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# *Godement's criterion for convergence of Eisenstein series*

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The point is a relatively simple presentation of what turns out to be a sharp estimate for the region of convergence of the relatively simple *Siegel-type* Eisenstein series [1] on classical groups. The argument is essentially due to Godement, reproduced for real Lie groups by Borel in AMS Proc. Symp. Pure Math. IX (The *Boulder Conference 1966*). The necessary small excursion into reduction theory can be extracted from Godement's *Domaines fondamentaux des groupes arithmetiques*, Sem. Bourb. 257 (1962-3).

- Affine heights: reduction theory
- Siegel-type Eisenstein series

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# 1. *Affine Heights: Reduction Theory*

**[1.1] Height functions** Let $k$ be a global field with adeles $\mathbb{A}$. For $v$-adic completion $k_v \approx \mathbb{R}$, let $h_v$ be the usual real Hilbert-space norm on $k_v^n \approx \mathbb{R}^n$. For $k_v \approx \mathbb{C}$, let $h_v$ be the *square* of the usual complex Hilbert-space norm on $k_v^n \approx \mathbb{C}^n$. For $k_v$ non-archimedean, let $h_v(x)$ be the sup of the $v$-adic norms of the coordinates of $x \in k_v^n$. The family of absolute values on all the $k_v$ makes the product formula holds, of course. These $h_v$ are **local height functions**. The (global) **height function** $h$ is

$$h(x) \;=\; \prod_v h_v(x_v) \qquad\qquad (\text{for } x = \{x_v\})$$

Sufficient conditions are given below for finiteness of this product.

The isometry groups $K_v$ of the height functions $h_v$ are as follows. For $k_v \approx \mathbb{R}$, the isotropy group is the standard orthogonal group $K_v = O(n, \mathbb{R})$. For $k_v \approx \mathbb{C}$, the isotropy group is the standard unitary group $K_v = U(n)$. For $k_v$ non-archimedean, the isotropy group is $K_v = GL(n, \mathfrak{o}_v)$, the group of matrices over the local integers $\mathfrak{o}_v$ in $k_v$, with determinant in the local units $\mathfrak{o}_v^\times$. Let

$$K \;=\; \prod_v K_v \;\subset GL(n, \mathbb{A})$$

Let $P$ be the (*parabolic*) subgroup of upper-triangular matrices. Recall the *Iwasawa decomposition*

$$GL(n, \mathbb{A}) \;=\; P_{\mathbb{A}} \cdot K$$

Now we identify a class of vectors with *finite height*. First, given $x \in k^n - \{0\}$, for all but finitely-many $v$ the components of the vector $x$ are all $v$-integral, and generate the local integers $\mathfrak{o}_v$. In particular, for all but finitely-many $v$ the $v^{th}$ local height $h_v(x)$ of $x \in k^n$ is 1, and the infinite product for $h(x)$ is a *finite* product.

Note that for each prime $v$ the group $K_v$ is *transitive* on the collection of vectors in $k_v^n$ with given norm.

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[1] The name *Siegel-type* is accurately descriptive, faithful to history, and widely used. A more technically suggestive term is *degenerate*, referring to the fact that the function *automorphized* from $GL(n)$ to $Sp(n)$ by forming the Eisenstein series is not nearly as complicated as an automorphic form on $GL(n)$ might be, being just a power of (the norm of) the determinant. In particular, that data from which the Eisenstein series is formed is not a *cuspform* on $GL(n)$ except for $n = 1$. Indeed, as a relatively trivial case of Langlands' general work *Functional equations satisfied by Eisenstein series*, SLN 544, these degenerate Eisenstein series on $Sp(n)$ occur as $(n-1)$-fold *residues* of non-degenerate Eisenstein series. This is another story.

Consider vectors to be *row* vectors, and let $GL(n, \mathbb{A})$ act on the *right* by matrix multiplication. Say that a non-zero vector $x \in \mathbb{A}^n$ is **primitive** if $x \in k^n \cdot GL(n, \mathbb{A})$.

### [1.1.1] Theorem:

• For idele $t$ of $k$ and primitive $x$, $h(tx) = |t| \cdot h(x)$, where $|t|$ is the idele norm. In particular, $k^\times$ preserves heights.
• For fixed $g \in GL(n, \mathbb{A})$ and for fixed $c > 0$ the set

$$\{x \in k^n : h(x \cdot g) < c\}$$

is finite modulo $k^\times$
• For a compact subset $C$ of $GL(n, \mathbb{A})$ there are positive constants $c, c'$ (depending only upon $C$) so that for all primitive vectors $x$ and for all $g \in C$ we have

$$c \cdot h(x) \le h(x \cdot g) \le c' \cdot h(x)$$

*Proof:* The first assertion is immediate, and the *product formula* shows that $k^\times$ leaves heights invariant.

For the second assertion, fix $g \in GL(n, \mathbb{A})$. Since $K$ preserves heights, via Iwasawa we may suppose that $g$ is in the group $P_\mathbb{A}$ of upper triangular matrices in $GL(n, \mathbb{A})$. Choose representatives $x = (x_1, \ldots, x_n)$ for non-zero vectors in $k^n$ modulo $k^\times$ such that, letting $\mu$ be the first index with $x_\mu \neq 0$, then $x_\mu = 1$. That is, $x$ is of the form

$$x \;=\; (0, \ldots, 0, 1, x_{\mu+1}, \ldots, x_n)$$

To illustrate the idea of the argument with a light notation, first consider $n = 2$, let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $x = (1, y)$. Thus,

$$x \cdot g = (1, y) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, b + yd)$$

From the definition of the local heights, at each $v$

$$\max(|a|_v, |b + yd|_v) \;\le\; h_v(xg)$$

so

$$|b + yd|_v \prod_{w \neq v} |a|_w \;\le\; \prod_{\text{all } w} h_w(xg) = h(xg)$$

Since $g$ is fixed, $a$ is fixed, and at almost all places $|a|_w = 1$. Thus, for $h(xg) < c$ there is a uniform constant $c'$ so that for all places $v$

$$|b + yd|_v \le c'$$

Since for almost all $v$ the residue class field cardinality $q_v$ is strictly greater than $c$, for almost all $v$

$$|b + yd|_v \;\le\; 1$$

Therefore, $b + yd$ lies in a compact subset $C$ of $\mathbb{A}$. Since $b, d$ are fixed, and since $k$ is discrete (and closed) in $\mathbb{A}$, the collection of images $\{b + dy : y \in k\}$ is discrete in $\mathbb{A}$. Thus, the collection of $y$ so that $b + dy$ lies in $C$ is finite, as desired.

For general $n$ and $x \in k^n$ such that $h(xg) < c$, let $\mu - 1$ be the least index such that $x_\mu \neq 0$. Adjust by $k^\times$ so that $x_\mu = 1$.From $h(xg) < c$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \;\le\; h(gx) < c \qquad \text{(for each } v\text{)}$$

For almost all $v$ we have $|g_{\mu-1,\mu-1}|_v = 1$, so there is a uniform constant $c'$ such that

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \;<\; c' \qquad\qquad \text{(for all } v\text{)}$$

For almost all places $v$ the residue field cardinality $q_v$ is strictly greater than $c'$, so for almost all $v$

$$|g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}|_v \le 1$$

Therefore, $g_{\mu-1,\mu} + x_\mu g_{\mu,\mu}$ lies in a compact subset $C$ of $\mathbb{A}$. Since $k$ is discrete, the collection of $x_\mu$ is finite.

Continue similarly to show that there are only finitely many choices for the other entries of $x$. Inductively, suppose that $x_i = 0$ for $i < \mu - 1$, and that $x_\mu, \ldots, x_{\nu-1}$ are fixed, and show that $x_\nu$ has only finitely many possibilities. Looking at the $\nu^{th}$ component $(xg)_\nu$ of $xg$,

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \prod_{w \neq v} |g_{\mu-1,\mu-1}|_w \;\le\; h(xg) \;\le\; c$$

For almost all $v$ we have $|g_{\mu-1,\mu-1}|_w = 1$, so there is a uniform constant $c'$ such that

$$|(xg)_\nu|_v = |g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \;<\; c' \qquad\qquad \text{(for all places } v\text{)}$$

For almost all places $v$ the residue field cardinality $q_v$ is strictly greater than $c'$, so

$$|g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}|_v \le 1 \qquad\qquad \text{(for almost all } v\text{)}$$

Therefore,

$$g_{\mu-1,\nu} + x_\mu g_{\mu,\nu} + \ldots + x_{\nu-1} g_{\nu-1,\nu} + x_\nu g_{\nu,\nu}$$

lies in the intersection of a compact subset $C$ of $\mathbb{A}$ with a (closed) discrete set, so lies in a finite set. Thus, the number of possibilities for $x_\nu$ is finite. By induction we obtain the finiteness.

For the *third* and last assertion, recall the Cartan decompositions

$$GL(n, k_v) \;= K_{\;v} \cdot A_v \cdot K_v$$

where $A_v$ is the subgroup of $GL(n, k_v)$ of diagonal matrices ($v$ archimedean or not). Since the map

$$\theta_1 \times a \times \theta_2 \longrightarrow \theta_1 a \theta_2$$

is not an injection, one cannot immediately infer that for a given compact set $C$ in $GL(n, k_v)$ the set

$$\{a \in A_v : \text{ for some } c \in C, \; c \in K_v a\, K_v\}$$

is compact. Since $K_v$ is compact, $C' = K_v \cdot C \cdot K_v$ is compact, and now $\theta_1 a \theta_2 \in C'$ with $\theta_i \in K_v$ implies $a \in C' \cap A_v$, which *is* compact.

Thus, any compact subset of $GL(n, \mathbb{A})$ is contained in a set

$$\{\theta_1 \delta \theta_2 : \theta_1, \theta_2 \in K, \delta \in C_D\}$$

where $C_D$ is a suitable *compact* set of diagonal matrices. Since $K$ preserves heights and since the set of primitive vectors is stable under $K$, the set of values

$$\{\frac{h(xg)}{h(x)} : x \text{ primitive, } g \in C\}$$

is contained in a set

$$\{\frac{h(x\delta)}{h(x)} : x \text{ primitive, } \delta \in C_D\}$$

for some compact set $C_D$ of diagonal matrices. Letting the diagonal entries of $\delta$ be $\delta_i$, we have

$$0 < \inf_{\delta \in C_D} \inf_i |\delta_i| \le \frac{h(x\delta)}{h(x)} \le \sup_{\delta \in C_D} \sup_i |\delta_i| < +\infty$$

This gives the desired bound. ///

# 2. *Siegel-type Eisenstein Series*

Let $V, \langle , \rangle$ be a non-degenerate alternating space over a totally real number field $k$, with $V$ of dimension $2n$. Let $G$ be the isometry group of this space. [2] Choose (*good*) maximal compact subgroups $K_v$ of the $k_v$-points $G_v$ for all primes $v$ of $k$. Let $h$ be a height function on the vectorspace $\bigwedge^n V$ invariant under the chosen maximal compact subgroups $K_v$. (Let $G$ act on $\bigwedge^n V$ in the natural fashion, on the right.) Fix a maximal totally isotropic subspace $V_o$ of $V$, and let $P$ be the parabolic subgroup [3] of $G$ stabilizing $V_o$.

As usual, take a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for $V$ so that for each $i$ the pair $e_i, f_i$ is a hyperbolic pair, and such that for distinct indices the hyperbolic planes spanned by these pairs are orthogonal in $V$. The standard maximal totally isotropic subspace $V_o$ is the span of $e_1, \ldots, e_n$.

Choose a non-zero vector $v_o$ in the one-dimensional subspace $\bigwedge^n V_o$ of $\bigwedge^n V$. For

$$p = \begin{pmatrix} m & * \\ 0 & m^{\top -1} \end{pmatrix} \in P_{\mathbb{A}}$$

we have

$$h(v_o p) = h((\det m)^{-1} \cdot v_o) = |\det m|^{-1} \cdot h(v_o)$$

Therefore, for $g \in G_{\mathbb{A}}$,

$$h(v_o(pg)) \;=\; |\det m|^{-1} \cdot h(v_o g)$$

Define a **Siegel-type Eisenstein series**

$$E_s(g) \;=\; \sum_{\gamma \in P_k \backslash G_k} h(v_o \gamma g)^{-s}$$

**[2.0.1] Theorem:** The series defining the Eisenstein series $E_s(g)$ is absolutely convergent for $\mathrm{Re}(s) > n+1$, and uniformly so for $g$ in compacts.

*Proof:* The modular function of the adele points $P_{\mathbb{A}}$ of the parabolic subgroup $P$ is

$$\Delta(\begin{pmatrix} m & * \\ 0 & m^{\top -1} \end{pmatrix}) \;=\; |\det m|^{n+1}$$

The defining property is

$$d(p'p) \;=\; \Delta(p')\, dp \qquad (dp \text{ is right Haar measure on } P_{\mathbb{A}}, \text{ and } p' \in P_{\mathbb{A}})$$

Also,

$$\Delta(p)^{-1}\, dp \;=\; \textit{left} \text{ Haar measure on } P_{\mathbb{A}}$$

In coordinates, a right Haar measure on $P_{\mathbb{A}}$ is

$$d(mn) = dn\, dm = |\det m|^{n+1} dm\, dn$$

where $dm$ is Haar measure on the Levi component $GL(V_o) \approx GL(n, k_v)$ of $P$ and $dn$ is Haar measure on the unipotent radical $N_{\mathbb{A}}$ of $P_{\mathbb{A}}$.

Note that for $p \in P_{\mathbb{A}}$

$$\Delta(p) = \left(\frac{h(v_o p)}{h(v_o)}\right)^{-(n+1)}$$

[2] Except for variations in details, the same argument works for all classical groups.

[3] This is the *Siegel-type* or *popular* parabolic.

It suffices to consider $s$ real. By reduction theory, for compact $C \subset G_{\mathbb{A}}$ there are constants $0 < c \leq c' < +\infty$ such that

$$c \cdot h(v) \;\leq\; h(vg) \;\leq\; c' \cdot h(v) \qquad\qquad \text{(for all } g \in C\text{, for all primitive } v \text{ in } \textstyle\bigwedge^n V)$$

Thus, convergence of the series is equivalent to convergence of the integral

$$\int_C E_s(g)\; dg$$

By discreteness of $G_k$ in $G_{\mathbb{A}}$, we can shrink $C$ so that, for $\gamma$ in $G_k$, if $\gamma C \cap C \neq \phi$ then $\gamma = 1$. Then

$$\int_C E_s(g)\; dg \;=\; \int_C \sum_{\gamma \in P_k \backslash G_k} h(v_o \gamma g)^{-s}\; dg \;=\; \int_{P_k \backslash G_k \cdot C} h(v_o g)^{-s}\; dg$$

Let $\mu > 0$ be the infimum of $h(v)$ over non-zero primitive $v$ in $\bigwedge^n V$. From reduction theory this infimum is attained, so is strictly positive. Then $\Gamma \cdot C$ is contained in

$$Y \;=\; \{x \in G_{\mathbb{A}} : h(v_o x) \geq c \cdot \mu\}$$

The set $Y$ is right $K$-stable, since $h$ is $K$-invariant. Thus,

$$\int_{P_k \backslash G_k\, C} h(v_o g)^{-s}\; dg \leq \int_{P_k \backslash Y} h(v_o g)^{-s}\; dg = \int_{P_k \backslash (P_{\mathbb{A}} \cap Y)} h(v_o g)^{-s}\; dp \qquad\qquad \text{(left Haar } dp)$$

using Iwasawa decompositions and

$$\int_{G_{\mathbb{A}}} f(g)\; dg \;=\; \int_{P_{\mathbb{A}}} \int_K f(p\theta)\; dp\; d\theta \qquad\qquad \text{(up to a constant, } d\theta \text{ Haar on } K)$$

A left Haar measure $dp$ on $P_{\mathbb{A}}$ is

$$dp \;=\; \Delta(m)^{-1} \cdot dn\; dm$$

By the definition of $h$, $Y$ is left $N_{\mathbb{A}}$-stable. As the induced measure on the compact quotient $N_k \backslash N_{\mathbb{A}}$ is *finite*, up to a constant the integral is

$$\int_{M_k \backslash (M_{\mathbb{A}} \cap Y)} h(v_o m)^{-s}\; \Delta(m)^{-1}\; dm$$

Now

$$M_{\mathbb{A}} = M^1 \times \mathbb{R}^+$$

where

$$M^1 = \{m \in M_{\mathbb{A}} : |\det m| = 1\} = \{m \in M_{\mathbb{A}} : \Delta(m) = 1\} \qquad\qquad \text{(idele norm } |\;|)$$

where $\mathbb{R}^+$ is a copy of the positive real numbers arranged as matrices

$$\begin{pmatrix} t^{1/n} & & & & & \\ & \ddots & & & & \\ & & t^{1/n} & & & \\ & & & t^{-1/n} & & \\ & & & & \ddots & \\ & & & & & t^{-1/n} \end{pmatrix} \in M_{\mathbb{A}}$$

Thus,

$$\Delta(t) \;=\; t$$

Therefore,

$$\begin{aligned} Y \cap M_{\mathbb{A}} \;&=\; M^1 \times \{t \in \mathbb{R}^+ : h(t^{-1}v_o) \geq c \cdot \mu\} \\ &=\; M^1 \times \{t \in \mathbb{R}^+ : \frac{h(t^{-1}v_o)}{h(v_o)} \;\geq\; \frac{c \cdot \mu}{h(v_o)}\} \\ &=\; M^1 \times \{t \in \mathbb{R}^+ : \Delta(t^{-1})^{1/(n+1)} \;\geq\; \frac{c \cdot \mu}{h(v_o)}\} \end{aligned}$$

The condition on $t$ can be rewritten as

$$\Delta(t) \;\leq; \Big(\frac{c \cdot \mu}{h(v_o)}\Big)^{-(n+1)}$$

Let $c''$ be the latter finite constant. Thus,

$$\int_{M_k \backslash (M_{\mathbb{A}} \cap Y)} h(v_o m)^{-s}\, \Delta(m)^{-1}\, dm \;=\; \int_{M_k \backslash M^1} 1\, dm \;\times\; \int_0^{c''} \Delta(t)^{\frac{s}{n+1}-1}\, \frac{dt}{t}$$

The volume of $M_k \backslash M^1$ is *finite*, so convergence is implied by convergence of the elementary

$$\int_0^{c''} \Delta(t)^{\frac{s}{n+1}-1}\, \frac{dt}{t} = \int_0^{c''} t^{\frac{s}{n+1}-1}\, \frac{dt}{t}$$

where $\Delta(t) = t$ is by choice of the copy of $\mathbb{R}^+$ inside $M_{\mathbb{A}}$. That is, convergence is assured for

$$\frac{s}{n+1} - 1 \;>\; 0$$

which is for $s > n+1$. ///