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Convergence of half-zeta integrals

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The point is to genuinely prove convergence of the half-zeta integrals

$$\int_{|y|\geq 1} |y|^s\, f(y)\, dy$$

with f a Schwartz function on the adeles, for *all* $s \in \mathbb{C}$. [1]

Since f is at worst a finite sum of monomials $\otimes_v f_v$, without loss of generality we take it to be such a monomial, with f_v Schwartz on k_v. Since f is Schwartz, for all N there is a constant C_N (depending on f) such that

$$|f(x)| \;\leq\; C_N \cdot \prod_v \sup(|x_v|_v, 1)^{-2N} \qquad \text{(for adele } x = \{x_v\}\text{)}$$

For an *idele* y define the **group norm** or **gauge** [2]

$$\nu(y) \;=\; \prod_v \sup\{|y_v|_v,\, |\frac{1}{y_v}|_v\}$$

Almost all factors on the right-hand side are 1, so there is no issue of convergence. Further, note that

$$\big(\sup\{a,\, 1\}\big)^2 \;=\; \sup\{a^2,\, 1\} \;=\; a \cdot \sup\{a,\, \frac{1}{a}\} \qquad \text{(for } a > 0\text{)}$$

Applying the latter equality to every factor,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} \;=\; |y|^{-N} \prod_v \sup(|y_v|_v,\, \frac{1}{|y_v|_v})^{-N} \;=\; |y|^{-N}\nu(y)^{-N}$$

Thus, on the set of ideles $\{|y| \geq 1\}$,

$$\prod_v \sup(|y_v|_v, 1)^{-2N} \;=\; |y|^{-N}\nu(y)^{-N} \;\leq\; \nu(y)^{-N} \qquad \text{(when } |y| \geq 1,\ N \geq 0\text{)}$$

Thus, with $\sigma = \mathrm{Re}\, s$, for every $N \geq 0$

$$\Big|\int_{|y|\geq 1} |y|^s\, f(y)\, dy\Big| \;\ll\; \int_{|y|\geq 1} |y|^\sigma\, \nu(y)^{-N}\, dy \;\ll\; \int_{\mathbb{J}} |y|^\sigma\, \nu(y)^{-N}\, dy \;=\; \prod_v \Big(\int_{k_v^\times} |y|^\sigma\, \sup(|y|, \frac{1}{|y|})^{-N}\, dy\Big)$$

For $N > |\sigma|$, the non-archimedean local integrals are absolutely convergent:

$$\int_{k_v^\times} |y|^\sigma\, \sup(|y|, \frac{1}{|y|})^{-N}\, dy \;=\; \sum_{\ell=0}^\infty q_v^{-\sigma-N} \;+\; \sum_{\ell=1}^\infty q_v^{\sigma-N}$$

$$=\; \frac{1}{1-q^{-\sigma-N}} + \frac{q^{\sigma-N}}{1-q^{\sigma-N}} \;=\; \frac{1-q^{-2N}}{(1-q^{-\sigma-N})\,(1-q^{\sigma-N})}$$

The archimedean integrals are convergent for similarly over-whelming reasons. For $2N > 1$ and $N > |\sigma| + 1$, the product over places is dominated by the Euler product for the completed zeta functions $\xi_k(N+\sigma)\xi_k(N-\sigma)/\xi_k(2N)$, which converges absolutely.

[1] In particular, we do not want to reduce to the classical viewpoint, as this would sacrifice the clarity and simplicity of the adelic set-up. Part of the point is that a genuine proof from an adelic viewpoint is clearer and easier than a classical one.

[2] The terminology *group norm* is mildly unfortunate, but standard. A group norm is *submultiplicative* and bounded below by 1, and is *not* in any sense *linear*. The better term *gauge* is also used.