_2(\mathbb{C}) \;=\; \{g \in GL_2(\mathbb{C}) : \det g = 1\}$$

since under the quotient map $GL_2(\mathbb{C}) \to GL_2(\mathbb{C})/Z$ the subgroup $SL_2(\mathbb{C})$ maps surjectively to the whole quotient $GL_2(\mathbb{C})/Z$. The special but well-known formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \;=\; \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

for the inverse of a two-by-two matrix implies that

$$SL_2(\mathbb{C}) \;=\; \{g \in GL_2(\mathbb{C}) : g^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$$

Thus, the subgroup in $SL_2(\mathbb{C})$ stabilizing the upper half-plane is

$$G' \;=\; \{g \in GL_2(\mathbb{C}) : g^\top \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$$

In particular, for such g we have $g^\top = g^*$, so the entries of g are *real*. Thus, we have shown that

$$\{g \in SL_2(\mathbb{C}) : g \cdot \mathfrak{H} = \mathfrak{H}\} \;=\; SL_2(\mathbb{R}) \;=\; \{g \in GL_2(\mathbb{R}) : \det g = 1\}$$

This is the group usually prescribed to act on the upper half-plane. Again, *arbitrary* complex scalar matrices act trivially on $\mathbb{P}^1$, so it is reasonable that we get a clearer determination of this stabilizing group by restricting attention to $SL_2(\mathbb{C})$.

The **Cayley map** $\mathbf{c} : \mathfrak{D} \to \mathfrak{H}$ is the holomorphic isomorphism given by the linear fractional transformation

$$\mathbf{c} \;=\; \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in SL_2(\mathbb{C})$$

If we use

$$SU(1,1) \;=\; \{g \in SL_2(\mathbb{C}) : g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

instead of $U(1,1)$ as the group stabilizing the disk $\mathfrak{D}$, then it is easy to check that

$$\mathbf{c}^{-1} \cdot SL_2(\mathbb{R}) \cdot \mathbf{c} \;=\; SU(1,1)$$

Thus, the disk and the upper half-plane are the same thing apart from coordinates. On the upper half-plane, the subgroup

$$\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\} \subset SL_2(\mathbb{R})$$

acts by affine transformations

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} (z) \;=\; a^2 z + b$$

In particular, this shows that $SL_2(\mathbb{R})$ is transitive on $\mathfrak{H}$. On the other hand, in the action of $SU(1,1)$ on the disk $\mathfrak{D}$, the subgroup

$$\{\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : |\mu| = 1\}$$

acts by affine transformations

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} (z) \;=\; \mu^2 \cdot z$$

This is the subgroup of $SU(1,1)$ fixing the point $0 \in \mathfrak{D}$. When the latter subgroup is conjugated by the Cayley element $\mathbf{c}$ to obtain the corresponding subgroup of $SL_2(\mathbb{R})$ fixing $\mathbf{c}(0) = i \in \mathfrak{H}$, consisting of elements

$$\mathbf{c} \cdot \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \cdot \mathbf{c}^{-1} \;=\; \begin{pmatrix} \frac{\mu+\mu^{-1}}{2} & \frac{\mu-\mu^{-1}}{2i} \\ -\frac{\mu-\mu^{-1}}{2i} & \frac{\mu+\mu^{-1}}{2} \end{pmatrix} \;=\; \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for $\theta \in \mathbb{R}$ such that $\mu = e^{i\theta}$.

The *special unitary group* is

$$SU(1,1) \;=\; \{g \in SL_2(\mathbb{C}) : g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\} \;=\; \{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |a|^2 - |b|^2 = 1\}$$

The Cayley element $\mathbf{c}$ conjugates $SL_2(\mathbb{R})$ and $SU(1,1)$ back and forth. Since $SL_2(\mathbb{R})$ acts transitively on the upper half-plane, $SU(1,1)$ acts transitively on the disk. Also, this can be seen directly by verifying that for $|z| < 1$ the matrix

$$\frac{1}{\sqrt{1-|z|^2}} \cdot \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$$

lies in $SU(1,1)$ and maps 0 to z.

The isotropy group in $SL_2(\mathbb{R})$ of the point $i \in \mathfrak{H}$ is the *special orthogonal group*

$$SO(2) \;=\; \{g \in SL_2(\mathbb{R}) : g^\top g = 1_2\} = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\}$$

The natural map

$$SL_2(\mathbb{R})/SO(2) \longrightarrow \mathfrak{H}$$

given by

$$g \cdot SO(2) \longrightarrow g(i)$$

is readily verified to be a diffeomorphism.

The disk is an instance of *Harish-Chandra's realization* of quotients analogous to $SL_2(\mathbb{R})/SO(2)$ as bounded subsets of spaces $\mathbb{C}^n$. The ambient complex projective space $\mathbb{P}^1$ in which both the bounded (disk) and unbounded (upper half-plane) models of $SL_2(\mathbb{R})/SO(2)$ are realized is an instance of the *compact dual*, whose general construction is due to Borel. The Cayley map from a bounded to an unbounded model was studied in general circumstances by Piatetski-Shapiro for the classical domains, and later by Koranyi and Wolf instrinsically.

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2. *Classical groups over* $\mathbb{C}$ *and over* $\mathbb{R}$

We list the classical *complex* groups and the classical *real* groups. Further, we tell the *compact* real groups, and indicate the *maximal compact* subgroups of the real groups.

There are very few different classical groups over $\mathbb{C}$, because $\mathbb{C}$ is algebraically closed and of characteristic zero. They are

$$GL_n(\mathbb{C}) \;=\; \text{invertible } n\text{-by-}n \text{ matrices with complex entries} \qquad \text{(general linear groups)}$$

$$Sp_n(\mathbb{C}) \;=\{g \in GL_{2n}(\mathbb{C}) : g^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}\} \quad \text{(symplectic groups)}$$

$$O(n, \mathbb{C}) \;=\; \{g \in GL_n(\mathbb{C}) : g^\top g = 1_n\} \qquad \text{(orthogonal groups)}$$

Sometimes what we've denoted $Sp_n(\mathbb{C})$ is written $Sp_{2n}(\mathbb{C})$. Also, for some purposes the orthogonal groups with n *odd* are distinguished from those with n *even*. Closely related to $GL_n(\mathbb{C})$ are

$$\text{(special linear groups)}\;\; SL_n(\mathbb{C}) \;=\; \{g \in GL_n(\mathbb{C}) : \det g = 1\}$$

and likewise related to orthogonal groups are

$$\text{(special orthogonal groups)}\;\; SO(n, \mathbb{C}) \;=\; O(n, \mathbb{C}) \cap SL_n(\mathbb{C})$$

It is not hard to show that

$$Sp_n(\mathbb{C}) \subset SL_{2n}(\mathbb{C})$$

A special case of the latter is $Sp_1(\mathbb{C}) = SL_2(\mathbb{C})$. E. Cartan's labels for these families are

type A_n	$SL_{n+1}(\mathbb{C})$
type B_n	$O(2n+1, \mathbb{C})$
type C_n	$Sp_n(\mathbb{C})$
type D_n	$O(2n, \mathbb{C})$

Over $\mathbb{R}$ there is a greater variety of classical groups, since $\mathbb{R}$ has the proper algebraic extension $\mathbb{C}$, and also is the center of the non-commutative division algebra $\mathbb{H}$, the Hamiltonian quaternions

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = -1 \quad ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j$$

Each family of *complex* group breaks up into several different types of *real* groups. We will not worry about determinant-one conditions.

Type A: The groups over $\mathbb{R}$ affiliated with the complex general linear groups $GL_n(\mathbb{C})$ include are the obvious *general linear groups*

$$GL_n(\mathbb{R}) \qquad GL_n(\mathbb{C}) \qquad GL_n(\mathbb{H})$$

over the real numbers, complex numbers, and Hamiltonian quaternions. To this family also belong the *unitary groups*

$$U(p,q) \;=\; \{g \in GL_{p+q}(\mathbb{C}) : g^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} g = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\}$$

of *signature* (p,q). As usual, often $U(n,0)$ and $U(0,n)$ are denoted $U(n)$.

Types B,D: The groups over $\mathbb{R}$ affiliated with the complex orthogonal groups $O(n,\mathbb{C})$ include $O(n,\mathbb{C})$ itself, and *real orthogonal groups*

$$O(p,q) \;=\; \{g \in SL_{p+q}(\mathbb{R}) : g^\top \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} g = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\}$$

of signature (p,q). As usual, $O(n,0)$ and $O(0,n)$ are denoted $O(n)$. There are also the *quaternion skew-hermitian* groups

$$O^*(2n) \;=\; \{g \in GL_n(\mathbb{H}) : g^*(i \cdot 1_n)g = i \cdot 1_n\}$$

where g^* denotes the conjugate-transpose with respect to the quaternion conjugation (entry-wise)

$$a + bi + cj + dk \;\longrightarrow\; a - bi - cj - dk$$

The i occurring in the definition of the group is the *quaternion* i, and is *not* in the center of the division algebra of quaternions.

Types C: The groups over $\mathbb{R}$ affiliated with the complex symplectic groups $Sp_n(\mathbb{C})$ include $Sp_n(\mathbb{C})$ itself, and *real symplectic groups*

$$Sp_n(\mathbb{R}) \;=\; \{g \in GL_{2n}(\mathbb{R}) : g^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}\}$$

and also the *quaternion hermitian groups*

$$Sp^*(p,q) \;=\; \{g \in GL_{p+q}(\mathbb{H}) : g^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} g = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\}$$

with *signature* (p,q). As usual, $Sp^*(n,0)$ and $Sp^*(0,n)$ are denoted $Sp^*(n)$.

The fact that the signatures (p,q) are the unique invariants for groups $O(p,q)$, $U(p,q)$, and $Sp^*(p,q)$ (and that the other groups have only *dimension* as invariant) is the **Inertia Theorem**.

There are minor variations of the above families, obtained in two ways. The first is by adding a determinant-one condition:

$$\begin{aligned}
SL_n(\mathbb{C}) &= \{g \in GL_n(\mathbb{C}) : \det g = 1\} \\
SL_n(\mathbb{R}) &= \{g \in GL_n(\mathbb{R}) : \det g = 1\} \\
SU(p,q) &= U(p,q) \cap SL_{p+q}(\mathbb{C}) \\
SL_n(\mathbb{H}) &= GL_n(\mathbb{H}) \cap SL_{2n}(\mathbb{C}) \\
SO(p,q) &= O(p,q) \cap SL_{p+q}(\mathbb{R}) \\
SO(n,\mathbb{C}) &= O(n,\mathbb{C}) \cap SL_n(\mathbb{C}) \\
SO^*(2n) &= O^*(2n) \cap SL_n n(\mathbb{H})
\end{aligned}$$

A bit of thought is necessary to make sense of $SL_n(\mathbb{H})$, since $\mathbb{H}$ is not commutative, and thus determinants of matrices with entries in $\mathbb{H}$ do not behave as simply as for entries from commutative rings. The second variations, for groups preserving a quadratic, hermitian, or alternating form, is to allow **similitudes**, meaning that the form is possibly changed by a scalar. For example,

$$\begin{aligned} GU(p,q) &= \{g \in GL_{p+q}(\mathbb{C}) : g^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} g = \mu(g) \cdot \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \text{ for some } \mu(g) \in \mathbb{R}^\times\} \\ GO(p,q) &= \{g \in GL_{p+q}(\mathbb{R}) : g^\top \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} g = \mu(g) \cdot \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \text{ for some } \mu(g) \in \mathbb{R}^\times\} \\ GSp_n(\mathbb{R}) &= \{g \in GL_{2n}(\mathbb{R}) : g^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} g = \mu(g) \cdot \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \text{ for some } \mu(g) \in \mathbb{R}^\times\} \end{aligned}$$

The list of the classical **compact** groups among the groups over $\mathbb{R}$ is short:

$$U(n) \qquad O(n) \qquad Sp^*(n)$$

Each of the classical groups over $\mathbb{R}$ has a maximal compact subgroup, unique up to conjugation (by a theorem of E. Cartan, which can be proven case-by-case for these groups). We list the *isomorphism classes* of the maximal compact subgroups in the format $G \supset K$, where G runs over groups over $\mathbb{R}$ and K is a maximal compact subgroup.

$$\begin{array}{ccc} GL_n(\mathbb{C}) & \supset & U(n) \\ GL_n(\mathbb{R}) & \supset & O(n) \\ GL_n(\mathbb{H}) & \supset & Sp^*(n) \\ U(p,q) & \supset & U(p) \times U(q) \\ O(n,\mathbb{C}) & \supset & O(n) \\ O(p,q) & \supset & O(p) \times O(q) \\ O^*(2n) & \supset & U(n) \\ Sp_n(\mathbb{C}) & \supset & Sp^*(n) \\ Sp_n(\mathbb{R}) & \supset & U(n) \\ Sp^*(p,q) & \supset & Sp^*(p) \times Sp^*(q) \end{array}$$

In a few of these cases it is non-trivial to understand the copy of the compact group. For example, the copy of $U(n)$ inside $Sp_n(\mathbb{R})$ is

$$U(n) \;\approx\; \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a + ib \in U(n)\} \;\subset\; Sp_n(\mathbb{R})$$

Similarly, if (relying upon an inertia theorem) we use skew form

$$\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

to define $O^*(2n)$, the analogously defined copy of $U(n)$ is a maximal compact in $O^*(2n)$.

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3. *The four families of self-adjoint cones*

A *cone* in a real vectorspace V is a non-empty open subset C closed under scalar multiplication by positive real numbers. A cone is *convex* when it is convex as a set, meaning as usual that for $x, y \in C$ and $t \in [0,1]$ then $tx + (1-t)y$ is also in C. Now suppose V has a positive-definite inner product $\langle , \rangle$. Let C be a convex cone. The *adjoint* cone C^* of C is

$$C^* \;=\; \{\lambda \in V : \langle v, \lambda \rangle > 0 : \text{ for all } v \in C\}$$

A convex cone C is *self-adjoint* when $C = C^*$. A cone is *homogeneous* when there is a group G of $\mathbb{R}$-linear automorphisms of the ambient vectorspace V stabilizing C and acting transitively on C. In that case, letting K be the isotropy group of a basepoint in C, we have a homeomorphism $C \approx G/K$. In the following descriptions we use the *orthogonal similitude* group $GO(Q)$ of a quadratic form on a real vector space V,

$$GO(Q) \;=\; \{g \in GL(V) : Q(gv) = \mu(g) \cdot Q(v) \;\text{ for all } v \in V\}$$

where $\mu(g) \in \mathbb{R}^\times$ depends on g.

There are four families of convex, self-adjoint, homogeneous cones, described as quotients G/K.

positive-definite symmetric n-by-n real matrices	$\approx$	$GL_n(\mathbb{R})/O(n)$
positive-definite hermitian n-by-n complex matrices	$\approx$	$GL_n(\mathbb{C})/U(n)$
positive-definite quaternion-hermitian n-by-n matrices	$\approx$	$GL_n(\mathbb{H})/Sp^*(n)$
interior of light cone, $\{(y_0, y_1, \ldots, y_n) : y_0^2 - (y_1^2 + \ldots + y_n^2) > 0,\; y_0 > 0\}$	$\approx$	$GO(1,n)^+/\{1\} \times GO(n)'$

where H^+ denotes the connected component of the identity in a topological group H, and

$$GO(n)' = \{k \in GO(n) : 1 \times k \in GO(n,1)^+\}$$

The actions of the indicated groups are as follows. Proofs of transitivity are given after the descriptions of the actions.

Elements g of $G = GL_n(\mathbb{R})$ act on positive definite symmetric real n-by-n matrices S by

$$g(S) \;=\; gSg^\top$$

The orthogonal group $O(n)$ is the isotropy group of the n-by-n identity matrix 1_n. Elements g of $G = GL_n(\mathbb{C})$ act on positive definite hermitian complex n-by-n matrices S by

$$g(S) \;=\; gSg^*$$

where g^* is conjugate-transpose. The unitary group $U(n)$ is the isotropy group of 1_n. Elements g of $G = GL_n(\mathbb{H})$ act on positive definite quaternion-hermitian n-by-n matrices S by

$$g(S) \;=\; gSg^*$$

where g^* is quaternion-conjugate-transpose. The symplectic group $Sp^*(n)$ is the isotropy group of 1_n. Elements g of $G = GO(n,1)^+$ act on vectors $y = (y_0, y_1, \ldots, y_n)$ by the natural linear action

$$g(y) \;=\; g \cdot y$$

The imbedded orthogonal similitude group $\{1\} \times GO(n)'$ is the isotropy group of $(1, 0, \ldots, 0)$. Note that it is necessary to restrict ourselves to the connected component of the identity to assure that we stay in the chosen half of the interior of the light cone.

Proof of transitivity (and stabilization of the cone) of the indicated group has a common pattern for the first three types of cones, using spectral theorems for symmetric and hermitian operators. We do just the real case. First, for a symmetric real n-by-n matrix S (positive definite or not), by the spectral theorem there is a matrix $g \in GL_n(\mathbb{R})$ so that $S' = gSg^\top$ is diagonal. Since S is positive definite, it must be that all the diagonal entries of S' are positive. Then there is a diagonal matrix h in $GL_n(\mathbb{R})$ so that $hS'h^\top = 1_n$. Thus, every positive definite symmetric matrix lies in the $GL_n(\mathbb{R})$-orbit of 1_n, so we have the transitivity.

To prove transitivity of the action of $O(n,1)^+$ on the light cone, first note that the imbedded copy of $GO(n)'$ is transitive on vectors $(y_0, y_1, \ldots, y_n)$ with $(y_1, \ldots, y_n)$ non-zero. Thus, every $GO(n)'$-orbit has a representative either of the form $(y_0, 1, 0, \ldots, 0)$ or $(y_0, 0, 0, \ldots, 0)$. In the former case, we may as well think in terms of $SO(1,1)$, which contains matrices

$$M_t \;=\; \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

with $t \in \mathbb{R}$. Given y_0 there is $t \in \mathbb{R}$ so that of $M_t \begin{pmatrix} y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_0' \\ 0 \end{pmatrix}$ with $y_0' > 0$. Thus, every orbit contains a vector $(y_0, 0, \ldots, 0)$ with $y_0 > 0$. Then dilation by a suitable positive real number (which is a similitude) yields $(1, 0, \ldots, 0)$. Thus, every orbit contains the latter vector, which proves the transitivity.

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4. *The four families of classical domains*

Let 1_n denote the n-by-n identity matrix, and for hermitian matrices M, N write $M > N$ to indicate that $M - N$ is positive definite. The four families of **bounded classical domains**, each diffeomorphic to a quotient G/K for a classical real group G and maximal compact K, are as follows.

$$\begin{array}{lcll}
\mathrm{I}_{p,q} & = & \{p\text{-by-}q \text{ complex matrix } z : 1_q - z^* z > 0\} & \approx \; U(p,q)/U(p) \times U(q) \\
\mathrm{II}_n & = & \{n\text{-by-}n \text{ complex matrix } z : z^T = -z \text{ and } 1_n - z^* z > 0\} & \approx \; O^*(2n)/U(n) \\
\mathrm{III}_n & = & \{n\text{-by-}n \text{ complex matrix } z : z^T = z \text{ and } 1_n - z^* z > 0\} & \approx \; Sp_n(\mathbb{R})/U(n) \\
\mathrm{IV}_n & = & \{z \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_n|^2 < \frac{1}{2}(1 + |z_1^2 + \ldots + z_n^2|) < 1\} & \approx \; O(n,2)/O(n) \times O(2)
\end{array}$$

The fourth family of domains is less amenable than the first three, since its description is relatively ungainly.

We prove that $U(p,q)$ stabilizes $I_{p,q}$ and acts transitively upon it by generalized linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (az+b)(cz+d)^{-1}$$

where a is p-by-p, b is p-by-q, c is q-by-p, and d is q-by-q. Then, letting $0 < A$ mean that A is positive-definite hermitian,

$$0 \; < \; 1_q - z^* z = \begin{pmatrix} z \\ 1_q \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ & 0 & 1_q \end{pmatrix} \begin{pmatrix} z \\ 1_q \end{pmatrix} = \begin{pmatrix} z \\ 1_q \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1_q \end{pmatrix}$$

$$= \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = (cz+d)^*(cz+d) - (az+b)^*(az+b)$$

For the right-hand side to be positive definite, it must be that $(cz+d)$ is non-singular, which also verifies that the apparent definition of the action by linear fractional transformations is plausible. Then, in similar fashion,

$$1_q - g(z)^* g(z) = \begin{pmatrix} g(z) \\ 1_q \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} g(z) \\ 1_q \end{pmatrix} = (cz+d)^{*-1} \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} (cz+d)^{-1}$$

$$= (cz+d)^{*-1} \begin{pmatrix} z \\ 1_q \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1_q \end{pmatrix} (cz+d)^{-1} = (cz+d)^{*-1}(1_q - z^* z)(cz+d)^{-1}$$

The latter quantity is positive definite, so the image $g(z)$ lies back in the specified domain.

The **unbounded models** of these domains are

$$\begin{array}{lll}
\mathrm{I}_{p \le q} & \{(z,u) : \frac{1}{i}(z - z^*) - u^* u > 0, \; z = q\text{-by-}q, \; u = (p-q)\text{-by-}q\} & (\text{for } p = q, \textit{hermitian upper half-space}) \\
\mathrm{II}_n & \{n\text{-by-}n \text{ quaternionic } z : iz - z^* i > 0\} & (\textit{quaternion upper half-space}) \\
\mathrm{III}_n & \{\text{symmetric complex } n\text{-by-}n \; z : -i(z - z^*) > 0\} & (\textit{Siegel upper half-space}) \\
\mathrm{IV}_n & \{(z,u) \in \mathbb{C} \times \mathbb{C}^n : \frac{1}{i}(z - \bar{z}) - |u|^2 > 0\} &
\end{array}$$

Note that the unbounded models $\mathrm{I}_{n,n}$, II_n, and III_n all are visibly **tube domains** in the sense that they are describable as

$$\text{domain } = \{z = x + iy : x \in V, \; y \in C\}$$

where C is a self-adjoint homogeneous convex open cone in a real vector space V. The domains IV_n are also tube domains, related to light cones. The domains $\mathrm{I}_{p,q}$ with $p \neq q$, including the case of the complex unit ball in $\mathbb{C}^n$ (which is $\mathrm{I}_{n,1}$), are not tube domains.

These coordinates are more convenient to verify that, for example, $Sp_n(\mathbb{R})$ stabilizes and acts transitively on the Siegel upper half-space $\mathfrak{H}_n$ by generalized linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; (az+b)(cz+d)^{-1}$$

using n-by-n blocks for matrices in $Sp_n(\mathbb{R})$. To verify that $cz+d$ is genuinely invertible, use the fact that matrices in $Sp(n,\mathbb{R})$ are *real*, and compute

$$z - z^* \;=\; \begin{pmatrix} z \\ 1_n \end{pmatrix}^* \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} z \\ 1_n \end{pmatrix} \;=\; \begin{pmatrix} z \\ 1_n \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1_n \end{pmatrix}$$

$$= \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \;=\; (cz+d)^*(az+b) - [(cz+d)^*(az+b)]^*$$

Thus, the latter expression is i times a positive-definite symmetric real matrix. Thus, for any non-zero 1-by-n complex matrix v,

$$0 \;\neq\; v^*\left((cz+d)^*(az+b) - [(cz+d)^*(az+b)]^*\right) v$$

In particular, $(cz+d)v \neq 0$. Thus, $cz+d$ is invertible, and the expression $(az+b)(cz+d)^{-1}$ makes sense. Next, we verify that $(az+b)(cz+d)^{-1}$ is symmetric. To this end, similarly compute

$$0 \;=\; z - z^\top = \begin{pmatrix} z \\ 1_n \end{pmatrix}^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} z \\ 1_n \end{pmatrix} \;=\; \begin{pmatrix} z \\ 1_n \end{pmatrix}^\top \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1_n \end{pmatrix}$$

$$= \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^\top \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \;=\; (cz+d)^\top(az+b) - [(cz+d)^\top(az+b)]^\top$$

$$= \;(cz+d)^\top[(az+b)(cz+d)^{-1} - ((az+b)(cz+d)^{-1})^\top](cz+d) \;=\; (cz+d)^\top[g(z) - g(z)^\top](cz+d)$$

Since $cz+d$ is invertible, it must be that $g(z)$ is symmetric. Next, we verify that $g(z)$ does lie in the Siegel upper half-space $\mathfrak{H}_n$. In the equality from above

$$z - z^* \;=\; (cz+d)^*(az+b) - [(cz+d)^*(az+b)]^*$$

left multiply by $(cz+d)^{*-1}$ and right multiply by $(cz+d)^{-1}$ to obtain

$$(cz+d)^{*-1}(z-z^*)(cz+d)^{-1} \;=\; (az+b)(cz+d)^{-1} - [(az+b)(cz+d)^{-1}]^* \;=\; g(z) - g(z)^*$$

Thus, since $g(z)$ is actually symmetric, $-i(g(z) - g(z)^*)$ is positive-definite symmetric real.

The bounded and unbounded models of domains G/K are both open subsets of a larger *compact* complex manifold called the **compact dual**, on which the *complexified* G acts by linear fractional transformations. The bounded model is mapped to the unbounded model by a special element in the complexified group, the **Cayley element c**. In the case of the domains $\mathrm{I}_{p\geq q}$, the picture is relatively easy to understand. Let

$$\Omega = \{\text{ complex } p\text{-by-}q \; v : \; \text{rank } v = q\}$$

and let $\check{D}$ be the quotient

$$\check{D} = \Omega / GL_q(\mathbb{C})$$

where $GL_q(\mathbb{C})$ acts on the right by matrix multiplication. The Ω coordinates on $\check{D}$ are *homogeneous coordinates*. (This is a Grassmannian variety.) The group $G_{\mathbb{C}} = GL_{p+q}(\mathbb{C})$ acts on the left on Ω by matrix multiplication, and certainly preserves the maximal rank. Thus, the bounded model of the $\mathrm{I}_{p,q}$ domain consists of points with homogeneous coordinates

$$\begin{pmatrix} z \\ 1_q \end{pmatrix} \quad \text{and} \quad 1_q - z^* z > 0$$

The right action of $GL_q(\mathbb{C})$ preserves the latter relation, so this set is well-defined via homogeneous coordinates. The unbounded model has a similar definition in these terms, being the set of points $\begin{pmatrix} z \\ u \\ 1_q \end{pmatrix}$ with z being q-by-q, u being $(p-q)$-by-q,

$$\begin{pmatrix} z \\ u \\ 1_q \end{pmatrix} \text{ so that } \frac{1}{i}(z-z^*)-u^*u>0$$

Again, the latter relation is stable under the right action of $GL_q(\mathbb{C})$, so this set is well-defined in homogeneous coordinates. Let

$$\mathbf{c} \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}}\cdot 1_q & 0 & \frac{i}{\sqrt{2}}\cdot 1_q \\ 0 & 1_{p-q} & 0 \\ \frac{i}{\sqrt{2}}\cdot 1_q & 0 & \frac{1}{\sqrt{2}}\cdot 1_q \end{pmatrix}$$

Of course if $p=q=n$ then this is simpler, and completely analogous to the $SL_2(\mathbb{R})$ situation:

$$\mathbf{c} \;=\; \frac{1}{\sqrt{2}}\cdot \begin{pmatrix} 1_n & i\cdot 1_n \\ i\cdot 1_n & 1_n \end{pmatrix}$$

In the case of the domains II_n the direct relation between the bounded and unbounded realizations is computationally more irksome, since the bounded model makes no mention of quaternions, while the unbounded model is most naturally described in such terms. The domain IV_n is even less convivial in these *ad hoc* terms.

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5. *Harish-Chandra's and Borel's realization of domains*

The general construction of this section clarifies the phenomena connected with the four families of classical domains. We will not prove anything, but only describe intrinsically what happens. We will verify that the construction duplicates the bounded and unbounded models for $Sp_n(\mathbb{R})$.

Let G be an almost simple semi-simple real Lie group, K a maximal compact subgroup of G, and suppose that the center of K contains a circle group Z (that is, a group isomorphic to $\mathbb{R}/\mathbb{Z}$). It turns out that the action of Z on the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ of G decomposes $\mathfrak{g}_\mathbb{C}$ into 3 pieces

$$\mathfrak{g}_\mathbb{C} \;=\; \mathfrak{p}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^-$$

where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{p}^\pm$ are $\chi^{\pm 1}$-eigenspaces for a non-trivial character χ of Z. It turns out that

$$G \subset \exp \mathfrak{p}^+ \cdot K_\mathbb{C} \cdot \exp(\mathfrak{p}^-)$$

where $K_\mathbb{C}$ is the complexification of K. Thus, letting

$$B \;=\; K_\mathbb{C} \exp \mathfrak{p}^-$$

we have

$$G/K \;\approx\; GB/B \;\subset\; G_\mathbb{C}/B$$

In fact, given $g \in G$, there is a unique $\zeta(g) \in \mathfrak{p}^+$ so that

$$g \;\in\; \exp\zeta(g)\cdot K_{\mathbb{C}}\cdot \exp\mathfrak{p}^-$$

The function

$$g \;\longrightarrow\; \zeta(g) \in \mathfrak{p}^+$$

is the **Harish-Chandra imbedding** of G/K in the complex vector space $\mathfrak{p}^+$. Given $g, h \in G$, the action $\zeta(h) \to g\cdot\zeta(h)$ of g on the point $\zeta(h)$ is defined by

$$g\cdot\exp\zeta(h) \in \exp(g\cdot\zeta(h))\cdot K_{\mathbb{C}}\cdot\exp\mathfrak{p}^-$$

We can show that the above description does really work in the case of $G = Sp_n(\mathbb{R})$, and produces the bounded model given earlier, namely

$$\mathrm{III}_n \;=\; \{\text{ symmetric } n\text{-by-}n \text{ complex matrices } z : 1_n - z^*z > 0\}$$

In the bounded model,

$$K \;=\; \{\begin{pmatrix} k & 0 \\ 0 & k^{\top -1}\end{pmatrix} : k \in U(n)\}$$

$$K_{\mathbb{C}} \;=\; \{\begin{pmatrix} k & 0 \\ 0 & k^{\top -1}\end{pmatrix} : k \in GL_n(\mathbb{C})\}$$

Then

$$\mathfrak{p}^+ \;=\; \{\begin{pmatrix} 0 & z \\ 0 & 0\end{pmatrix} : z = z^\top\}$$

$$\mathfrak{p}^- \;=\; \{\begin{pmatrix} 0 & 0 \\ \zeta & 0\end{pmatrix} : z = z^\top\}$$

Then given $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ try to solve for z, ζ, and $h \in GL_n(\mathbb{C})$ such that

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \;=\; \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & h^{\top -1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \zeta & 1\end{pmatrix}$$

This is

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \;=\; \begin{pmatrix} h + zh^{\top -1}\zeta & zh^{\top -1} \\ h^{\top -1}\zeta & h^{\top -1}\end{pmatrix}$$

One finds that $h = d^{\top -1}$, and

$$z \;=\; bd^{-1} \;=\; \begin{pmatrix} a & b \\ c & d\end{pmatrix}(0) \quad \text{(with linear fractional action)}$$

and also $\zeta = d^{-1}c$. (This evidently requires that d is invertible.) That is,

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \;=\; \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} d^{\top -1} & 0 \\ 0 & d\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1\end{pmatrix}$$

To compute the action of $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ on a point z, we do a similar computation:

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \;=\; \begin{pmatrix} a & az+b \\ c & cz+d\end{pmatrix}$$

which has the decomposition

$$\begin{pmatrix} a & az+b \\ c & cz+d\end{pmatrix} \;=\; \begin{pmatrix} 1 & (az+b)(cz+d)^{-1} \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} (cz+d)^{\top -1} & 0 \\ 0 & cz+d\end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ (cz+d)^{-1}c & 1\end{pmatrix}$$

and thus (assuming $cz+d$ is invertible) the action is

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}(z) = (az+b)(cz+d)^{-1}$$