Spectral identities and integral moments

(joint with Diaconu and Goldfeld)

Idea: *integral moments* over *families* of L-functions arise as *coefficients* in automorphic spectral decompositions.

Example of decomposition: For cuspform f on GL_n, complex s, cuspform F on GL_{n-1}

$$L(s, f \otimes F) \;=\; s, \overline{F}^{\;th} \text{ component of}$$

$$f \text{ restricted to } \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \approx GL_{n-1}$$

by

$$L(s, f \otimes F) \;=\; \int |\det h|^{s-\frac{1}{2}}\; F(h) \;\cdot\; f\begin{pmatrix} h & \\ & 1 \end{pmatrix} dh$$

• Exploit non-commutativity of reductive groups to produce non-trivial **spectral identities**... with **positivity** property, involving (integral) **second moments**.

Sample application: Subconvex bounds.

Example: *Theorem: t-aspect* **subconvexity** for $L(\frac{1}{2} + it, f)$ for GL_2 *over number fields.* (Diaconu-PG 2006)

Recipe for identities: With Euler-Gelfand subgroup H of G

$$\begin{array}{ccccc} & & G \times G & & \\ & \nearrow & & \nwarrow & \\ H \times H & & & & G^\Delta \\ & \nwarrow & & \nearrow & \\ & & H^\Delta & & \end{array}$$

Automorphic distribution $u =$ integrate along $H_k^\Delta \backslash H_{\mathbb{A}}^\Delta$

$$u \;=\; \begin{cases} \displaystyle\sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} F \otimes F^\vee & (\text{along } H \times H) \\[2ex] \displaystyle\sum\!\!\!\!\!\!\!\!\int_{F \text{ on } G} (H\text{-period of } F) \cdot F & (\text{along } G^\Delta) \end{cases}$$

Apply to $f \otimes f^\vee$ on $G \times G$, *regularizing* as needed, have
harsh spectral identity

$$\sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} |\langle f, F \rangle_H|^2 \;=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } G} \langle \overline{F}, 1 \rangle_H \;\cdot\; \langle |f|^2, F \rangle_G$$

Note: *Harsh* because in this form archimedean contributions are of exponential decay, swamping interesting phenomena.

Note: The Euler-Gelfand condition is approximately that

$$\langle f, F \rangle_H \;=\; \text{Euler product}$$

Deform u in two ways:
- spread out to classical function on G^Δ
- undo residue: 1 is residue of Eisenstein series.

Examples of second-moment identities thus obtainable:

Any Rankin-Selberg configuration...

Hecke-type L-functions for $GL_n \times GL_{n-1}$:
$H = GL_{n-1}$ and $G = GL_n$

Rankin-Selberg L-functions for $GL_n \times GL_n$:
$H = GL_n$ and $G = GL_n \times GL_n$

Triple product L-functions for $SL_2 \times SL_2 \times SL_2$:
$H = SL_2 \times SL_2 \times SL_2$ and $G = Sp_3$

Triple product L-functions *one cuspform fixed*:
$H = SL_2 \times SL_2$ and $G = Sp_2$

Standard L-functions on classical groups:
$H = \mathrm{Isom}\langle , \rangle$ and $G = \mathrm{Isom}(\langle , \rangle \oplus -\langle , \rangle)$

Note: overcoming *regularization* problems is essential.

Packets of cuspidal-data Eisenstein series regularize.

Degenerate Eisenstein series appear as *residues*.

Notion of *Schwartz space* and *tempered* automorphic distributions.

Caution: Not *every* second-moment asymptotic (with reasonable error term) yields *subconvex estimates*...

Examples: $GL_2(\mathbb{Q})$ symmetric powers, eigenvalue aspect convexity bound

$$L(\tfrac{1}{2}, \mathrm{Sym}^n f) \ll \begin{cases} |\mu_f|^{\frac{n}{4}+\varepsilon} & \text{(for } n \text{ even)} \\ |\mu_f|^{\frac{n+1}{4}+\varepsilon} & \text{(for } n \text{ odd)} \end{cases} \qquad \forall \varepsilon > 0$$

Best reasonable asymptotic-with-error (Lindelöf + Weyl)

$$\sum_{|\mu_f| \le T} |L(\tfrac{1}{2}, \mathrm{Sym}^n f)|^2 \;=\; T^2\, P(\log T) + O(T^{2-\mathrm{small}})$$

gives

$$|L(\tfrac{1}{2}, \mathrm{Sym}^n f)| \;\ll\; T^{1-\mathrm{small}}$$

For $n = 2$, *fails* to break convexity.

For $n \geq 3$, *breaks* convexity.

Note: The convexity bound is not not known for large n.

For $n \geq 5$, this asymptotic is too good, probably unattainable.

Further, it seems that for $n \geq 3$ an average of $|L(\frac{1}{2}, \mathrm{Sym}^n f)|^2$ does not appear *naturally*, that is, is not *spectrally complete*.

Examples: $GL_2(\mathbb{Q})$ triple products, eigenvalue aspect... analytic conductor

$$Q(f,g,h) \;=\; \prod_{\text{signs}} \Big(1 + \big|\pm\mu_f \pm \mu_g \pm \mu_h\big|\Big) \qquad\qquad \text{(eight factors)}$$

Convexity bound

$$|L(\tfrac{1}{2}, f\otimes g\otimes h)| \;\ll\; Q(f,g,h)^{\frac{1}{4}+\varepsilon} \qquad \forall\varepsilon{>}0$$

Failure: when *all three* vary,

$$\sum_{|\mu_f|,|\mu_g|,|\mu_h|\le T} |L(\tfrac{1}{2}, f\otimes g\otimes h)|^2 \;=\; T^6\, P(\log T) + O(T^{6-\text{small}})$$

fails to break convexity: for $|\mu_f|, |\mu_g|, |\mu_h|$ all $\sim T$, prove

$$|L(\tfrac{1}{2}, f\otimes g\otimes h)| \;\ll\; T^{3-\text{small}}$$

while $Q(f,g,h)^{\frac{1}{4}} \ll (T^8)^{\frac{1}{4}} = T^2$

Success: if only *two* vary,

$$\sum_{|\mu_g|,|\mu_h|\le T} |L(\tfrac{1}{2}, f\otimes g\otimes h)|^2 \;=\; T^4\, P(\log T) + O(T^{4-\text{small}})$$

breaks convexity when not only $|\mu_g|$ and $|\mu_h|$ are $\sim T$, but also $|\mu_g \pm \mu_h|$ *both* $\sim T$.

That is, have subconvex bound *away from* **conductor-dropping**.

First deformation: to classical function on G^Δ

Deform distribution u on $G_k^\Delta\backslash G_\mathbb{A}^\Delta$ given by integration-along $H_k^\Delta\backslash G_\mathbb{A}^\Delta$ into integration on $G_k^\Delta\backslash G_\mathbb{A}^\Delta$ against a left $Z_k H_k$-invariant classical function φ on $G_\mathbb{A}$ and *wind up*

$$\mathfrak{P}^\varphi(g) \;=\; \sum_{\gamma\in Z_k H_k\backslash G_k} \varphi(\gamma\cdot g)$$

Prescription: $\varphi = \bigotimes_v \varphi_v$ and

$$\varphi_v(g) \;=\; \begin{cases} 1 & (\text{for } g = zhk,\ z\in Z_v,\ h\in H_v,\ k\in K_v) \\ \\ 0 & (\text{for } g\notin Z_v H_v K_v) \end{cases} \qquad\qquad (v<\infty)$$

$$\varphi_v(zh\theta k) \;=\; \Phi_v(\theta) \qquad (z\in Z_v,\ h\in H_v,\ k\in K_v,\ \theta\in\Theta_v) \qquad (v|\infty)$$

where Φ_v is suitable function on Θ_v, submanifold *transverse* to H_v in G_v.

Example: With $G = GL_n$ and $H = GL_{n-1}$, Iwasawa decomposition suggests Θ = unipotent radical of $n-1,1$ parabolic.

Thus, *canonical* **trivial** deformation at finite places, although we reserve the possibility of non-trivial p-adic deformations *also*. (cf. Letang)

Somewhat canonical deformation at $v|\infty$: Let Ω be Casimir descended to G_v/K_v, take $\lambda\in\mathbb{C}$, and specify $\varphi_v = \varphi^\lambda$ by PDE on G_v

$$(\Omega-\lambda)\,\varphi_v^\lambda \;=\; u_v$$

where u_v is integration along H_v, and require invariance: left by H_v, right by K_v.

Spectral decomposition of $\mathfrak{P}^\varphi$

Over $\mathbb{Q}$, computing by projecting, the F^{th} spectral component is

$$\int_{Z_\mathbb{A} G_k \backslash G_\mathbb{A}} F \cdot \mathfrak{P}^{\varphi^\lambda} = \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} F \cdot \varphi^\lambda$$

$$= \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} \frac{\Omega - \lambda}{\lambda_F - \lambda} F \cdot \varphi^\lambda = \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} F \cdot \frac{\Omega - \lambda}{\lambda_F - \lambda} \varphi^\lambda$$

by integrating by parts. Then this is

$$\frac{1}{\lambda_F - \lambda} \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} F \cdot \Big(u_v \otimes \bigotimes_{v<\infty} \varphi_v\Big) = \frac{1}{\lambda_F - \lambda} \cdot \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} F$$

$$= \frac{H\text{-period of } F}{\lambda_F - \lambda}$$

Note: For H large in G, *period non-vanishing* condition is non-trivial. Local necessary condition

$$\mathrm{Hom}_{G_v}(\pi_{F,v},\ \mathrm{Ind}_{H_v}^{G_v} 1) \neq 0$$

Globally non-trivial also: example, with $H = GL_{n-1}$ inside $G = GL_n$, for $n \geq 3$ cuspforms have vanishing H-period.

Note: *trailing* poles of $\mathfrak{P}^\lambda$ appear at eigenvalues of cuspforms with non-vanishing periods.

Note: GL_2 case suggests that poles are insensitive to choices of data!

Moment expansion after first deformation:

Spherical cuspform f on G, initial unwinding

$$\int \mathfrak{P}^\lambda \cdot |f|^2 \;=\; \int_{Z_\mathbb{A} H_k \backslash G_\mathbb{A}} \varphi^\lambda \cdot |f|^2$$

$$= \int_{H_\mathbb{A} \backslash G_\mathbb{A}} \varphi^\lambda(g) \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f(hg)\, f^\vee(hg)\, dh\, dg$$

Expand $h \to f(hg)$ along H

$$f(hg) \;=\; \sum\!\!\!\!\!\!\int_{F \text{ on } H} F(h) \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f(\eta g)\, \overline{F}(\eta)\, d\eta$$

$$= \sum\!\!\!\!\!\!\int_{F \text{ on } H} F(h)\, \langle g \cdot f,\, F\rangle_H$$

with right-translation action of $g \in G_\mathbb{A}$ on functions f. Thus,

$$\int \mathfrak{P}^\lambda \cdot |f|^2$$

$$= \sum\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_\mathbb{A} \backslash G_\mathbb{A}} \varphi^\lambda(g) \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} F(h)\, \langle g \cdot f,\, F\rangle_H\, f^\vee(hg)\, dh\, dg$$

$$= \sum\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_\mathbb{A} \backslash G_\mathbb{A}} \varphi^\lambda(g) \cdot |\langle g \cdot f,\, F\rangle_H|^2\, dg$$

As φ^λ is non-trivially deformed only at ∞, in the integral over $H_\mathbb{A} \backslash G_\mathbb{A}$ the element $g = \{g_v\}$ can be taken in H_v except at ∞. Thus, with the expected *entanglement* at archimedean places, **moment expansion:**

$$\int \mathfrak{P}^\lambda \cdot |f|^2 \;=\; \sum\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_\infty \backslash G_\infty} \varphi^\lambda_\infty(g_\infty) \cdot |\langle g_\infty \cdot f,\, F\rangle_H|^2\, dg_\infty$$

Second deformation: undoing residues

Note: Typically, convergence of $\mathfrak{P}^\varphi$ requires further deformation on $H_\mathbb{A}$ itself. This can be subsumed in more sweeping *second* deformation of the 1 on $H_\mathbb{A}$ to *Eisenstein series* on $H_\mathbb{A}$.

Let Q be minimal parabolic in H, E^β minimal-parabolic Eisenstein series on H attached to vector $\eta^\beta = \bigotimes_v \eta_v^\beta$ with η_v^β in β^{th} principal series on H_v.

E^β has a constant residue: deform 1 into E^β.

At $v < \infty$, canonical *trivial* deformation of η_v by right K_v-invariance to φ_v on G_v.

At $v|\infty$, over $\mathbb{Q}$ for example, make *semi-canonical* deformation of η_v by right K_v-invariance, left Q_v-equivariance, solving PDE

$$(\Omega - \lambda)\, \varphi_v^{\lambda,\beta} \;=\; \eta_v^\beta$$

Set

$$\varphi^{\lambda,\beta} \;=\; \varphi_\infty^{\lambda,\beta} \,\otimes\, \bigotimes_{v<\infty} \varphi_v^\beta$$

Overlying Poincaré series

$$\mathfrak{Q}^{\lambda,\beta}(g) \;=\; \sum_{\gamma\in Z_k Q_k \backslash G_k} \varphi^{\lambda,\beta}$$

The original Poincaré series $\mathfrak{P}$ is *essentially* a residue of $\mathfrak{Q}$.

Overlying identity is expansion of $\int \mathfrak{Q}^{\lambda,\beta} \cdot |f|^2$ in two different ways, as above with $\int \mathfrak{P}^{\lambda,\beta} \cdot |f|^2$.

Example: of **overlying identity**:

With $H = GL_2 \subset G = GL_3$, for simplicity suppressing *first* deformation: $\mathfrak{Q}$ is really just E^β on $H = GL_2$, and for cuspform f on GL_3

$$\int E^\beta \cdot f \;=\; GL_2 \times GL_3 \;\text{ Hecke-type integral}$$

Moment expansion of $\int \mathfrak{Q} \cdot |f|^2$ begins

$$\sum\!\!\!\!\!\!\int_{F_1, F_2 \text{ on } GL_2} \Lambda(\beta, F_1 \otimes \overline{F}_2) \;\cdot\; \Lambda(\tfrac{1}{2}, f \otimes F_1)\, \overline{\Lambda}(\tfrac{1}{2}, f \otimes F_2)$$

Spectral expansion of $\mathfrak{Q}$ has non-trivial cuspidal components: for F cuspform on GL_3

$$\int \mathfrak{Q} \cdot F \;=\; \Lambda(\tfrac{1}{2}, F \otimes E^\beta) \;=\; \Lambda(\tfrac{1}{2} + \beta_1, F) \cdot \Lambda(\tfrac{1}{2} + \beta_2, F)$$

Taking residue at $\beta = 1$ annihilates *off-diagonal* terms in *moment* expansion and annihilates *cuspidal* terms on *spectral side*.

This recovers identity for $\int \mathfrak{P} \cdot |f|^2$.

More generally: for $GL_{n-1} \subset GL_n$, a similar second deformation, undo-ing a single residue, gives Rankin-Selberg convolutions on GL_{n-1} as coefficients in the moment expansion. Undoing $n-2$ residues recovers non-trivial cuspidal components in the spectral expansion.

Asymptotics of weights in moments

$$\int \mathfrak{P}^\lambda \cdot |f|^2 \;=\; \not\!\!\!\sum\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_\infty \backslash G_\infty} (g_\infty) \cdot |\langle g_\infty \cdot f,\, F\rangle_H|^2 \, dg_\infty$$

Explicit classical computations barely possible for $GL_1 \subset GL_2$. For $F = 1$ on GL_1, the F^{th} integral becomes

$$\int_{-\infty}^{\infty} |L(\tfrac{1}{2} + it, f)|^2 \, \mathrm{wt}(t)\, dt$$

where with $\Phi^\lambda(x) \sim (1+x^2)^{-\lambda/2}$, for example, $\mathrm{wt}(t)$ is

$$\int\int\int \Phi^\lambda(x)\, \psi\big((y-y')x\big)\, W_{F,\infty}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \overline{W}_{F,\infty}\begin{pmatrix} y' & \\ & 1 \end{pmatrix} \Big(\frac{y}{y'}\Big)^{it}$$

Qualitative computation in the simple case of $GL_1 \subset GL_2$: *asymptotics* extracted without (futile) demand for further detail. Let $y = e^u$ and $y' = e^v$. Note essential interchange by Fourier of

$$A^\lambda(x) \;=\; \frac{1}{(1-ix)^\lambda} + \frac{1}{(1+ix)^\lambda} \qquad\qquad B^\lambda(\xi) \;=\; |\xi|^{\lambda-1}\, e^{-|\xi|}$$

Integrate first in x:

$$\widehat{\Phi}^\lambda(e^u - e^v) \sim \widehat{A}^\lambda(e^u - e^v)$$

rough along $e^u = e^v$, rapidly decreasing in $|e^u - e^v|$. Multiplication by $W(e^u)\, W(e^v)$ adds decay, leaves diagonal roughness. Integral against $e^{it(u-v)}$ is Fourier, then restriction to anti-diagonal. Fourier returns something *asymptotically* A^λ on anti-diagonal, rapidly decreasing on diagonal:

$$A^\lambda(t_1 - t_2) \;\cdot\; (\text{rapidly decreasing})(t_1 + t_2)$$

Restriction to anti-diagonal $(t, -t)$ gives essentially

$$A^\lambda(t) \;\sim\; |t|^{-\lambda}$$

Spectral decomposition, (cancellation of) poles

Beyond L^2: singular terms... Recall:

Classic example: decomposition of $E_\alpha \cdot E_\beta$ on GL_2 with $\alpha = \frac{1}{2} + ia$ and $\mathrm{Re}\beta > 1$. With

$$E_\alpha = y^\alpha + c_\alpha y^{1-\alpha} + \ldots$$

guided by constant terms,

$$F = E_\alpha \cdot E_\beta - \Big(E_{\alpha+\beta} + c_\alpha \, E_{(1-\alpha)+\beta} \Big)$$

not only L^2 but integrable against $E_{\frac{1}{2}+it}$.

Integrate F against *truncated* $\wedge^T E_s$, unwind, let $T \to +\infty$

$$\int F \cdot \wedge^T E_s = \int_0^\infty c_P F \cdot \begin{cases} y^s & (\text{for } 0 < y < T) \\ -c_s \, y^{1-s} & (\text{for } T < y) \end{cases} \frac{dy}{y^2}$$

In constant term $c_P F$ insufficiently-decreasing terms cancel

$$c_P F = \sum_{\xi \neq 0} W_\xi^\alpha W_{-\xi}^\beta + y^\alpha \cdot c_\beta \, y^{(1-\beta)} + c_\alpha y^{1-\alpha} \, c_\beta \, y^{(1-\beta)}$$

$$- c_{\alpha+\beta} y^{1-(\alpha+\beta)} - c_\alpha \cdot c_{(1-\alpha)+\beta} \, y^{\alpha-\beta}$$

Limit of higher part is Rankin-Selberg $\Lambda(s, E^\alpha \otimes E^\beta)/\xi(2s)$, normalized to

$$\frac{\xi\big(s+\alpha+\beta-1\big) \cdot \xi\big(s-\alpha+\beta\big) \cdot \xi\big(s+\alpha-\beta\big) \cdot \xi\big(s+1-\alpha-\beta\big)}{\xi(2s) \cdot \xi(2\alpha) \cdot \xi(2\beta)}$$

Other part of integral as in Maass-Selberg

$$\int \Big(c_{\alpha+\beta} y^{1-\alpha-\beta} + c_\alpha \, c_{(1-\alpha)+\beta} y^{\alpha-\beta} \Big) \cdot \begin{cases} y^s & (0 < y < T) \\ c_s \, y^{1-s} & (0 < y < T) \end{cases} \frac{dy}{y^2}$$

$$= \; c_{\alpha+\beta} \frac{T^{s-\alpha-\beta}}{s-\alpha-\beta} - c_{\alpha+\beta} \, c_s \, \frac{T^{(1-s)-\alpha-\beta}}{(1-s)-\alpha-\beta}$$

$$+ c_\alpha c_{(1-\alpha)+\beta} \frac{T^{s-(1-\alpha)-\beta-1}}{s-(1-\alpha)-\beta-1}$$

$$- c_\alpha c_{(1-\alpha)+\beta} c_s \, \frac{T^{-s-(1-\alpha)-\beta}}{-s-(1-\alpha)-\beta}$$

All this goes to 0 as $T \to +\infty$ for $\mathrm{Re}\,\alpha$ fixed, $\mathrm{Re}\,s$ fixed, and $\mathrm{Re}\,\beta$ sufficiently large. Thus,

$$\lim_T \int \Big(E_\alpha \cdot E_\beta \; - \; \Big(E_{\alpha+\beta} + c_\alpha \, E_{(1-\alpha)+\beta} \Big) \Big) \cdot \wedge^T E_s \; = \; \frac{\Lambda(s, E_\alpha \otimes E_\beta)}{\xi(2s)}$$

Whole integral converges for $\alpha = \frac{1}{2} + ia$ and $s = \frac{1}{2} + it$ and $\mathrm{Re}\,\beta > 1$, so identity holds in that range, namely

$$\int \Big(E_{\frac{1}{2}+ia} \cdot E_\beta \; - \; \Big(E_{\frac{1}{2}+ia+\beta} + c_{\frac{1}{2}+ia} \, E_{\frac{1}{2}+ia+\beta} \Big) \Big) \cdot E_{\frac{1}{2}+it}$$

$$= \; \frac{\Lambda(\frac{1}{2} + it, E_{\frac{1}{2}+ia} \otimes E_\beta)}{\xi(1+2it)}$$

Cuspidal components are directly Rankin-Selberg integrals, so:

$\Longrightarrow$ **Spectral expansion** with two singular terms:

$$E_\alpha \cdot E_\beta \;=\; E_{\alpha+\beta} + c_\alpha E_{(1-\alpha)+\beta} \;+\; \frac{1}{2\pi}\int \frac{\Lambda(\frac{1}{2}+it, E_\alpha \otimes E_\beta)}{\xi(1+2it)} \cdot E_{\frac{1}{2}+it}$$

$$+\; \sum_F \frac{\Lambda(\frac{1}{2}+ia, \overline{F} \otimes E_\beta)}{\langle F, F\rangle} \cdot F \qquad (\text{with } \alpha = \tfrac{1}{2}+it \text{ and } \mathrm{Re}\beta \gg 1)$$

Contour dance: Unobviously, this expression fails for $\mathrm{Re}\beta < 1$. Must be more circumspect to move to $\beta = \frac{1}{2} + ib$ from $\mathrm{Re}\beta > 1$. *This example is useful because it admits alternative computation.* In

$$\int \frac{\xi\big(it+ia+\beta\big)\xi\big(it-ia+\beta\big)\xi\big(it+ia+1-\beta\big)\xi\big(it-ia+1-\beta\big)}{\xi(1+2it)\;\xi(1+2ia)\;\xi(2\beta)} E_{\frac{1}{2}+it}$$

(1a) Move β to $\mathrm{Re}\beta = 1+\varepsilon$. (1b) Move $s = \frac{1}{2}+it$ to $\frac{1}{2}+2\varepsilon+it$. For $a \neq 0$, each of the two factors

$$\xi\big(s-\tfrac{1}{2}+ia+1-\beta\big)\;\xi\big(s-\tfrac{1}{2}-ia+1-\beta\big)$$

catches a pole of ξ at 0, that is, at $s = \beta - \frac{1}{2} + ia = \beta - 1 + \alpha$ and $s = \beta - \frac{1}{2} - ia = \beta - 1 + (1-\alpha)$, with residues multiples of

$$E_{\beta-1+\alpha} \qquad \text{and} \qquad E_{\beta-1+(1-\alpha)}$$

(2a) Move β to $\mathrm{Re}\beta = 1-\varepsilon$. (2b) Move s back to $\frac{1}{2}+it$. Each of

$$\xi\big(s-\tfrac{1}{2}+ia+\beta\big)\;\xi\big(s-\tfrac{1}{2}-ia+\beta\big)$$

catches a pole of ξ at 1, that is, at $s = -\beta + \frac{1}{2} \pm ia$, with residues multiples of

$$E_{-\beta+\alpha} \qquad \text{and} \qquad E_{-\beta+1-\alpha}$$

(3) *Now* move β to $\beta = \frac{1}{2} + ib$.

$\Longrightarrow$ **Spectral expansion** with *four* singular terms: using functional equation of E_s

$$E_\alpha \cdot E_\beta \;=\; E_{\alpha+\beta} + c_\alpha E_{(1-\alpha)+\beta} + c_\beta E_{\alpha+1-\beta} + c_\alpha c_\beta E_{1-\alpha+1-\beta}$$

$$+\ \frac{1}{2\pi}\int \frac{\Lambda(\frac{1}{2}+it, E_\alpha \otimes E_\beta)}{\xi(1+2it)} \cdot E_{\frac{1}{2}+it}$$

$$+\ \sum_F \frac{\Lambda(\frac{1}{2}+ia, \overline{F} \otimes E_\beta)}{\langle F, F\rangle} \cdot F \qquad (\alpha = \tfrac{1}{2}+it \text{ and } \beta = \tfrac{1}{2}+ib)$$

Check that this procedure is *correct*, hence *necessary*, by repeating earlier computation but with $\alpha = \frac{1}{2}+ia$ and $\beta = \frac{1}{2}+ib$.

Similar, more complicated continuation necessary with Poincaré series $\mathfrak{P}^{\alpha,\lambda}$.

This is how pole of $E_{1+\alpha}$ at $\alpha = 0$ in spectral expansion of $\mathfrak{P}^{\alpha,\lambda}$ is *cancelled* by *continuous part.*

Relevant fragment E_1^* of Eisenstein series $E_{1+\alpha}$ in $\mathfrak{P}^{0,\lambda}$ is *zero-order term* in Laurent expansion near $\alpha = 0$. *Not* eigenfunction for Casimir Ω, but satisfies

$$\Omega\, E_1^* \;=\; (\text{non-zero constant}) \qquad \text{and} \qquad \Omega^2\, E_1^* \;=\; 0$$

$\Longrightarrow$ leading *constant* in main term of asymptotic is essentially

$$\int E_1^* \cdot |f|^2$$

Naive approach to GL_3

Imagining that the *actual* spectral relation for $GL_3 \times GL_2$ produces an asymptotic-with-error for *classical* moments... and proceeding naively thereafter... **what should we get?**

Bonus: introduces **conductor dropping**.

For f on GL_3, F on GL_2, $t \in \mathbb{R}$, analytic conductor $Q_{t,F,f}$ is of degree 6 in t, with archimedean data μ_F of F, archimedean data ν_1, ν_2, ν_3 of f with $\sum_j \nu_j = 0$

$$Q_{t,F,f} \;=\; \prod_{j=1,2,3} \Big(1 \;+\; \big|t + \mu_F + \nu_j\big|\Big)\Big(1 \;+\; \big|t - \mu_F + \nu_j\big|\Big)$$

$$\sim\; \Big(\big(1 \;+\; \big|t + \mu_F\big|\big) \cdot \big(1 \;+\; \big|t - \mu_F\big|\big)\Big)^3 \qquad\qquad \text{(for } f \text{ fixed)}$$

$$|L(\tfrac{1}{2} + it, f \otimes F)| \;\ll\; \begin{cases} Q_{t,F,f}^{1/4+\varepsilon} & \text{(convexity)} \\ \\ Q_{t,F,f}^{\varepsilon} & \text{(Lindelöf)} \end{cases}$$

Two extremes: *conductor dropping* where one of $|t \pm \mu_F|$ small. *Away* from conductor dropping is where t, μ_F, and both $|t \pm \mu_F|$ all large.

Away from... : $Q \le T$ implies $|t| \ll T^{1/6}$, $|\mu_F| \ll T^{1/6}$, and with Lindelöf and Weyl

$$\Big(\text{sum of such } L\text{'s with } Q \le T\Big) \;\ll\; T^{1/6} \cdot (T^{1/6})^2 \cdot T^{\varepsilon} \;=\; T^{\frac{1}{2}+\varepsilon}$$

Near... : Take $|t - \mu_F|$ small. $Q \le T$ implies $|\mu_F| \ll T^{1/3}$ with t nearby. With Lindelöf and Weyl

$$\Big(\text{sum of such } L\text{'s with } Q \le T\Big) \;\ll\; 1 \cdot (T^{1/3})^2 \cdot T^{\varepsilon} \;=\; T^{\frac{2}{3}+\varepsilon}$$

Under *naive optimism*, might be reasonable to prove

$$\not\!\!\!\sum\!\!\!\!\!\!\int_{t,F\,:\,Q\le T} \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2 \;=\; T^{\frac{2}{3}}\, P(\log T) + O(T^{\frac{2}{3}-\mathrm{small}})$$

Then, under various hypotheses (fixed F or fixed t),

$$\left|L(\tfrac{1}{2}+it, f\otimes F)\right| \;\ll\; Q^{\frac{1}{3}-\mathrm{small}} \qquad\qquad (\text{with } Q = Q_{t,F,f})$$

But this is worse than

$$|L(\tfrac{1}{2}+it, f\otimes F)| \;\ll\; Q^{\frac{1}{4}+\varepsilon} \qquad\qquad (\text{convexity})$$

Aggressive optimism might suggest an error term comparable to away-from-conductor-dropping:

$$\not\!\!\!\sum\!\!\!\!\!\!\int_{t,F:Q\le T} \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2 = T^{\frac{2}{3}}\, P(\log T) + \ldots + O(T^{\frac{1}{2}-\mathrm{small}})$$

If so, then, under various hypotheses, subconvex bound

$$\left|L(\tfrac{1}{2}+it, f\otimes F)\right| \;\ll\; Q^{\frac{1}{4}-\mathrm{small}} \qquad\qquad (\text{with } Q = Q_{t,F,f})$$

Too much to hope for?

Typical: conductor-dropping gives dominant portion of moment, *masking* natural sub-families.

For *either* fixed F or fixed t, (f fixed throughout) conductor-dropping cannot occur.

Conductor-dropping occurs in *full(er)* spectral family, sabotaging naive treatment of simpler sub-families.

Overcome hurdle of conductor-dropping?

Integral moments can *discount* conductor-dropping regime, and/or *enhance* away-from-dropping regime.

Futile example: By same heuristic, naive modification of classical moments should satisfy

$$\sum\!\!\!\!\!\!\!\int_{t,F\,:\,Q\leq T} Q^a \cdot \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2 \;=\; T^{a+\frac{2}{3}}\, P(\log T) + O(T^{a+\frac{2}{3}-\text{small}})$$

On subfamilies yields *same* non-subconvex bound

$$\left|L(\tfrac{1}{2}+it, f\otimes F)\right| \;\ll\; Q^{\frac{1}{3}-\text{small}}$$

Same non-subconvex bound follows from asymptotics-with-error for integral moments

$$\sum\!\!\!\!\!\!\!\int_{t,F} Q^{-w} \cdot \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2$$

Better example: optimistically, by same heuristic, might prove

$$\sum\!\!\!\!\!\!\!\int_{t,F\,:\,Q\leq T} \frac{1}{1+|t|} \cdot \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2 \;=\; T^{\frac{1}{3}}\, P(\log T) + O(T^{\frac{1}{3}-\text{small}})$$

and on subfamilies

$$\frac{1}{1+|t|} \cdot \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2 \;\ll\; Q^{\frac{1}{3}-\text{small}}$$

For fixed F, have $Q \sim t^{1/6}$, and obtain *subconvex*

$$\left|L(\tfrac{1}{2}+it, f\otimes F)\right| \;\ll\; |t|^{\frac{1}{2}} \cdot Q^{\frac{1}{6}-\text{small}} \;=\; Q^{\frac{1}{12}+\frac{1}{6}-\text{small}} \;=\; Q^{\frac{1}{4}-\text{small}}$$

Can our Poincaré series be made to do anything like this?

The weight function $\eta(t, F)$ in integral moment

$$\sum\!\!\!\!\!\!\int_{t,F\,:\,Q\le T} \eta(t,F)\cdot \left|L(\tfrac{1}{2}+it, f\otimes F)\right|^2$$

must be allowed to be function not only of

$$Q \;\sim\; \Big((1+|t-\mu_F|)\cdot(1+|t+\mu_F|)\Big)^3$$

but more general *symmetric* function of $|t+\mu_F|$ and $|t-\mu_F|$.

Plausible example: by same heuristic, might prove

$$\sum\!\!\!\!\!\!\int_{t,F\,:\,Q\le T} \frac{\left|L(\frac{1}{2}+it, f\otimes F)\right|^2}{1+|t-\mu_F|+|t+\mu_F|} \;=\; T^{\frac{1}{3}}\,P(\log T)+O(T^{\frac{1}{3}-\mathrm{small}})$$

and on subfamilies

$$\frac{\left|L(\frac{1}{2}+it, f\otimes F)\right|^2}{1+|t-\mu_F|+|t+\mu_F|} \;\ll\; Q^{\frac{1}{3}-\mathrm{small}}$$

For fixed F, have $Q\sim t^{1/6}$, $|t\pm\mu_F|\sim t$, and obtain *subconvex*

$$\left|L(\tfrac{1}{2}+it, f\otimes F)\right| \;\ll\; Q^{\frac{1}{4}-\mathrm{small}} \qquad\qquad (t\text{-aspect})$$

More possibilities

In fact, there is no mandate to think only in terms of averages with bounds described in terms of the analytic conductor Q.

True, the trivial/convexity bound naturally arises in that form, and there is interest in surpassing that bound on its own terms.

Nevertheless, we are not constrained to describe all phenomena in those terms.

For example, *in place of* $GL_3 \times GL_2$ moments

$$\sum\!\!\!\!\!\!\!\int_{t,F\,:\,Q \le T} \left|L(\tfrac{1}{2}+it, f \otimes F)\right|^2$$

which include *conductor-dropping* complications, we could consider moments of a different shape,

$$\sum\!\!\!\!\!\!\!\int_{|t| \le T,\ F : |\mu_F| \le T} \left|L(\tfrac{1}{2}+it, f \otimes F)\right|^2$$

An asymptotic with power-saving in an error term would *break convexity.*

This method would break convexity for $GL_n \times GL_{n-1}$.

All such nearly-classical moments are potential targets for *integral* moments produced by spectral identities.

At this point, due to the difficulty of understanding the asymptotic phenomena produced in spectral identities for moments, it is not clear which of these targets can be hit.