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Proof of a simple case of the Siegel-Weil formula

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First, I confess I never understood Siegel's arguments for his *mass formula* relating positive definite quadratic forms and Eisenstein series. Of course, Siegel's context did not separate things into *local* and *global* arguments.

On the other hand, while current technique is arguably much more sophisticated, the questions addressed are commensurately more complicated, so that simplification of a proof of a basic Siegel-Weil formula may get lost in more difficult issues. For example, the work of Kudla-Rallis on *regularization* addresses much more delicate questions than the simple equality of holomorphic Eisenstein series and linear combinations of theta series in the region of convergence.

Here I will use by-now-standard methods to prove a modern form of a Siegel-Weil formula for SL_2 over totally real number fields k.

[0.0.1] Theorem: *(Vague version of Siegel-Weil)* Certain holomorphic Eisenstein series of weights 4ℓ (with $\ell = 1, 2, 3, \ldots$) are certain linear combinations of holomorphic theta series attached to certain totally positive definite quadratic forms of dimension 8ℓ.

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1. *Weil/oscillator representations*

We must define a Weil representation for $O(Q) \times SL_2$, where for simplicity we only use certain 8ℓ-dimensional quadratic forms.

Fix a non-degenerate quadratic form Q over a number field k, with dimension 8ℓ divisible by 8, and so that $Q = Q_1 \oplus Q_1$ where Q_1 is a non-degenerate quadratic form with dimension divisible by 4, and **discriminant** a square. (The discriminant of a 4ℓ-dimensional quadratic form is just the determinant of the matrix of the form with respect to a choice of basis.) This hypothesis on the quadratic form simplifies the following discussion.

Fix a non-trivial (continuous) character ψ on the adeles $\mathbb{A}$ of k, trivial on k viewed as sitting inside $\mathbb{A}$. Let $G = SL_2(k)$ and $H = O(Q)$. Let v be a place of k. The local **Weil** or **oscillator representation** ρ of $G_v \times H_v$ is defined on the vector space $\mathscr{S}(8\ell \times 1)$ of Schwartz-Bruhat functions on the space of 8ℓ-by-1 matrices with entries in k_v. For $h \in H$, the action is

$$\rho(h)f\,(t) \;=\; f(h^{-1}t)$$

for Schwartz-Bruhat function f and 8ℓ-by-1 matrix t. (The inverse assures associativity.)

The definition of ρ_v on G_v is done in pieces. For

$$n \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_v$$

the unipotent radical of the standard parabolic subgroup

$$P \;=\; \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

the action is

$$\rho(n)f\,(t) \;=\; \psi(\frac{x}{2}\,Q(t))\,f(t)$$

For

$$m \;=\; \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in M_v$$

where M_v is the standard Levi component of P_v (consisting of all such elements m)

$$\rho(m)f\,(t) \;=\; |a|_v^{4\ell}\, f(ta)$$

The exponent of $|a|$ is half the dimension. Finally, for the Weyl element

$$w \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

define

$$\rho(w)f\,(t) \;=\; \hat{f}(t)$$

where $\hat{f}$ is the Fourier transform

$$\hat{f}(t) \;=\; \int_{k^{8\ell\times 1}} \psi(-\langle \xi, t\rangle)\, f(\xi)\, d\xi$$

where the integral is over the 8ℓ-by-1 matrices with entries in k, and the measure is an additive Haar measure normalized as usual, described in greater detail below. The Weil repn ρ is defined on arbitrary group elements by using the (spherical) Bruhat decomposition of G_v.

[1.0.1] Remark: This is a peculiar way to define a representation. It is not clear that ρ is a group homomorphism on G_v. Work is required to verify that *extending by Bruhat decomposition* is well-defined. Verification that ρ is a group homomorphism is not trivial, but can be done directly.

[1.0.2] Remark: Without the simplifying hypotheses on the quadratic form, the definition of the Weil representation on the Levi component must be adjusted by a finite-order character, and the definition on the Weyl element must be adjusted by an eighth root of unity.

For each place v of k define a *local* Weil representation ρ_v as sketched just above (where we suppressed the subscript). The **global Weil representation** is on the space of Schwartz-Bruhat functions on the space of 8ℓ-by-1 matrices with entries in the adeles $\mathbb{A}$ of k. This Schwartz-Bruhat space is usually described as a **restricted tensor product** of the local Schwartz-Bruhat spaces, so almost everywhere one has a distinguished local Schwartz-Bruhat function f_{vo}. So that the global/adelic Fourier transform is well-defined on the global/adelic Schwartz-Bruhat space, almost everywhere these distinguished vectors must be their own Fourier transform. The usual way to do this, at absolutely unramified non-archimedean places v of k, take f_{vo} to be the characteristic function of 8ℓ-by-1 matrices with entries in the local integers $\mathfrak{o}_v$.

To be sure that the character ψ and the measures $d\xi$ are their own Fourier transforms almost everywhere is as in Iwasawa-Tate theory.

[1.0.3] Remark: We must also check, for *continuity*, that almost everywhere locally at places v

$$\rho(h_o)f_{vo} \;=\; f_{vo}$$

where h_o lies in a specified (maximal) compact subgroup H_{vo} of H_v. We may as well take H_{vo} to be the elements of H_v with entries in the local integers. Due to our simplifying hypotheses on the quadratic form, at almost all primes v the local Witt index of Q is maximal, meaning that there is a 4ℓ-dimensional maximal isotropic subspace, and this choice of H_{vo} works out.

[1.0.4] Remark: For subsequent applications, the number field k will be totally real, and the quadratic form Q positive definite at every archimedean place of k. For the basic set-up this is not necessary.

2. *Theta correspondences*

Let ρ be the Weil representation as above on the product $G_{\mathbb{A}} \times H_{\mathbb{A}}$ of $G = SL_2$ and orthogonal group $H = O(Q)$. Let φ be an adelic Schwartz-Bruhat function, and define the associated **theta kernel**

$$\Theta_\varphi(g,h) \;=\; \sum_x \rho(g,h)\varphi(x)$$

where x ranges over 8ℓ-by-1 matrices with entries in k. It is a little exercise to see that this theta kernel is *continuous* in both g and h in the respective adele groups, and is $SL_2(k) \times O(Q)_k$*-invariant*. The latter invariance uses adelic Poisson summation.

We can use the theta kernel as we might *any* kernel function to define two maps, called **theta liftings** or **theta correspondences**. For f a compactly-supported continuous function on $O(Q)_k \backslash O(Q)_{\mathbb{A}}$, define the **theta lift**

$$\Theta_\varphi(f)(g) \;=\; \int_{O(Q)_k \backslash O(Q)_{\mathbb{A}}} f(h)\,\Theta_\varphi(g,h)\,dh$$

Similarly, for f a compactly-supported continuous function on $SL_2(k)\backslash SL_2(\mathbb{A})$, define the **theta lift** *in the other direction* by

$$\Theta_\varphi(f)(h) \;=\; \int_{SL_2(k)\backslash SL_2(\mathbb{A})} f(g)\,\Theta_\varphi(g,h)\,dg$$

In both cases the image is a continuous function on the target adele group, and is left-invariant under $SL_2(k)$ or $O(Q)_k$.

That is, at least in this crude sense, the theta liftings (depending on the Schwartz-Bruhat function φ) map automorphic forms on $O(Q)$ to automorphic forms on SL_2, and vice-versa. We'll be concerned only with the first case, which roughly maps automorphic forms on the orthogonal group to automorphic forms on SL_2.

[2.0.1] Remark: The general restriction to compactly-supported continuous functions is a non-trivial and too-constrictive condition for general application, since automorphic forms have compact support only on compact arithmetic quotients. However, in the case we care about for the Siegel-Weil formula this restrictive hypothesis will be met.

In particular, for orthogonal groups, by **reduction theory**, the arithmetic quotient $O(Q)_k\backslash O(Q)_{\mathbb{A}}$ is compact if and only if the quadratic form is k-**anisotropic**. Thus, for globally anisotropic quadratic forms the associated arithmetic quotient of the orthogonal group is indeed compact, and the theta lifting assuredly makes sense as an operator sending automorphic forms on the orthogonal group to automorphic forms on SL_2.

Specifically, for the example of the Siegel-Weil formula here, take k **totally real**, and the quadratic form **positive definite** at all archimedean places. This assures that the quadratic form is anisotropic at every real place, so is globally anisotropic.

[2.0.2] Remark: Recall that the general form of the Hasse-Minkowski theorem asserts that a quadratic form is globally anisotropic if and only if it is locally anisotropic *everywhere*. Of course, one direction of this implication is trivial.

In particular, we are interested in precisely identifying the image $\Theta_\varphi(1)$ of the identically-one function 1 on the orthogonal group. As noted, this function is compactly-supported when the quadratic form is anisotropic.

3. *Siegel's Eisenstein series*

Let P be the standard parabolic subgroup of SL_2 consisting of upper-triangular matrices, as usual. Define characters χ_s for $s \in \mathbb{C}$ on $P_{\mathbb{A}}$ by

$$\chi_s \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \;=\; |a|^{2s}$$

A continuous $\mathbb{C}$-valued right-K-finite[1] function $\varepsilon(g)$ on the adele group $SL_2(\mathbb{A})$ with the left equivariance property

$$\varepsilon(pg) \;=\; \chi_s(p)\,\varepsilon(g)$$

for $p \in P(\mathbb{A})$ is in the (adelic) **principal series** I_s. Consider the **Eisenstein series** attached to the **kernel** ε by

$$E(g) \;=\; \sum_{\gamma \in P_k \backslash SL_2(k)} \varepsilon(\gamma g)$$

This series is nicely convergent for $\mathrm{Re}(s) > 1$, so defines an automorphic form.

[3.0.1] Remark: This convergence is not so hard to verify here in this special case of SL_2, but is not elementary in general.

By the definition of continuity and K-finiteness on the adele group the kernel ε is a finite sum of products

$$\varepsilon(\{g_v\}) \;=\; \bigotimes{}'_v \; \varepsilon_v(g_v)$$

For a totally real number field k, when $2 < s = \ell \in \mathbb{Z}$, and when the infinite-prime vectors ε_v are (constant multiples of) the special form

$$\varepsilon_v(\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) \;=\; |a|^\ell \, e^{i\ell\theta}$$

the Eisenstein series is **holomorphic** of weight ℓ in two (equivalent) senses: First, the classical automorphic forms attached to this Eisenstein series are all literally holomorphic of weight ℓ. Second, the representation of $SL_2(k_v)$ generated by such ε_v is a **holomorphic discrete series** representation of weight ℓ; it is irreducible.

[3.0.2] Remark: By definition of K-finiteness, for almost all finite primes v, ε_v is right K_v-invariant. For such v, ε_v is the standard **spherical vector** in the **principal series representations** of $SL_2(k_v)$ consisting of all left P_v, χ_s-equivariant (right K_v-finite) functions on $SL_2(k_v)$.

[3.0.3] Remark: For $2 < s = k \in \mathbb{Z}$ the finite-prime principal series are *irreducible*. (And, as noted above, the holomorphic discrete series subrepresentation of these infinite-prime principal series are also irreducible). Neither of these assertions is trivial to verify, but both are standard.

[3.0.4] Remark: Also, for $1 < \mathrm{Re}(s)$ the finite-prime principal series representations are definitely *not* **unitarizable**. This is also not trivial, but is standard.

[1] A function f on a group is *right K-finite* for a subgroup K when the collection of *right translates* $R_x f$, defined by $R_x f(y) = f(yx)$, is a finite-dimensional space of functions.

4. *Eisenstein series from the Weil representation*

One way to make an Eisenstein series from the Weil representation is as follows. This does not directly connect to theta series or the theta correspondence, but sets up that comparison.

Let φ be in the Schwartz-Bruhat space $\mathscr{S}(8\ell \times 1)$ on 8ℓ-by-1 matrices over the adeles $\mathbb{A}$ of the number field k. Keep the earlier hypotheses on the quadratic form Q (and hence on the Weil representation). The function

$$\varepsilon(g) \;=\; (\rho(g)\varphi)(0)$$

is readily seen to be in the adelic principal series I_s with $s = 4\ell$. Define an Eisenstein series by

$$E_\varphi(g) \;=\; \sum_{\gamma \in P_k \backslash SL_2(k)} (\rho(\gamma g)\varphi)(0)$$

[4.0.1] **Remark:** The Schwartz-Bruhat function φ is a finite sum of tensor products of local Schwartz-Bruhat functions φ_v, almost all of which are the characteristic function of the set of 8ℓ-by-1 matrices with entries in the local integers $\mathfrak{o}_v$. From this, a little computation with the Weil representation shows that the corresponding ε_v really is right K_v-invariant.

5. *Holomorphic discrete series*

At least for the most classical applications, we want automorphic forms to be *holomorphic*. That is, we want automorphic forms to generate *holomorphic discrete series* locally at archimedean primes.

Therefore, we take k *totally real* (has only *real* archimedean completions), since $SL_2(\mathbb{R})$ *has* holomorphic discrete series (discussed momentarily) while $SL_2(\mathbb{C})$ has no such representations. For present purposes it is not necessary to know why $SL_2(\mathbb{C})$ *fails* to have holomorphic discrete series, only to understand that $SL_2(\mathbb{R})$ *does* have such.

For less obvious reasons, we take *totally positive definite* quadratic form Q, that is, is positive definite locally at all the archimedean places of k. Otherwise, it will turn out that we will not have created *holomorphic* automorphic forms. It is not easy to see why this positive-definiteness is *necessary*, but it is relatively easy to verify that taking Q totally positive definite is *sufficient*, insofar as the discussion below *succeeds*.

For the present discussion, let

$$c \;=\; \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} / \sqrt{2}$$

be the usual **Cayley element**. The significance of this element can be understood in a variety of ways, but we choose a relatively elementary viewpoint. The linear fractional transformation

$$z \to c(z) \;=\; \frac{z+i}{iz+1}$$

maps the complex unit disk to the upper half-plane. For a real archimedean prime v

$$K_v \;=\; SO(2) \;=\; c \cdot \{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\} \cdot c^{-1} \;=\; \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\}$$

Thus, on the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ of $SL_2(\mathbb{R})$, K_v has three eigenspaces

$$\mathfrak{k}_\mathbb{C} \;=\; \text{trivial eigenspace} \;=\; c \cdot \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\} \cdot c^{-1}$$

$$\mathfrak{p}^+ = c \cdot \{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\} \cdot c^{-1}$$

$$\mathfrak{p}^- = c \cdot \{\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}\} \cdot c^{-1}$$

Choose

$$R = c \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot c^{-1} = \text{raising operator}$$

$$L = c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot c^{-1} = \text{lowering operator}$$

Justification for the names will be clear shortly. Let any element α in the Lie algebra $\mathfrak{g}$ of the group $SL_2(\mathbb{R})$ act by the standard infinitesimal action on any (*smooth*) vector x in a representation π:

$$d\pi(\alpha)x = \frac{d}{dt}\Big|_{t=0} \pi(e^{t\alpha})x$$

The *smoothness* hypothesis on x is as usual that the function

$$g \longrightarrow \pi(g)x$$

is infinitely-differentiable on the real Lie group $SL_2(\mathbb{R})$. Let R and L, or any element in the *complexification* of the Lie algebra, act on a (*smooth*) vector x in a by the complexification: suppose that the element is $\alpha + i\beta$ with α and β in the (real) Lie algebra, and put

$$d\pi(\alpha + i\beta)x = d\pi(\alpha)x + i\, d\pi(\beta)x$$

We may suppress the $d\pi$ notation.

Let x be a vector in a representation π of $SL_2(\mathbb{R})$ with the **weight-ℓ** K_v-equivariance property

$$\pi \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} x = e^{i\ell\theta} \cdot x$$

[5.0.1] Proposition: For a smooth weight-ℓ vector x, Rx has weight $\ell + 2$ and Lx has weight $\ell - 2$.

Proof: Both assertions are immediate from the hypothesis and from the fact that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} R \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = e^{2i\theta} \cdot R$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} L \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = e^{-2i\theta} \cdot L$$

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[5.0.2] Remark: Thus, R *raises* weights and L *lowers* weights.

[5.0.3] Definition: A (smooth) vector x in a representation is **holomorphic** if it is annihilated by L, and is an $SO(2) = K_v$ eigenvector.

[5.0.4] Remark: It is supposedly well known by now that a classical automorphic form living on a complex domain is holomorphic in the elementary sense if and only if when converted to an automorphic form on a Lie group (or on an adele group) it is holomorphic in this representation-theoretic sense.

The defining property of the **weight ℓ holomorphic discrete series** representation π_ℓ of $SL_2(\mathbb{R})$ (with $\ell \geq 2$) is that it is generated by a **holomorphic vector** x_o. The **lowest K-type** of π_ℓ is ℓ (or, really, the representation of K_v indexed by ℓ). (The K-type is not so easily indexed for larger groups, of course.) In a typical abuse of language, the complex-linear span of x_o in π_ℓ is also called the **lowest K-type** of π_ℓ.

[5.0.5] Theorem: For $\ell \geq 2$ there exists a weight-ℓ holomorphic discrete series representation of $SL_2(\mathbb{R})$, unique up to isomorphism. (We won't prove this here, but it is not difficult.) ///

[5.0.6] Proposition: For totally real fields k, at infinite primes v, for (totally) positive-definite quadratic form Q, for φ_v the Gaussian (attached to Q)

$$\varphi_v(x) \;=\; e^{-\pi Q[x]}$$

for any 8ℓ-by-1 matrix x the function

$$g \to \rho(g)\varphi(x)$$

is *holomorphic*, and generates a *holomorphic discrete series* representation of weight 4ℓ. It lies in the *lowest K-type* 4ℓ of the holomorphic discrete series $\pi_{4\ell}$.

[5.0.7] Corollary: The corresponding ε_v is *holomorphic*, and generates a *holomorphic discrete series* representation of weight 4ℓ. It lies in the *lowest K-type* 4ℓ of that holomorphic discrete series.

[5.0.8] Remark: That is, with such choice of Schwartz-Bruhat functions at infinite primes, the Eisenstein series arising from φ via the Weil representation really is a holomorphic automorphic form.

[5.0.9] Corollary: The corresponding

$$g \to \Theta_\varphi(g,h) \;=\; \sum_x \rho(g,h)\varphi\,(x)$$

is *holomorphic* for any $h \in H_v$.

[5.0.10] Remark: That is, with such choice of Schwartz-Bruhat functions at infinite primes, the theta series arising from φ really are holomorphic automorphic forms.

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6. *Outline of proof of Siegel-Weil formula*

We suppose that the number field k is totally real, and that the quadratic form Q is 8ℓ-dimensional, with further specifics as above at archimedean primes. In particular, Q is positive-definite at all infinite primes. Choose a Schwartz-Bruhat function φ in $\mathscr{S}(8\ell \times 1)$ with the infinite-prime factors φ_v being the standard Gaussians. Let $\mathbf{1}$ denote the identically-one function on the adelized orthogonal group $O(Q)$. Then

[6.0.1] Theorem: *(Siegel-Weil)*

$$E_\varphi \;=\; \Theta_\varphi(\mathbf{1})$$

That is, the theta lift of the constant function $\mathbf{1}$ on the orthogonal group is 'the' Eisenstein series on SL_2.

Proof: *(Sketch)* Of course the goal is to prove that the difference

$$E_\varphi - \Theta_\varphi(\mathbf{1})$$

is the identically-zero function.

Compute directly that, due to the holomorphy, the two Bruhat-cell parts of the constant term of E_φ simplify: the big-cell term is simply 0. The little-cell term is directly computed to be equal to the constant term of the automorphic form $\Theta_\varphi(\mathbf{1})$. And both E_φ and $\Theta_\varphi(\mathbf{1})$ are holomorphic. Thus, the difference

$$E_\varphi - \Theta_\varphi(\mathbf{1})$$

is holomorphic, of moderate growth, and with constant term 0. By the simple way that holomorphic automorphic forms behave, this assures that $E_\varphi - \Theta_\varphi(\mathbf{1})$ is square-integrable, so is a cuspform.

If it were the case that at some finite prime v the Eisenstein series E_φ and the theta-lift $\Theta_\varphi(\mathbf{1})$ *locally* generated isomorphic irreducible representations of $SL_2(k_v)$, then the difference $E_\varphi - \Theta_\varphi(\mathbf{1})$ would also generate an isomorphic copy of that irreducible representation of $SL_2(k_v)$. We know that almost everywhere the Eisenstein series E_φ generates the irreducible principal series $I_{4\ell}$. Below we will sketch a proof that almost everywhere $\Theta_\varphi(\mathbf{1})$ generates that same irreducible. Granting this, we find that the difference $E_\varphi - \Theta_\varphi(\mathbf{1})$ generates $I_{4\ell}$ almost everywhere.

Since the difference $E_\varphi - \Theta_\varphi(\mathbf{1})$ is square-integrable and locally generates the irreducible representation $I_{4\ell}$ of $SL_2(k_v)$ for at least one finite prime v, if this difference were not identically zero then the local representation $I_{4\ell}$ would be unitarizable. We have noted that it is not, so we must conclude that the difference $E_\varphi - \Theta_\varphi(\mathbf{1})$ is 0. This proves the Siegel-Weil formula. ///

[6.0.2] Remark: We don't get any contradiction to unitariness by looking at archimedean primes, because the holomorphic discrete series *are* unitary.

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7. *Computing some Jacquet modules*

To complete a reasonable sketch of a proof of Siegel-Weil, even in our simple case, we should verify that at almost all primes the theta-lift $\Theta_\varphi(\mathbf{1})$ generates the *principal series* $I_{4\ell}$ at almost all finite primes. This is true regardless of φ, although the finite set of exceptional primes changes depending upon the choice of φ.

To verify this, we compute the Jacquet module of the representation of $SL_2(k_v)$ generated by the trivial-representation co-isotype for $O(Q)_v$ of the Weil representation ρ. (Note that the Jacquet module is the trivial-representation co-isotype for the standard unipotent radical N_v in $SL_2(k_v)$.)

[7.0.1] Lemma: Almost everywhere locally, the trivial-representation co-isotype for $O(Q)_v$ of the local Weil representation ρ_v is *$I_{4\ell}$-isotypic.*

Proof: We may suppose that the residue field characteristic is not 2. By Frobenius reciprocity, the assertion of the lemma is equivalent to the assertion that, as a P_v-representation, the trivial-representation co-isotype for $O(Q)_v \times N_v$ of ρ_v is a sum of copies of the one-dimensional representation

$$\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \to |a|^{4\ell}$$

That is, the trivial-representation co-isotype for $O(Q)_v \times N_v$ of ρ_v consists entirely of vectors which are left P_v-equivariant by the character $\chi_{4\ell}$.

This terminology is standard, if not common: the trivial-representation co-isotype of a representation space (σ, V) of a group G is the smallest quotient space

$$q : (\sigma, V) \to (\tau, W)$$

through which every intertwining operator from (σ, V) to the trivial G-representation factors. If all representations at hand were unitary, this quotient would be isomorphic to a subrepresentation, but in general this is not so. The Jacquet module is the trivial-representation co-isotype for the unipotent radical N_v, and the theta-lift of $\mathbf{1}$ generates the trivial representation of $O(Q)_v$, so the intertwining operator

$$\rho \to \Theta_\varphi(\mathbf{1})$$

must indeed factor through the trivial-representation co-isotype for $O(Q)_v \times N_v$.

Also we must recall that we have an exact sequence of Schwartz-Bruhat spaces

$$0 \to S(Y) \to S(X) \to S(X-Y) \to 0$$

for any totally disconnected space X and open subset $Y \subset X$. In this context, $S(X)$ is the collection of locally constant compactly-supported complex-valued functions on a totally-disconnected space X. The proof of this assertion is not difficult, and goes back at least to Bruhat's early work circa 1960.

Let

$$q : \rho \to \text{ trivial co-isotype of representation gen'd by } \Theta_\varphi(\mathbf{1})$$

be the map to the trivial representation co-isotype of the $O(Q)_v \times N_v$ representation generated by $\Theta_\varphi(\mathbf{1})$.

First, we claim that any Schwartz-Bruhat function f with support on non-isotropic vectors in Q maps to 0 under q. Specifically, on the support of f the function $x \to Q[x]$ is certainly continuous, being polynomial, and non-zero by assumption, so *uniformly* bounded away from 0 by the compactness of support of f. Thus, $\mathrm{ord}_v\, Q[x]$ is bounded from above for x in the support of f. Say $\mathrm{ord}_v\, Q[x] \le n$. Thus, for any fixed x in the support of f, the function $t \to \psi(t \cdot Q[x]/2)$ on $\varpi^{-n-1}\mathfrak{o}_v$ is a non-trivial character on $\varpi^{-n-1}\mathfrak{o}_v$, hence has integral 0 over $\varpi^{-n-1}\mathfrak{o}_v$. By standard apparatus for Jacquet modules, this implies that f maps to 0 in the quotient map to the Jacquet module.

Let Z be the set of isotropic vectors in Q. By the previous claim, the map q to the co-isotype factors through $S(Z)$. That is, the co-isotype is a quotient of $S(Z)$.

By Witt's theorem, the collection of isotropic vectors for Q falls into exactly two $O(Q)_v$-orbits: $\{0\}$ and all non-zero isotropic vectors. (By the local results for the Hasse-Minkowski theorem, since $8\ell > 4$ there *do* exist non-zero Q-isotropic vectors locally at all finite primes v.)

From this, invoking the exact sequence above, (and using Frobenius Reciprocity) we see that the trivial-representation co-isotype π fits into a short exact sequence

$$0 \to S(Z - \{0\})_1 \to \pi \to S(\{0\}) \to 0$$

of P_v-representations (trivial representation of $O(Q)_v$), where $S(Z)_1$ is shorthand for the $O(Q)_v$ trivial co-isotype of the Schwartz-Bruhat space $S(Z-\{0\})$ on the open subset $Z - \{0\}$ of Z.

In particular, we **claim** that every $O(Q)_v$-invariant distribution on $S(Z)$ is a linear combination of the functional

$$u_o(f) \;=\; f(0)$$

and the functional

$$u_1(f) \;=\; \int_Q f(x)\, dx$$

where we lift f back to a Schwartz-Bruhat function on the whole vector space Q of the quadratic form, over k_v. Certainly these two functionals are both $O(Q)_v$-invariant and are linearly independent. Thus, from the previous computation, it must be the the trivial-representation $O(Q)_v$ co-isotype is the *direct sum* of these two trivial representations of $O(Q)_v$.

It remains to identify the representation of the Levi component M_v of the parabolic P_v of $SL_2(k_v)$ which occurs in these two spaces. On the subspace in the co-isotype corresponding to the functional u_o, the Levi component of $SL_2(k_v)$ acts by the character $\chi_{4\ell}$ coming from the definition of the Weil representation: with $m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,

$$u_o(mf) \;=\; u_o(|a|^{4\ell}\, f(*a)) \;=\; |a|^{4\ell}\, u_o(f(*a)) \;=\; |a|^{4\ell}\, f(0\cdot a) \;=\; |a|^{4\ell}\, f(0) \;=\; |a|^{4\ell}\, u_o(f)$$

On the subspace corresponding to the functional u_1, the effect of such m is a little messier to verify but turns out to be $|a|^{1-4\ell}$.

Thus, finally, we see that the Jacquet module for $SL_2(k_v)$ of the theta kernel is

$$\chi_{4\ell} \oplus \chi_{1-4\ell}$$

Noting that we hadn't done any normalizing, this is none other than the Jacquet module of the principal series. ///