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Distribution $|\det x|^s$ on p-adic matrices

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Let F be a p-adic field with integers $\mathfrak{o}$, local parameter ϖ, and residue field cardinality q. Let $A = M_n(F)$ be the F-vectorspace of n-by-n matrices over k, and $G = GL_n(F)$. Let u_s be the tempered distribution

$$u_s(f) \;=\; \int_G |\det x|^s\, f(x)\, d^\times x \qquad\qquad \text{(Schwartz function } f \text{ on } A\text{, for } \mathrm{Re}(s) \gg 1)$$

where $d^\times x$ denotes a Haar measure on G. Up to constants, for (additive) Haar measure d^+x on A, $d^\times x = d^+x/|\det x|^n$. For brevity, write $|x|$ for $|\det x|$ when possible.

[0.1] Convergence

The integral defining u_s converges absolutely in $\mathrm{Re}(s) > n-1$:

Recall the Iwasawa decomposition $G = P \cdot K$ with P the parabolic subgroup of upper-triangular matrices. Since K is *open* in G, Haar measure on G restricted to K is Haar measure on K. Recall the integral formula

$$\int_G f(g)\, dg \;=\; \int_P \int_K f(pk)\, dp\, dk \qquad\qquad \text{(up to normalization, with } \textit{left} \text{ Haar measure on } P)$$

In Levi-Malcev coordinates $NM = P$, with N the unipotent radical and M diagonal matrices, up to normalization, left Haar measure on P is

$$d\left(\begin{pmatrix} 1 & x_{12} & \ldots & & x_{1n} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}\begin{pmatrix} y_1 & & & & \\ & y_2 & & & \\ & & \ddots & & \\ & & & y_{n-1} & \\ & & & & y_n \end{pmatrix}\right)$$

$$=\quad dx_{12}\, dx_{13} \ldots dx_{n-1,n}\, \frac{dy_1}{|y_1|^{1+(n-1)}} \frac{dy_2}{|y_2|^{1+(n-3)}} \frac{dy_3}{|y_n|^{1+(n-5)}} \cdots \frac{dy_n}{|y_n|^{1-(n-1)}}$$

with additive Haar measures in the coordinates. For χ the characteristic function of $M_n(\mathfrak{o})$, and $\mathrm{Re}(s) \gg 1$, up to normalization constants,

$$u_s(\chi) \;=\; \int_G \chi(g)\, |\det x|^s\, dg \;=\; \int_P \chi(p)\, |\det p|^s\, dp \;=\; \prod_j \int_{\mathfrak{o}\cap k^\times} \prod_{i<j} \Big(\int_{y_j^{-1}\mathfrak{o}} 1\, dx_{ij} \Big)\, \frac{|y_j|^s\, dy_j}{|y_j|^{1+(n-(2j-1))}}$$

$$= \prod_j \int_{\mathfrak{o}\cap k^\times} |y_j|^{s-(j-1)} \frac{dy_j}{|y_j|^{1+(n-(2j-1))}} \;=\; \prod_j \int_{\mathfrak{o}\cap k^\times} |y_j|^{s-(n-j)} \frac{dy_j}{|y_j|} \;=\; \prod_j \frac{1}{1-q^{-(s-(n-j))}}$$

Thus, $u_s(\chi)$ converges absolutely in $\mathrm{Re}(s) > n-1$, admits a meromorphic continuation, and definitely blows up as $s \to (n-1)^+$. Using the homogeneity of u_s, the integral expresion for $u_s(f)$ for any Schwartz function f is dominated by the integral for $u_s(\chi)$, so u_s gives a tempered distribution in $\mathrm{Re}(s) > n-1$.

[0.2] Meromorphic continuation and residues of $u_s(\chi)$

The outcome of the computation of $u_s(\chi)$ to understand convergence also gives a meromorphic continuation in s, with simple poles (and non-zero residues) at $s = 1, 0$ (and at points differing by integer multiples of $2\pi i/\log q$ from these).

[0.3] Meromorphic continuation of $u_s(f)$

For an arbitrary Schwartz function f the value $u_s(f)$ can be meromorphically continued similarly, as follows. First, since u_s is right K-invariant, first average f on the right over K, and then use a Levi-Malcev decomposition:

$$\int_G |\det x|^s\, f(x)\, d^\times x \;=\; \int_P |\det p|^s \Big(\int_K f(xk)\, dk \Big)\, d_{\mathrm{left}} p \;=\; \int |m_1|^{s-(n-1)} \ldots |m_n|^{s-(1-n)} f^K(nm)\, dn\, dm$$

where f^K is the averaged f. Since f^K is itself a Schwartz function, it is a finite linear combination of *monomials*

$$\varphi(x) \;=\; \prod_{ij} \varphi_{ij}(x_{ij})$$

of Schwartz functions φ_{ij} in the coordinates x_{ij}. Of course, the support of φ_{ij} must include 0 for $i > j$, or else $\varphi(p) = 0$. For $i < j$, the relevant integral is

$$\int_F \varphi_{ij}(x_{ij}m_j)\, dx_{ij} \;=\; |m_j|^{-1} \int_F \varphi_{ij}(x_{ij})\, dx_{ij}$$

The whole is

$$\prod_{i>j} \varphi_{ij}(0) \times \prod_{i<j} \int_F \varphi_{ij}(x_{ij})\, dx_{ij} \times \prod_i \int_{F^\times} \varphi_{ii}(m_i)\, |m_i|^{s-(n-2i)-i}\, d^\times m_i$$

$$= \prod_{i>j} \varphi_{ij}(0) \times \prod_{i<j} \int_F \varphi_{ij}(x_{ij})\, dx_{ij} \times \prod_i \int_{F^\times} \varphi_{ii}(t)\, |t|^{s-(n-i)}\, d^\times t$$

The first two products are constants. Each integral in the last product is an Iwasawa-Tate local zeta integral: when the support of φ_{ii} does *not* include 0, it is a polynomial in q^{-s}, and when the support of φ_{ii} *does* include 0, the zeta integral is a sum of a polynomial in q^{-s} and a constant multiple of $\frac{1}{1-q^{-(s-j)}}$.

Finite sums of such expressions admit meromorphic continuations with poles at most at $s = n-1, n-2, \ldots, 2, 1, 0$ (and points differing from these by integer multiples of $2\pi i/\log q$). Poles, if any, are simple.

Thus, $v_s(f) = (1 - q^{s-(n-1)}) \ldots (1 - q^{-s}) \cdot u_s(f)$ has a meromorphic for every Schwartz function f. That is, v_s is *weakly* holomorphic. Weak holomorphy implies (strong) holomorphy for vector-valued functions with values in a quasi-complete locally convex topological vector space. Tempered distributions are such. Thus, v_s is a holomorphic tempered-distribution-valued function of $s \in \mathbb{C}$. In particular, the *residues* of u_s at poles are tempered distributions.

[0.4] Support of residues

For f a Schwartz function with support inside G, the meromorphic scalar-valued function $u_s(f)$ is *entire*. Thus, the residues of u_s at $s = n-1, n-2, \ldots, 0$ are tempered distributions supported on the set $A^{<n}$ of matrices of less-than-full rank.

[0.5] Uniqueness and existence of equivariant distributions

The standard argument shows that, given $s \in \mathbb{C}$, there is a unique tempered distribution on G such that $u(AxB) = |\det A \cdot \det B|^s \cdot u(x)$, since G acts transitively on itself. The tempered distribution is given by the integral for u_s in the range of convergence, and by meromorphic continuation otherwise.

Let

$$G^1 \;=\; \{g \in G \;:\; |\det g| = 1\}$$

The product $G^1 \times G^1 K$ acts transitively on the set A_r of matrices of a given rank $r < n$ by $(g \times h)(x) = g^{-1}xh$. The isotropy group of

$$E_r \;=\; \begin{pmatrix} 1_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}$$

is

$$H_r = \{\begin{pmatrix} A & * \\ 0 & D \end{pmatrix} \times \begin{pmatrix} a & 0 \\ c & D \end{pmatrix} : D \in GL_{n-r}(\mathfrak{o}),\; a, A \in GL_r(F),\; |\det a| = |\det A| = |\det D|^{-1}\} \subset G^1 \times G^1$$

Both H_r and $G^1 \times G^1$ are *unimodular*, so there is a unique $G^1 \times G^1$ -invariant measure on $A_r \approx G^1 \times G^1/H_r$, and integration against this measure gives the *unique* $G^1 \times G^1$-invariant distribution on Schwartz functions supported on the set $A^{\geq r}$ of matrices of rank $\geq r$.

At the same time, $G^1 \times K$ is already transitive on A_r, so up to scalars there is unique $G^1 \times K$-invariant measure and corresponding distribution. We can easily write a formula for it in terms of Euclidean coordinates, namely

$$u^{(r)}(f) \;=\; \int_K \int_{F^{r\times r}} \int_{F^{(n-r)\times r}} f(\begin{pmatrix} x_{11} & 0 \\ x_{21} & 0_{n-r} \end{pmatrix})\, dx_{21}\, dx_{11}\, dk$$

By the uniqueness of $G^1 \times K$-invariant functional, this integral formula must also be a $G^1 \times G^1$-invariant functional. Similarly, the $K \times G^1$-invariant form of that integral must give the same functional:

$$u^{(r)}(f) \;=\; \int_K \int_{F^{r\times r}} \int_{F^{r\times(n-r)}} f(k \cdot \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0_{n-r} \end{pmatrix} \cdot k)\, dx_{12}\, dx_{11}\, dk$$

Equality up to constants follows from uniqueness, and the constant is 1 because the two integrals agree on the characteristic function of Λ.

These integrals converge absolutely, so extend to tempered distributions on the whole Schwartz space.

Changing variables in the first integral expression, the equivariance under the full group $G = GL_n$ is

$$u^{(r)}\Big(x \to f(Ax)\Big) \;=\; |\det A|^{-r} \cdot u^{(r)}(f) \qquad\qquad (\text{for } A \in GL_n(F))$$

Changing variables in the second integral expression,

$$u^{(r)}\Big(x \to f(xB)\Big) \;=\; |\det B|^{-r} \cdot u^{(r)}(f) \qquad\qquad (\text{for } B \in GL_n(F))$$

The residue of u_s at $s = r < n$ is supported on the set $A^{\le r}$ of matrices of rank $\le r$, and has the same equivariance under $G \times G$ as does $u^{(r)}$. Suggesting that, up to a constant, the distribution $u^{(r)}$ is the residue of u_s at $s = r < n$.

Indeed, on Schwartz functions supported on $A^{\ge r}$, the uniqueness result just above does show that the residue of u_s at r is a constant multiple of $u^{(r)}$.

The integral expression for $u^{(r)}$ specifies it on the whole Schwartz space. The appearance of the residue as a residue specifies it on the whole Schwartz space. The difference v of suitable multiples vanishes on Schwartz functions supported on $A^{\ge r}$. This difference restricted to Schwartz functions supported on $A^{\ge r-1}$ is $G^1 \times G^1$-invariant, so must be a multiple of $u^{(r-1)}$. However, the $G \times G$-equivariance does not match that of $u^{(r-1)}$, so this restriction to $A^{\ge r-1}$ is 0. Similarly, the restriction of v to Schwartz functions supported on $A^{\ge r-2}$ must be a multiple of $u^{(r)}$, and the equivariance forces it to be 0. Continuing, we find that the residue of u_s at $s = r < n$ is a multiple of $u^{(r)}$.

[0.6] Non-extendability of $|\det x|^1$ for GL_2 In the small example $G = GL_2(F)$, a relatively elementary argument shows that $u_1(f) = \int_G |x|^1\, f(x)\, d^\times x$ has *no* extension from Schwartz functions supported on G to the whole space of Schwartz functions on A. Specifically, we claim that any tempered distribution u with the homogeneity property

$$u(R_g f) \;=\; |\det g|^{-1} \cdot u(f) \qquad\qquad (\text{with } (R_g f)(x) = f(xg))$$

is supported on $A^{\le 1}$. This will follow from the Hecke operator identity (proven below)

$$\mathrm{ch}_K \;=\; \mathrm{ch}_\Lambda + q \cdot \mathrm{ch}_{\varpi\Lambda} - T_p(\mathrm{ch}_\Lambda)$$

where ch_X is the characteristic function of a set, $\Lambda = M_2(\mathfrak{o})$, and T_p is the Hecke operator incarnated as

$$(T_p f)(x) \;=\; \int_G \mathrm{ch}_{D_1}(g^{-1}) \cdot f(xg)\, dg \qquad\qquad (\text{with } D_n = \{:\, g \in M_2(\mathfrak{o}),\ |\det g| = q^{-n}\})$$

Indeed, a tempered distribution u with the indicated homogeneity property restricted to Schwartz functions on G is $c \cdot u_1$ for some constant c, by uniqueness. We show that $c = 0$. The interaction of Hecke operator and u is easily determined:

$$u(T_p f) \;=\; u\Big(\int_G \mathrm{ch}_{D_1}(g^{-1}) \cdot R_g f \, dg\Big) \;=\; \int_G \mathrm{ch}_{D_1}(g^{-1}) \cdot u(R_g f)\, dg \;=\; \int_G \mathrm{ch}_{D_1}(g) \cdot u(f) \cdot |\det g|^{-1}\, dg$$

$$= \; u(f) \cdot q^{-1} \cdot \int_G \mathrm{ch}_{D_1}(g)\, dg \;=\; u(f) \cdot q^{-1}(q+1)$$

Applying u to the identity of characteristic functions gives

$$u(\mathrm{ch}_K) \;=\; u(\mathrm{ch}_\Lambda) + q \cdot u(\mathrm{ch}_{\varpi\Lambda}) - u(T_p(\mathrm{ch}_\Lambda))$$

and then

$$\begin{aligned} c \cdot u_1(\mathrm{ch}_K) \;&=\; u(\mathrm{ch}_\Lambda) + q \cdot q^{-2} \cdot u(\mathrm{ch}_\Lambda) - q^{-1}(q+1) u(\mathrm{ch}_\Lambda) \\ &=\; u(\mathrm{ch}_\Lambda) \cdot \Big(1 + q \cdot q^{-2} - q^{-1}(q+1)\Big) \;=\; u(\mathrm{ch}_\Lambda) \cdot 0 \end{aligned}$$

yielding $c = 0$. To prove the identity

$$\mathrm{ch}_K \;=\; \mathrm{ch}_\Lambda + q \cdot \mathrm{ch}_{\varpi\Lambda} - T_p(\mathrm{ch}_\Lambda)$$

we explicate the action of T_p on the functions ch_{D_n}, since

$$\mathrm{ch}_\Lambda \;=\; \sum_{n \geq 0} \mathrm{ch}_{D_n} + \mathrm{ch}_{A^{\leq 1}}$$

As in the classical context,

$$(T_p \mathrm{ch}_{D_n})(x) \;=\; \int_G \mathrm{ch}_{D_1}(g^{-1}) \cdot \mathrm{ch}_{D_n}(xg)\, dg$$

is certainly 0 unless $x \in D_{n+1}$. Left modulo K, D_{n+1} has representatives

$$\begin{pmatrix} \varpi^i & b \\ 0 & \varpi^{n+1-i} \end{pmatrix} \qquad\qquad (\text{with } b \bmod \varpi^{n+1-i})$$

Similarly for g right modulo K, equivalently, for $g^{-1} \in D_1$ left modulo K, take representatives

$$g^{-1} \;=\; \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}, \; \begin{pmatrix} 1 & b' \\ 0 & \varpi \end{pmatrix} \qquad\qquad (\text{with } b' \bmod \varpi)$$

and then

$$g \;=\; \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \; \begin{pmatrix} 1 & -b'/\varpi \\ 0 & 1/\varpi \end{pmatrix} \qquad\qquad (\text{with } b' \bmod \varpi)$$

and

$$xg \;=\; \begin{pmatrix} \varpi^{i-1} & b \\ 0 & \varpi^{n+1-i} \end{pmatrix}, \quad \begin{pmatrix} \varpi^i & b\varpi^{-1} - b'\varpi^{i-1} \\ 0 & \varpi^{n-i} \end{pmatrix} \qquad\qquad (\text{with } 0 \leq i \leq n+1)$$

Given x, the integral produces the value 1 for g such that $xg \in M_2(\mathfrak{o})$. The first case gives a value 1 exactly for $i \geq 1$. The second family gives 1 for $1 \leq i \leq n$ and $b \in \varpi\mathfrak{o}$, or for $i = 0$ and $b = b' \bmod \varpi$. In summary,

$$T_p \mathrm{ch}_{D_n} \begin{pmatrix} \varpi^i & b \\ 0 & \varpi^{n+1-i} \end{pmatrix} \;=\; \begin{cases} 1+q & \text{for } 1 \leq i \leq n \text{ and } b \in \varpi\mathfrak{o} \\ 1 & \text{for } 1 \leq i \leq n \text{ and } b \notin \varpi\mathfrak{o} \\ 1 & \text{for } i = 0 \\ 1 & \text{for } i = n+1 \end{cases} \qquad (0 \leq i \leq n+1 \text{ and } b \bmod \varpi^{n+1-i})$$

That is,

$$T_p \mathrm{ch}_{D_n} \;=\; \begin{cases} q \cdot \mathrm{ch}_{\varpi D_{n-1}} + \mathrm{ch}_{D_{n+1}} & (\text{for } n \geq 1) \\ \mathrm{ch}_{D_1} & (\text{for } n = 0) \end{cases}$$

Ignoring singular matrices since the outcome $T_p \mathrm{ch}_\Lambda$ is guaranteed to be a Schwartz function,

$$T_p \mathrm{ch}_\Lambda \;=\; T_p\Big(\mathrm{ch}_{D_o} + \sum_{n\geq 1} \mathrm{ch}_{D_n}\Big) \;=\; \mathrm{ch}_{D_1} + \sum_{n\geq 1} \Big(q \cdot \mathrm{ch}_{\varpi D_{n-1}} + \mathrm{ch}_{D_{n+1}}\Big) \;=\; q\, \mathrm{ch}_{\varpi\Lambda} + \mathrm{ch}_\Lambda - \mathrm{ch}_K$$

since $D_o = K$. This rearranges to the asserted identity, from which the non-extendability follows.
