Examples
in
automorphic spectral theory

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Some of this is joint work with A. Diaconu, D. Letang, and A. DeCelles, but they should not be blamed for any speculations or inaccuracies herein.

My interest in spectral identities comes from the demonstrated possibility of proving non-trivial number-theoretic results, such as sub-convexity bounds for L-functions.

An auxiliary point is that the zonal spherical functions on *complex* G are elementary, allowing vastly easier understanding of asymptotics in parameters... necessary for serious applications.

First/standard example:
motivating automorphic Sobolev spaces

Spectral expansions for $L^2(SL_2(\mathbb{Z})\backslash\mathfrak{H})$

$$f \;=\; \sum_{\text{cfm } F} \langle f, F\rangle \cdot F \qquad \text{(cuspidal component)}$$

$$+\; \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \qquad \text{(residual spectrum)}$$

$$+\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E_s\rangle \cdot E_s \; ds \qquad \text{(continuous)}$$

But... **locally uniformly pointwise?**

Surely ok ... for sufficiently smooth f, ... *how could it not?... proof?...* to justify manipulation as though it had *pointwise* meaning?

How hard could it be? ... to pointwise estimate cuspforms and Eisenstein series?
Recall for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$

$$E_s(i) \;=\; 2\,\zeta_{\mathbb{Q}(i)}(s)$$

So optimal pointwise estimates for E_s flirt with **Lindelöf!** Cuspforms surely subtler. ... Alternatives?

Typical application: meromorphic continuation

$f = f_z$ depending on parameter $z \in \mathbb{C}$, L^2 for $\mathrm{Re}(z) \gg 1$, supposedly meromorphic continuation via spectral expansion?

Continues to $\mathrm{Re}(z) > \frac{1}{2}$ when $\langle f_z, F\rangle$ and $\langle f_z, E_s\rangle$ dominated by values for $\mathrm{Re}(z) \gg 1$.

Discrete spectral components continue beyond $\mathrm{Re}(z) = \frac{1}{2}$ off poles of $\langle f_z, F\rangle$.

Continuous part may continue beyond $\mathrm{Re}(z) = \frac{1}{2}$, but with non-obvious poles, or possible *branch points* generally.

Infinitely-many branch points definitely occur for automorphic Green's functions: Hilbert modular surfaces, $SL_3(\mathbb{Z})$, generally.

Hyperbolic two-space, three-space

$\mathfrak{H} \approx SL_2(\mathbb{R})/SO(2) \approx$ hyperbolic two-space

$SL_2(\mathbb{C})/SU(2) \approx$ hyperbolic three-space

Split component

$$A^+ \;=\; \{a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \;:\; r \geq 0\}$$

Cartan decomposition $G = KA^+K$.

Invariant metric on G/K

$$d(gK, hK) \;=\; r \qquad \text{where} \qquad h^{-1}g \;\in\; Ka_rK$$

$$\text{Haar} \;=\; |\sinh r|\, dk\, dr\, dk' \qquad\qquad (\text{for } SL_2(\mathbb{R}))$$

$$\text{Haar} \;=\; |\sinh r|^2\, dk\, dr\, dk' \qquad\qquad (\text{for } SL_2(\mathbb{C}))$$

Laplacian Δ is Casimir on right K-invariant functions

Eigenvalue $\lambda_s = s(s-1)$ on s^{th} **principal series**

Radial Laplacian on $F(ka_r k') = f(r)$

$$\Delta F \;=\; f'' + \coth r \cdot f' \qquad \text{(for } SL_2(\mathbb{R})\text{)}$$

$$\Delta F \;=\; f'' + 2\coth r \cdot f' \qquad \text{(for } SL_2(\mathbb{C})\text{)}$$

$$\Delta F = f'' + (n-1)\coth r\, f' \qquad \text{(hyperbolic } n\text{-space)}$$

Spherical functions: smooth K-bi-invariant Δ-eigenfunctions

Non-trivial general fact: every eigenspace on G/K is image of **unramified principal series**

Convenient elementariness of spherical functions for $SL_2(\mathbb{C})$... for all **complex** groups. s^{th} **spherical function**

$$\varphi_s(r) \;=\; \frac{\sinh(2s-1)r}{(2s-1)\,\sinh r}$$

On unitary line

$$\varphi_{\frac{1}{2}+it}(r) \;=\; \frac{\sin 2tr}{2t\,\sinh r}$$

Spherical transform (All this is *local*!)

$$\widetilde{f}(\tfrac{1}{2}+i\xi) \;=\; \int_G f\cdot\overline{\varphi}_{\frac{1}{2}+i\xi} \;=\; \int_G f\cdot\varphi_{\frac{1}{2}-i\xi}$$

Spherical inversion

$$f \;=\; \int_{-\infty}^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi$$

where $\mathbf{c}(\frac{1}{2}+i\xi) \;=\; \xi^{-1}$ because it's a *complex* group. Plancherel:

$$\int_G f\cdot F \;=\; \int_0^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\widetilde{F}(\tfrac{1}{2}+i\xi)\cdot\xi^2\,d\xi$$

Convergent expansion in (global) spherical $-(\frac{3}{2}+\varepsilon)$ **Sobolev space**

$$\delta = \int_{-\infty}^{\infty}\widetilde{\delta}(\tfrac{1}{2}+i\xi)\,\varphi_{\frac{1}{2}+i\xi}\,\xi^2\,d\xi \;=\; \int_{-\infty}^{\infty}\varphi_{\frac{1}{2}+i\xi}\,\xi^2\,d\xi$$

Analogous to Fourier expression of delta:

$$\delta(x) = \int_{\mathbb{R}} e^{2\pi i x\xi}\,d\xi$$

Latter is Fourier inversion.

Free-space fundamental solution
= **Green's function** = solution u_z to

$$(\Delta - \lambda_z)^2 \, u_z \;=\; \delta^{\text{free}} \qquad\qquad (\text{on } G/K)$$

with $\lambda_z = z(z-1)$. Spherical transform: spectral decomposition diagonalizes differential operators

$$(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2 \, \widetilde{u}_z(\tfrac{1}{2}+i\xi) \;=\; \widetilde{\delta}^{\text{free}}(\tfrac{1}{2}+i\xi) \;=\; 1$$

$$\widetilde{u}_z(\tfrac{1}{2}+i\xi) \;=\; \frac{1}{(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2}$$

Spherical inversion

$$u_z \;=\; \int_{-\infty}^{\infty} \varphi_{\frac{1}{2}+i\xi} \, \frac{\xi^2 \, d\xi}{(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2}$$

By residues, up to constant,

$$u_z \;=\; \frac{r \, e^{-(2z-1)\,r}}{(2z-1)\,\sinh r}$$

For $\Gamma = SL_2(\mathbb{Z}[i])$ **Poincaré series**
= automorphic Green's function
= automorphic fundamental solution

$$\mathfrak{P}_z(g,h) \;=\; \sum_{\gamma\in\Gamma} u_z^{\text{free}}(g^{-1}\gamma h)$$

via *gauges*: converges absolutely, uniformly on compacts in $G \times G$. **As function of** g alone, or of h alone, in $L^2(\Gamma\backslash G/K)$, for $\mathrm{Re}(z) \gg 1$.

Automorphic L^2 expansion in h, $\mathrm{Re}(z) \gg 1$ for *meromorphic continuation* in parameter z:

$$\mathfrak{P}_z(g,h) \;=\; \frac{1/\langle 1,1\rangle}{(\lambda_1-\lambda_z)^2} \qquad \text{(residual)}$$

$$+ \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F-\lambda_z)^2} \qquad \text{(cuspidal)}$$

$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)\;ds}{(\lambda_s-\lambda_z)^2} \quad \text{(continuous)}$$

Visible poles at λ_F. Continuous-part poles?

Not integrate Poincaré series against Eisenstein series to determine continuous spectral components: would *fail* to demonstrate $\mathrm{Sob}(\frac{3}{2}+\varepsilon)$ convergence, thus *failing* to prove *pointwise* convergence of spectral expansion.

Rather, expand **automorphic delta** δ^{afc} in $-(\frac{3}{2}+\varepsilon)$ **global automorphic Sobolev space**

$$\begin{aligned} \delta_g^{\mathrm{afc}}(h) \;&=\; 1/\langle 1,1\rangle && \text{(residual)} \\ &+ \sum_F F(g)\,\overline{F}(h) && \text{(cuspidal)} \\ &+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_s(g)\,E_{1-s}(h)\;ds && \text{(continuous)} \end{aligned}$$

Analogous to Fourier series of periodic delta:

$$\sum_{n\in\mathbb{Z}} \delta_n = \sum_n e^{2\pi i n x}$$

Latter is Poisson summation.

Spectral expansions diagonalize invariant differential operators. Solve

$$(\Delta - \lambda_z)^2 u_z^{\mathrm{afc}} \;=\; \delta^{\mathrm{afc}}$$

by *dividing* by $(\lambda - \lambda_z)^2$.

Automorphic solution to differential equation, expansion convergent in $\mathrm{Sob}^{\mathrm{afc}}(\frac{5}{2} - \varepsilon)$:

$$u_z^{\mathrm{afc}}(g,h) \;=\; \frac{1}{\langle 1,1\rangle(\lambda_1 - \lambda_z)^2} \qquad \text{(residual)}$$

$$+ \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F - \lambda_z)^2} \qquad \text{(cuspidal)}$$

$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)}{(\lambda_s - \lambda_z)^2}\,ds \quad \text{(continuous)}$$

Two automorphic fundamental solutions of $(\Delta - \lambda_z)^2$: wound-up freespace and directly-automorphic.

Global automorphic Sobolev theory [sic] shows *no* automorphic solutions of *homogeneous* equation in $\mathrm{Sob}^{\mathrm{afc}}(-\infty)$: *uniqueness*.

Thus, spectral expansion of $\mathfrak{P}_z$ is the *obvious thing*, above, *and* expansion converges *not* just in $L^2(\Gamma\backslash G/K)$, but also **pointwise**, since

$$\mathrm{Sob}^{\mathrm{afc}}(-(\tfrac{3}{2}+\varepsilon)+4) \;=\; \mathrm{Sob}^{\mathrm{afc}}(\tfrac{5}{2}-\varepsilon)$$

$$\subset\; \mathrm{Sob}^{\mathrm{afc}}(\tfrac{3}{2}+\varepsilon) \;\subset\; C^o(\Gamma\backslash G/K)$$

Standard estimates (to set up global automorphic Sobolev)

η = char function of ball radius $1/T$, small, compute $L^2(\Gamma\backslash G/K)$ norm of

$$q_h(g) \;=\; \sum_{\gamma\in\Gamma} \eta(h^{-1}\gamma g)$$

Bessel's inequality and elementary estimates on principal series (an archimedean *local* issue!) uniformly locally in g,

$$\sum_{|\lambda_F|\le T} |F(g)|^2 + \frac{1}{2\pi}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} |E_{\frac{1}{2}+it}(g)|^2 \, dt \ll_C T^{\frac{3}{2}}$$

Partial summation: $\delta^{\rm afc}$ in ${\rm Sob}^{\rm afc}(-\frac{3}{2}-\varepsilon)$, and locally uniform *pointwise* convergence of $\mathfrak{P}_z$.

Corollary: In ${\rm Re}(z) > \frac{1}{2}$ poles of $\mathfrak{P}_z$ only at finitely-many $s_F \in (\frac{1}{2}, 1)$.

Must be sure that poles of sum+integral are poles of terms! *Not* a general truth: power series? Dirichlet series?

Meromorphic cont'n beyond $\mathrm{Re}(z) = \frac{1}{2}$:

Estimate: discrete part obvious poles.

Continuous: (*dance of contours*) adds terms!

For $\mathrm{Re}(z) < \frac{1}{2}$

$$\mathfrak{P}_z(g,h) \;=\; \frac{1/\langle 1,1\rangle}{\lambda_z^2} \;+\; \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F - \lambda_z)^2}$$

$$+\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)\,ds}{(\lambda_s - \lambda_z)^2}$$

$$-\,2\,\frac{(E_z \otimes E_{1-z})'}{(2z-1)^2} + 4\,\frac{E_z \otimes E_{1-z}}{(2z-1)^3}$$

E_z poles $z = \frac{\rho}{2}$ for non-trivial zeros ρ of $\zeta_k(s)$.

Note: meromorphic cont'n to $\mathrm{Re}(z) < \frac{1}{2}$ of $L^2(\Gamma\backslash G/K)$ function $\mathfrak{P}_z$ is immediately *not* L^2, not in $\mathrm{Sob}^{\mathrm{afc}}(-\infty)$, because $E_s \notin \mathrm{Sob}^{\mathrm{afc}}(-\infty)$.

Watch out: for number field $SL_2(\mathfrak{o})$ or for $SL_n(\mathbb{Z})$ with $n > 2$, infinitely-many Eisenstein series enter, mero cont'n of continuous spectrum has infinitely-many branch points.

Summary, first example: With r geodesic distance gK to hK in G/K,

$$u_z^{\rm free}(g,h) = \frac{r\,e^{-(2z-1)\,r}}{(2z-1)\,\sinh r}$$

Spectral identity for $\mathrm{Re}(z) \gg \frac{1}{2}$

$$\sum_{\gamma\in\Gamma} u_z^{\rm free}(h^{-1}\gamma g) \;=\; u_z^{\rm afc}(g,h)$$

$$= \frac{1}{\langle 1,1\rangle(\lambda_1-\lambda_z)^2} \qquad \text{(residual)}$$

$$+ \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F-\lambda_z)^2} \qquad \text{(cuspidal)}$$

$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)}{(\lambda_s-\lambda_z)^2}\,ds \;\; \text{(continuous)}$$

Applications? Fun: apply Perron, exact formula for lattice points... More serious: integral kernel for second moment of Rankin-Selberg L-functions, asymptotics...?

Higher-rank example: $G = SL_n(\mathbb{C})$

$\mathfrak{a}$ = Lie algebra standard split component A. $\langle,\rangle$ non-degenerate $\mathbb{R}$-valued symmetric $\mathrm{Ad}\,G$-invariant $\mathbb{R}$-bilinear form on $\mathfrak{g}$. $\mathfrak{a}^* \approx \mathfrak{a}$ via $\langle,\rangle$. Polynomial π^+ on $\mathfrak{a}^*$

$$\pi^+(\lambda) \;=\; \prod_{\gamma>0} \langle \gamma, \lambda \rangle \qquad\qquad (\text{with } \lambda \in \mathfrak{a}^*)$$

not counting multiplicities. ρ = half-sum of positive roots, **c**-function

$$\mathbf{c}(\rho + i\lambda) \;=\; \frac{\pi^+(\rho)}{\pi^+(i\lambda)} \;=\; \frac{\prod_{\gamma>0} \langle \gamma, \rho \rangle}{\prod_{\gamma>0} \langle \gamma, i\lambda \rangle}$$

spherical functions

$$\varphi_{\rho+i\lambda}(a) \;=\; \mathbf{c}(\rho+i\lambda) \cdot \frac{\sum_w \operatorname{sgn} w \cdot a^{iw\lambda}}{\sum_w \operatorname{sgn} w \cdot a^{w\rho}}$$

$$=\; \mathbf{c}(\rho+i\lambda) \cdot \frac{\sum_w \operatorname{sgn} w \cdot e^{i\langle w\lambda, \log a\rangle}}{\prod_{\gamma>0} 2\sinh \frac{1}{2}\langle \gamma, \log a\rangle}$$

Spherical transform

$$\widehat{f}(\lambda) \;=\; \int_G f(g)\cdot\overline{\varphi}_{\rho+i\lambda}(g)\,dg$$

Spherical inversion on W-invariant functions

$$f^\vee(a) \;=\; \int_{\mathfrak{a}^*} f(\lambda)\,\varphi_{\rho+i\lambda}(a)\;\frac{d\lambda}{|\mathbf{c}(\rho+i\lambda)|^2}$$

Plancherel

$$|f^\vee|^2_{L^2(G)} \;=\; |f|^2_{L^2(\mathfrak{a}^*)}$$

Rewritten

$$f^\vee(a) \;=\; \frac{(-i)^d\;|W|}{\prod_{\gamma>0}\langle\gamma,\rho\rangle}\times\frac{1}{\prod_{\gamma>0}2\sinh\frac{1}{2}\langle\gamma,\log a\rangle}$$

$$\times\int_{\mathfrak{a}^*} f(\lambda)\,e^{i\langle\log a,\lambda\rangle}\,\pi^+(\lambda)\,d\lambda$$

With Ω =Casimir, to solve

$$[\Omega - (z^2 - \langle\rho,\rho\rangle)]^\nu \, u_z^{\text{free}} \;=\; \delta^{\text{free}}$$

Recall

$$\Omega\varphi_{\rho+i\lambda} \;=\; \langle i\lambda, i\lambda\rangle - \langle\rho,\rho\rangle \;=\; -\langle\lambda,\lambda\rangle - \langle\rho,\rho\rangle$$

differential equation becomes equation in $\lambda \in \mathfrak{a}^*$

$$\big[-\langle\lambda,\lambda\rangle - z^2\big]^\nu \, \widehat{u}_z^{\text{free}} \;=\; \widehat{\delta^{\text{free}}} \;=\; 1$$

Solutions in *spherical Sobolev* by spherical inversion

$$u_z^{\text{free}}(a) \;=\; \text{const}\cdot\frac{1}{\prod_{\gamma>0}\sinh\frac{1}{2}\langle\gamma,\log a\rangle}$$

$$\times\int_{\mathfrak{a}^*}\frac{e^{i\langle\log a,\lambda\rangle}\;\;\pi^+(\lambda)\;\;d\lambda}{\big[\langle\lambda,\lambda\rangle + z^2\big]^\nu}$$

Very unobviously allows explicit evaluation for all complex groups, and *elementary!* for $\dim\mathfrak{a}$ *odd.* [A. DeCelles, PG, ... others?]

For complex groups, $\pi^+(\lambda)$ is harmonic for $\Delta = \sum_i e_i\, e_i^*$, with basis e_i of $\mathfrak{a}^*$, dual basis e_i^*. Proof for SL_n: by Leibniz

$$\Delta\pi^+(\lambda) \;=\; \sum_{\beta\neq\gamma} \frac{\langle\beta,\gamma\rangle}{\langle\lambda,\beta\rangle\,\langle\lambda,\gamma\rangle} \;\cdot\pi^+(\lambda)$$

α_{ij} standard positive root $\alpha_{ij}(x) = x_i - x_j$. Group *pairs* of distinct positive roots β,γ into (disjoint) *groups of three*, α_{ij},α_{jk} and α_{ik},α_{jk} and α_{ij},α_{ik}, indexed by $1\le i<j<k\le n$. For all $i<j<k$,

$$\frac{\langle\alpha_{ij},\alpha_{ik}\rangle}{\langle -,\alpha_{ij}\rangle\,\langle -,\alpha_{ik}\rangle} + \frac{\langle\alpha_{ij},\alpha_{jk}\rangle}{\langle -,\alpha_{ij}\rangle\,\langle -,\alpha_{jk}\rangle}$$

$$+\frac{\langle\alpha_{ik},\alpha_{jk}\rangle}{\langle -,\alpha_{ik}\rangle\,\langle -,\alpha_{jk}\rangle} \;=\; 0$$

Allows application of Hecke's identity, after set-up by gamma identity

$$\frac{1}{A^\nu} \;=\; \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-tA}\,t^\nu\;\frac{dt}{t}$$

Integral becomes $1/\Gamma(\nu)$ times

$$\int_0^\infty \int_{\mathfrak{a}^*} e^{i\langle\lambda,\log a\rangle}\; e^{-t(\|\lambda\|^2+z^2)}\, t^\nu\, \pi^+(\lambda)\; \frac{dt}{t}\, d\lambda$$

Let $r = \mathbb{R}$-rank of $G = \mathbb{R}$-dim $\mathfrak{a}$,
$d =$ no. pos. roots each mult 2.
Replace λ by $\lambda/\sqrt{t}$, apply Hecke, spherical inversion is

$$\frac{i^{-d}}{\Gamma(\nu)} \cdot \pi^+(\log a) \cdot \int_0^\infty e^{-tz^2}\, t^{\nu-\frac{r}{2}-d}\, e^{-\frac{1}{t}\|\log a\|^2}\; \frac{dt}{t}$$

$$= \;\text{const} \cdot \pi^+(\log a) \cdot \int_{\mathbb{R}} \frac{e^{i\lambda_1\|\log a\|}}{(\lambda_1^2+z^2)^{\nu-d-\frac{r-1}{2}}} d\lambda_1$$

As $G/K = 2d + r$, $\delta \in \mathrm{Sob}(-(d + \frac{r}{2} + \varepsilon))$. For continuous fundamental solution, need

$$-(d + \frac{r}{2} + \varepsilon) + 2\nu \;>\; d + \frac{r}{2}$$

$\nu > d + \frac{r}{2}$ suffices. For r odd, take

$$\nu \;=\; d + \frac{r+1}{2}$$

Miraculously, exponent in denominator is

$$\nu - d - \frac{r-1}{2} \;=\; d + \frac{r+1}{2} - d - \frac{r-1}{2} \;=\; 1$$

resulting integral is (up to constant)

$$\int_{\mathbb{R}} \frac{e^{i\lambda_1 \| \log a \|}}{\lambda_1^2 + z^2} d\lambda_1 \;=\; \frac{e^{-z\| \log a \|}}{z}$$

Thus, integral part of spherical inversion is

$$\text{const} \cdot \pi^+(\log a) \cdot \frac{e^{-z\| \log a \|}}{z}$$

Up to constant, free-space fundamental solution

$$u_z^{\text{free}}(a) \;=\; \frac{\pi^+(\log a)}{\prod_{\gamma>0} \sinh \frac{1}{2}\langle \gamma, \log a \rangle} \cdot \frac{e^{-z\| \log a \|}}{z}$$

$$= \;\left(\prod_{\gamma>0} \frac{\langle \gamma, \log a \rangle}{\sinh \frac{1}{2}\langle \gamma, \log a \rangle} \right) \cdot \frac{e^{-z\| \log a \|}}{z}$$

Automorphic solution Warm-up: SL_3

$L^2(SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/K)$ decomposition

$$f = \sum_F \langle f, F\rangle \cdot F \qquad \text{(cuspidal)}$$

$$+ \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \qquad \text{(residual)}$$

$$+ \sum_{F \text{ cfm } GL_2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E^{2,1}_{F,s}\rangle \cdot E^{2,1}_{F,s}\, ds \qquad \text{(2,1 para)}$$

$$+ \int_{\rho+i\mathfrak{a}^*} \langle f, E_\lambda\rangle \cdot E_\lambda\, d\lambda \qquad \text{(min parabolic)}$$

2,1 and 1,2 are *associated parabolics*, Eisenstein series interchanged by functional equations, no poles $\mathrm{Re}(s) \geq \frac{1}{2}$, by Maaß-Selberg or by Poisson-summation.

(Thus), *residual* spectrum just constants.

Convergence in L^2 sense... pointwise?

Identical for $SL_3(\mathbb{Z}[i])$.

Warm-up: $L^2(SL_4(\mathbb{Z})\backslash SL_4(\mathbb{R})/K)$ or $SL_2(\mathbb{Z}[i])$

$$f \quad = \quad \sum_F \langle f, F\rangle \cdot F \qquad \text{(cuspidal)}$$

$$+ \quad \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \qquad \text{(min para residual)}$$

$$+ \sum_{F \text{ cfm } GL_2} \langle f, S_F\rangle \cdot S_F \quad \text{(2,2 residual=Speh forms)}$$

$$+ \sum_{F,F' \text{ cfm } GL_2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E^{2,2}_{F,F',s}\rangle \cdot E^{2,2}_{F,F',s}\, ds \quad \text{(2,2 para)}$$

$$+ \sum_{F \text{ cfm } GL_3} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E^{3,1}_{F,s}\rangle \cdot E^{3,1}_{F,s}\, ds \quad \text{(3,1 assoc)}$$

$$+ \sum_{F \text{ cfm } GL_2} \int_{\rho + i\mathfrak{a}^*_{2,1,1}} \langle f, E^{2,1,1}_{F,\lambda}\rangle \cdot E^{2,1,1}_{F,s}\, d\lambda \quad \text{(2,1,1 assoc)}$$

$$+ \int_{\rho + i\mathfrak{a}^*_{\min}} \langle f, E_\lambda\rangle \cdot E_\lambda\, d\lambda \qquad \text{(min parabolic)}$$

Non-trivial: only Speh residues from $E^{2,2}_{F,\overline{F},s}$

Not from $3, 1$ nor $2, 1, 1$. (Not obvious.)

Warm-up: SL_5 decomposition in L^2

All Eis have cuspidal data and $s \sim \lambda \in \rho + i\mathfrak{a}_P^*$

Cuspidal.

Minimal parabolic, L^2 residues *constants* (!?)

4,1 (and 1,4) Eisenstein series, no residues in right half-plane (Maaß-Selberg or Poisson)

3,2 (and 2,3) Eisenstein series, no residues in right half-plane (Maaß-Selberg)

2,2,1 (and 2,1,2 and 1,2,2) have Speh residues for $F, \overline{F}$ data, no others (!?)

3,1,1 (and 1,3,1, and 1,1,3) no residues in positive chamber (!?)

2,1,1,1 (and 1,2,1,1, and 1,1,2,1 and 1,1,1,2) no residues in positive chamber (!?)

Standard estimates, compact periods
... Sobolev...

Rewrite (spherical) automorphic spectral expansion more succinctly/portably

$$f \;=\; \int_{\xi\in\Xi} \langle f, F_\xi\rangle \cdot F_\xi \; d\xi \qquad\qquad (\Delta F_\xi = \lambda_\xi)$$

For $H \subset G$, with *compact* $\Gamma_H\backslash H$, with $X = G/K$, $X_H = H/K_H$, as usual, form

$$\eta = \text{char fcn ball rad } \tfrac{1}{T} \text{ in } \Gamma_H\backslash X$$

as usual make

$$q(g) \;=\; \sum_{\gamma\in\Gamma_H\backslash\Gamma} \eta(\gamma g)$$

Computing L^2 norm, Bessel inequality, easy *local* estimates on principal series,

$$\int_{\xi:\lambda_\xi\le T^2} \Big|\int_{\Gamma_H\backslash X_H} F\Big|^2 \;\ll\; T^{\dim X - \dim X_H}$$

That is, $\int_{\Gamma_H\backslash X_H}$ is in

$$\mathrm{Sob}^{\mathrm{afc}}\big(-\Big(\tfrac{\dim X - \dim X_H}{2} + \varepsilon\Big)\big)$$

Example: $\mathbb{R}$-anisotropic H, maybe $\{1\}$.

$$\delta^{\rm afc} \;=\; \int_{\Gamma_H\backslash X_H}$$

and solve?

$$(\Delta - \lambda_z)^\nu\, u_z^{\rm afc} \;=\; \delta^{\rm afc} \qquad (\text{on } \Gamma\backslash X)$$

In $\mathrm{Sob}^{\rm afc}(-\frac{1}{2}\dim X - \varepsilon)$

$$\delta^{\rm afc} \;=\; \int_\Xi F_\xi(e)\, F_\xi\; d\xi$$

Solve PDE on spectral side by dividing: in $\mathrm{Sob}^{\rm afc}(2\nu - \frac{1}{2}\dim X - \varepsilon)$

$$u_z^{\rm afc} \;=\; \int_\Xi \frac{\overline{F}_\xi(e)}{(\lambda_\xi - \lambda_z)^\nu}\, F_\xi\; d\xi$$

Local Sobolev gives *convergence* in $C^o(\Gamma\backslash X)$ topology for $2\nu - \frac{1}{2}\dim X - \varepsilon > \frac{1}{2}\dim X$, that is, for

$$2\nu \;>\; \dim X + \varepsilon$$

Not merely $u_z^{\rm afc} \;\in\; C^o(\Gamma\backslash X) \;=\; C^o(X)^\Gamma$, but *convergence of spectral expansion in that topology.*

C^o business allows comparison, proof that (at first for large ν)

$$\sum_{\gamma \in \Gamma} u_z^{\text{free}}(\gamma\, g) \;=\; \int_\Xi \frac{\overline{F}_\xi(e)}{(\lambda_\xi - \lambda_z)^\nu}\, F_\xi(g)\, d\xi$$

Subsums of left-hand side not automorphic. Nonsense to claim convergence in *any* global function space on $\Gamma\backslash X$.

Rather, both sides converge in *local* C^o top, namely, proj lim topology of sups on *compacts*.

Or, similarly, *local* C^k topology.

Translating by $h \in G$, in C^o,

$$\sum_{\gamma \in \Gamma} u_z^{\text{free}}(h^{-1}\gamma\, g) \;=\; \int_\Xi \frac{\overline{F}_\xi(h)\, F_\xi(g)}{(\lambda_\xi - \lambda_z)^\nu}\, d\xi$$

If adelic, $u_z^{\text{free}} = \bigotimes_v u_{z,v}^{\text{free}}$, with $u_{z,v}^{\text{free}}$ as above for archimedean v.

Finite-prime $u_{z,v}^{\text{free}}$ *can* be *dummies*, namely, char fcns K_v. *Or...*

Or finite-prime $u_{z,v}^{\text{free}}$ can be Hecke operators.

mapping to local factor of L-function $L_v(z, -, \rho)$, at finitely-many or at all-but-finitely-many v.

For non-trivial at finite set $v \in S$, can have

$$\sum_{\gamma \in \Gamma} u_z^{\text{free}}(h^{-1}\gamma\, g) \qquad\qquad (S \text{ data implied})$$

$$= \int_{\Xi} \frac{\prod_{v \in S} \lambda_v(F) \cdot \overline{F}_\xi(h)\, F_\xi(g)}{(\lambda_\xi - \lambda_z)^\nu}\, d\xi$$

Or, non-trivial everywhere but bad primes S,

$$\sum_{\gamma \in \Gamma} u_z^{\text{free}}(h^{-1}\gamma\, g) \qquad\qquad (v < \infty \text{ data implied})$$

$$= \int_{\Xi} \frac{L^S(z, F, \rho) \cdot \overline{F}_\xi(h)\, F_\xi(g)}{(\lambda_\xi - \lambda_z)^\nu}\, d\xi$$

Meromorphic continuation of latter up to critical line/hyperplane immediate from estimates above. Beyond, discrete spectrum clear. Others need *contour dance*. Even-more-critical line(s)?!