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Standard compact periods for Eisenstein series

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For $\Gamma = SL_2(\mathbb{Z})$, the spherical Eisenstein series

$$E_s(z) \;=\; \sum_{c,d} \frac{y^s}{|cz+d|^{2s}} \qquad\qquad (c, d \text{ relatively prime}, \; z \in \mathfrak{H})$$

takes meaningful values at CM-points, points $z \in \mathfrak{H}$ such that $k = \mathbb{Q}(z)$ is quadratic over $\mathbb{Q}$. Especially, when $\mathfrak{o} = \mathbb{Z}[z]$ is the ring of algebraic integers in the quadratic field, *and* when $\mathfrak{o}$ has *class number one,*

$$E_s(z) \;=\; \frac{\zeta_k(s)}{\zeta_{\mathbb{Q}}(2s)}$$

When $\mathfrak{o}$ has larger class number, linear combinations of CM-point values corresponding to the ideal classes yield the corresponding ratio.

This example was understood in the 19th century, and is the simplest in well-known families of special values and *periods* of Eisenstein series.

The next case in order of increasing complexity is that of integrals of the same E_s along hyperbolic geodesics in $\mathfrak{H}$ that have compact images in $\Gamma\backslash\mathfrak{H}$. These were considered by Hecke and Maass. In fact, from a contemporary viewpoint the CM-point value example and the hyperbolic geodesic periods example are identical, as is made clear in the first section,

The first three examples are instances of periods of Eisenstein series on orthogonal groups $O(n+1)$ with rational rank 1, along orthogonal groups $O(n)$ with compact arithmetic quotients.

Another family extending the small examples is periods of degenerate Eisenstein series along anisotropic tori.

Periods of Eisenstein series are interesting prototypes for periods of cuspforms, often unwinding more completely than the corresponding integrals for cuspforms. In that context, periods attached to degenerate Eisenstein series are inevitably misleading to some degree. Nevertheless, there is a correct indication that sharp estimates on periods are often connected to Lindelöf hypotheses and other very serious issues.

See the bibliography for background and some pointers to contemporary literature.

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1. $\ell^\times \subset GL_2(k)$: *CM-point values, hyperbolic geodesic periods*

This example is essentially due to Hecke and Maass.

Let ℓ be a quadratic field extension of a global field k of characteristic not 2. Let $G = GL_2(k)$, and $H \subset G$ a copy of $\ell^\times$ inside G, by specifying the isomorphism in

$$\ell^\times \;\subset\; \mathrm{Aut}_k(\ell) \;\approx\; \mathrm{Aut}_k(k^2) \;=\; GL_2(k)$$

Let P be the standard parabolic of upper-triangular elements in G, and

$$E_s(g) \;=\; \sum_{\gamma \in P_k\backslash G_k} \varphi(\gamma \cdot g)$$

where φ is the everywhere-locallly-spherical vector

$$\varphi(\begin{pmatrix} a & * \\ & d \end{pmatrix} \cdot k) \;=\; \left|\frac{a}{d}\right|^s \qquad\qquad \text{(for } k \text{ in maximal compact in } G_{\mathbb{A}})$$

With Z the center of G, we want to evaluate

$$\text{period of } E_s \text{ along } H \;=\; \int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s$$

The subgroup P_k is the isotropy group of a k-line $k \cdot e$ for a fixed non-zero $e \in k^2 \approx \ell$. The group G_k is transitive on these k-lines, so $P_k\backslash G_k \approx \{k - \text{lines}\}$ The critical-but-trivial point is that the action of $\ell^\times$ on ℓ is transitive on non-zero elements. Thus, $P_k \cdot \ell^\times = GL_2(k)$. That is, the period integral unwinds

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \int_{Z_{\mathbb{A}} (P_k \cap H_k) \backslash H_{\mathbb{A}}} \varphi \;=\; \int_{Z_{\mathbb{A}} \backslash H_{\mathbb{A}}} \varphi$$

since $H \cap P = Z$. Since $\varphi = \bigotimes_v \varphi_v$ factors over primes, the unwound period integral factors over primes

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \int_{Z_{\mathbb{A}} H_{\mathbb{A}}} \varphi \;=\; \prod_v \int_{Z_v \backslash H_v} \varphi_v$$

The cleanest way to evaluate the local integrals is to use an integral presentation of the local spherical vector φ_v akin to better-known archimedean devices involving the Gamma function. That is, present φ_v in terms of Iwasawa-Tate local zeta integrals

$$\varphi_v(g) \;=\; \frac{1}{\zeta_{k,v}(2s)} \cdot |\det g|_v^s \cdot \int_{k_v^\times} |t|_v^{2s} \cdot \Phi_v(t \cdot e \cdot g)\, dt \qquad\qquad \text{(with } \zeta_{k,v}(2s) = \textstyle\int_{k_v^\times} |t|_v^{2s}\, \Phi_v(t\, e)\, dt)$$

for Φ_v a K_v-invariant Schwartz function on $k_v^2 \approx \ell_v$. The leading local zeta factor gives the normalization $\varphi_v(1) = 1$ at $g = 1$. Then

$$\int_{Z_v \backslash H_v} \varphi_v \;=\; \frac{1}{\zeta_{k,v}(2s)} \cdot \int_{k_v^\times \backslash \ell_v^\times} |\det h|_v^s \cdot \int_{k_v^\times} |t|_v^{2s} \cdot \Phi_v(t \cdot e \cdot h)\, dt\, dh$$

Since

$$|\det h|_{k_v} \;=\; |N_{\ell_v/k_v} h|_{k_v} \;=\; |h|_{\ell_v}$$

the local factor of the period becomes

$$\frac{1}{\zeta_{k,v}(2s)} \cdot \int_{k_v^\times \backslash \ell_v^\times} |h|_{\ell_v}^s \cdot \int_{k_v^\times} |t|_{\ell_v}^s \cdot \Phi_v(t \cdot e \cdot h)\, dt\, dh \;=\; \frac{1}{\zeta_{k,v}(2s)} \cdot \int_{k_v^\times \backslash \ell_v^\times} \int_{k_v^\times} |t \cdot h|_{\ell_v}^s \cdot \Phi_v(t \cdot e \cdot h)\, dt\, dh$$

$$=\; \frac{1}{\zeta_{k,v}(2s)} \cdot \int_{\ell_v^\times} |h|_{\ell_v}^s \cdot \Phi_v(e \cdot h)\, dt\, dh \;=\; \frac{1}{\zeta_{k,v}(2s)} \cdot \zeta_{\ell,v}(s)$$

Thus, the product of all the local factors of the period is

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \frac{\xi_\ell(s)}{\xi_k(2s)} \qquad\qquad \text{(with } E_s \text{ on } GL_2(k))$$

with ξ including the gamma factors.

[1.0.1] Remark: In this normalization, the unitary line is $\mathrm{Re}(s) = \frac{1}{2}$, and

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_{\frac{1}{2}+it} \;=\; \frac{\xi_\ell(\frac{1}{2}+it)}{\xi_k(1+2it)}$$

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2. $B^\times \subset GL_2(\ell)$

This is a Galois-twisted version of the diagonal imbedding $GL_2 \subset GL_2 \times GL_2$ from the classical Rankin-Selberg situation.

Let B be a quaternion division algebra over a number field k. By Fujisaki's lemma, $k_\mathbb{A}^\times B^\times \backslash B_\mathbb{A}^\times$ is *compact*. Let ℓ be a quadratic extension of k *splitting* B. Define a k-group by $G = GL_2(\ell)$, and let H be the image of $B^\times$ inside G by choice of an isomorphism

$$B^\times \;\subset\; \mathrm{Aut}_\ell(B) \;\approx\; \mathrm{Aut}_\ell(\ell^2) \;=\; GL_2(\ell)$$

Let P be the standard parabolic in G, and

$$E_s(g) \;=\; \sum_{\gamma \in P_\ell \backslash G_\ell} \varphi(\gamma \cdot g)$$

where φ is the everywhere-locally-spherical vector

$$\varphi(\begin{pmatrix} a & * \\ & d \end{pmatrix} \cdot k) \;=\; \left|\frac{a}{d}\right|_\ell^s \qquad\qquad \text{(for } k \text{ in maximal compact in } G_\mathbb{A}\text{)}$$

The norm is the product-formula normalization of the norm for ℓ. With Z the center of G, we want to evaluate

$$\text{period of } E_s \text{ along } H \;=\; \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_s$$

The subgroup P_k is the isotropy group of an ℓ-line $\ell \cdot e$ for a fixed non-zero $e \in \ell^2 \approx B$. The group G_k is transitive on these ℓ-lines, so $P_k \backslash G_k \approx \{\ell - \text{lines in } B\}$ The action of $B^\times$ on B is transitive on non-zero elements. Thus, $P_k \cdot B^\times = GL_2(\ell)$. That is, the period integral unwinds *partly*

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_s \;=\; \int_{Z_\mathbb{A}(P_k \cap H_k) \backslash H_\mathbb{A}} \varphi \;=\; \int_{k_\mathbb{A}^\times \ell^\times \backslash B_\mathbb{A}^\times} \varphi$$

since $B \cap P \approx \ell^\times$. Since ℓ is not central in B, this copy of ℓ is *not* the central copy in $GL_2(\ell)$.

We do have the factorization $\varphi = \bigotimes_v \varphi_v$ over primes, but the integral itself needs further unwinding.

Further unwinding of the integral, as well as evaluation of the local integrals, is facilitated by an integral presentation of the spherical vectors φ_v, in terms of Iwasawa-Tate/Tamagawa/Godement-Jacquet local zeta integrals

$$\varphi_v(g) \;=\; \frac{1}{\zeta_{\ell,v}(2s)} \cdot |\det g|_{\ell,v}^s \cdot \int_{\ell_v^\times} |t|_{\ell,v}^{2s} \cdot \Phi_v(t \cdot e \cdot g)\, dt \qquad\qquad \text{(with } \zeta_{\ell,v}(2s) = \textstyle\int_{\ell_v^\times} |t|_{\ell,v}^{2s}\, \Phi_v(t\, e)\, dt\text{)}$$

for Φ_v a K_v-invariant Schwartz function on $\ell_v^2 \approx B_v$. The leading local zeta factor gives the normalization $\varphi_v(1) = 1$ at $g = 1$. With $\Phi = \bigotimes \Phi_v$, the period integral is

$$\int_{k_\mathbb{A}^\times \ell^\times \backslash B_\mathbb{A}^\times} \varphi \;=\; \frac{1}{\xi_\ell(2s)} \int_{k_\mathbb{A}^\times \ell^\times \backslash B_\mathbb{A}^\times} |\det h|_\ell^s \cdot \int_{\ell_\mathbb{A}^\times} |t|_\ell^{2s} \cdot \Phi(t \cdot e \cdot h)\, dt\, dh$$

$$= \frac{1}{\xi_\ell(2s)} \int_{k_\mathbb{A}^\times \ell_\mathbb{A}^\times \backslash B_\mathbb{A}^\times} \Big(\int_{k_\mathbb{A} \ell^\times \backslash \ell_\mathbb{A}^\times} 1 \Big) |\det h|_\ell^s \cdot \int_{\ell_\mathbb{A}^\times} |t|_\ell^{2s} \cdot \Phi(t \cdot e \cdot h)\, dt\, dh$$

because the inner integral already produces a left $\ell_\mathbb{A}^\times$-invariant function of $h \in B_\mathbb{A}^\times$. What's left of the integral does now factor into local contributions:

$$\frac{\mathrm{vol}\,(k_\mathbb{A}^\times \ell^\times \backslash \ell_\mathbb{A}^\times)}{\xi_\ell(2s)} \int_{Z_\mathbb{A} \ell_\mathbb{A}^\times \backslash B_\mathbb{A}^\times} |\det h|_\ell^s \cdot \int_{\ell_\mathbb{A}^\times} |t|_\ell^{2s} \cdot \Phi(t \cdot e \cdot h)\, dt\, dh \;=\; \frac{\mathrm{vol}\,(k_\mathbb{A}^\times \ell^\times \backslash \ell_\mathbb{A}^\times)}{\xi_\ell(2s)} \int_{B_\mathbb{A}^\times} |\det h|_\ell^s \cdot \Phi(t \cdot e \cdot h)\, dh$$

Since

$$|\det h|_\ell \;=\; |N^{\mathrm{red}} h|_\ell \;=\; |N^{\mathrm{red}} h|_k^2 \qquad \text{(for } h \in B\text{, central simple over } k\text{)}$$

the period integral is

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_s \;=\; \frac{\mathrm{vol}\,(k_\mathbb{A}^\times \ell^\times \backslash \ell_\mathbb{A}^\times)}{\xi_\ell(2s)} \cdot \xi_B(2s) \qquad \text{(with } E_s \text{ on } GL_2(\ell)\text{)}$$

The zeta function of the quaternion algebra B over k is expressible in terms of that of k, at argument $2s$ as in the above,

$$\xi_B(2s) \;=\; \frac{\xi_k(2s) \cdot \xi_k(2s-1)}{\prod_{v \text{ rfd}} \zeta_{k,v}(2s-1)} \qquad \text{(product over places } v \text{ ramified for } B\text{)}$$

Also, the volume is

$$\mathrm{vol}\,(k_\mathbb{A}^\times \ell^\times \backslash \ell_\mathbb{A}^\times) \;=\; \frac{\mathrm{Res}_{s=1}\xi_\ell(s)}{\mathrm{Res}_{s=1}\xi_k(s)} \;=\; \Lambda_k(1, \chi_{\ell/k}) \qquad \text{(with } \chi_{\ell/k} \text{ the quadratic character of } \ell/k\text{)}$$

Thus, the period is

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_s \;=\; \frac{\Lambda_k(1, \chi_{\ell/k})\, \xi_k(2s)\, \xi_k(2s-1)}{\xi_\ell(2s) \prod_{v \text{ rfd}} \zeta_{k,v}(2s-1)}$$

$$= \frac{\Lambda_k(1, \chi_{\ell/k})\, \xi_k(2s)\, \xi_k(2s-1)}{\xi_k(2s)\, \Lambda_k(2s, \chi_{\ell/k}) \prod_{v \text{ rfd}} \zeta_{k,v}(2s-1)} \;=\; \frac{\Lambda_k(1, \chi_{\ell/k})\, \xi_k(2s-1)}{\Lambda_k(2s, \chi_{\ell/k}) \cdot \prod_{v \text{ rfd}} \zeta_{k,v}(2s-1)} \qquad (E_s \text{ on } GL_2(\ell))$$

[2.0.1] **Remark:** In this normalization, the unitary line is $\mathrm{Re}(s) = \frac{1}{2}$, and

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_{\frac{1}{2}+it} \;=\; \frac{\Lambda_k(1, \chi_{\ell/k})\, \xi_k(2it)}{\Lambda_k(1+2it, \chi_{\ell/k}) \cdot \prod_{v \text{ rfd}} \zeta_{k,v}(2it)} \qquad (E_{\frac{1}{2}+it} \text{ on } GL_2(\ell))$$

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3. $(B \otimes_k \ell)^\natural \subset GSp^*(1,1)$

Let B be a quaternion division algebra over a number field k, *not* split by a quadratic extension ℓ of k. Let $\ell = k(\omega)$, where $\mathrm{tr}_{\ell/k}(\omega) = 0$. Let θ be the *main involution* of B, and extend it ℓ-linearly to $C = B \otimes_k \ell$. Define a B-valued quaternion-hermitian form on the two-dimensional B-vectorspace C by

$$\langle x, y \rangle \;=\; (1 \otimes \mathrm{tr}_{\ell/k})\big(\omega \cdot y^\theta \cdot x\big) \;\in\; B \qquad \text{(for } x, y \in C\text{)}$$

Choice of a right B-basis for C gives an isomorphism

$$GL_2(B) \;\approx\; \mathrm{Aut}_B(C)$$

The similitude group $G = GSp^*(1,1)$ of the form $\langle , \rangle$ is

$$G \;=\; \{g \in GL_2(B) \;:\; \langle gx, gy\rangle = \nu(g) \text{ for } \nu(g) \in k^\times\} \;\sim\; GSp^*(1,1)$$

The subgroup

$$H \;=\; \{h \in C^\times \;:\; h^\theta h \in k^\times\} \qquad\qquad \text{(as opposed to lying in } \ell^\times)$$

of $C^\times$ imbeds in G, since for $h \in H$

$$\langle hx, hy\rangle \;=\; (1 \otimes \mathrm{tr}_{\ell/k})\big(\omega \cdot (hy)^\theta \cdot hx\big) \;=\; (1 \otimes \mathrm{tr}_{\ell/k})\big(\omega \cdot y^\theta \cdot h^\theta h \cdot x\big) \;=\; (h^\theta h)(1 \otimes \mathrm{tr}_{\ell/k})\big(\omega \cdot y^\theta \cdot h^\theta h \cdot x\big)$$

since the trace is k-linear and $h^\theta h$ is in k, not merely in ℓ.

Let P be a proper k-parabolic in G, characterized as the stabilizer of a fixed isotropic B-line $B \cdot e$in C. The simplest choice is $e = 1$, so $B = B \cdot 1$. Define an Eisenstein series

$$E_s(g) \;=\; \sum_{\gamma \in P_\ell \backslash G_\ell} \varphi(\gamma \cdot g)$$

where φ is an everywhere-spherical vector presented by an integral,

$$\varphi(g) \;=\; \frac{1}{\xi_B(2s)} |\det g|^s \int_{B^\times_{\mathbb{A}}} |\det t|^{2s} \cdot \Phi(t \cdot e \cdot g)\, dt$$

where, if necessary, det denotes a suitable reduced norm. The leading factor arranges that $\varphi(1) = 1$. With Z the center of G, we want to evaluate

$$\text{period of } E_s \text{ along } H \;=\; \int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s$$

To begin unwinding the period integral, H_k must be transitive on isotropic B-lines. Indeed, for $0 \neq x \in C$ such that $\langle x, x\rangle = 0$, then $x^\theta x \in k^\times$, so $x \in H$. Thus, $x \cdot x^{-1} = 1$, proving the transitivity. Thus,

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \int_{Z_{\mathbb{A}} (P_k \cap H_k) \backslash H_{\mathbb{A}}} \varphi \;=\; \int_{Z_{\mathbb{A}} B^\times \backslash C^\natural_{\mathbb{A}}} \varphi \qquad\qquad (\text{with } C^\natural = \{h \in C : h^\theta h \in k^\times\})$$

The integral presentation of φ gives

$$\frac{1}{\xi_B(2s)} \int_{Z_{\mathbb{A}} B^\times \backslash C^\natural_{\mathbb{A}}} |\det h|^s \int_{B^\times_{\mathbb{A}}} |\det t|^{2s} \cdot \Phi(t \cdot e \cdot h)\, dt\, dh$$

$$= \frac{1}{\xi_B(2s)} \int_{Z_{\mathbb{A}} B^\times_{\mathbb{A}} \backslash C^\natural_{\mathbb{A}}} \Big(\int_{Z_{\mathbb{A}} B^\times \backslash B^\times_{\mathbb{A}}} 1\Big) |\det h|^s \int_{B^\times_{\mathbb{A}}} |\det t|^{2s} \cdot \Phi(t \cdot e \cdot h)\, dt\, dh$$

$$= \frac{\mathrm{vol}\,(Z_{\mathbb{A}} B^\times \backslash B^\times_{\mathbb{A}})}{\xi_B(2s)} \int_{C^\natural_{\mathbb{A}}} |\det h|^s\, \Phi(e \cdot h)\, dh \;=\; = \frac{\mathrm{vol}\,(Z_{\mathbb{A}} B^\times \backslash B^\times_{\mathbb{A}}) \cdot \xi_{C^\natural}(s)}{\xi_B(2s)}$$

The zeta function of $C^\natural$ is only slightly more complicated than the zeta function of a quaternion algebra, and, up to a finite product from ramified primes,

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \mathrm{vol}\,(Z_{\mathbb{A}} B^\times \backslash B^\times_{\mathbb{A}}) \cdot \big(\text{finite}\big) \cdot \frac{\xi_k(2s)\, \xi_k(2s-2)\, \xi_\ell(2s-1)}{\xi_k(2s) \xi_k(2s-1)}$$

Cancelling where possible,

$$\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} E_s \;=\; \mathrm{vol}\,(Z_{\mathbb{A}} B^\times \backslash B^\times_{\mathbb{A}}) \cdot \big(\text{finite}\big) \cdot \xi_k(2s-2) \cdot \Lambda_k(2s-1, \chi_{\ell/k})$$

[3.0.1] **Remark:** In this normalization, the unitary line is $\mathrm{Re}(s) = \frac{3}{2}$, and the period is

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_{\frac{3}{2}+it} \;=\; \big(\mathrm{const}\big) \cdot \big(\mathrm{finite}\big) \cdot \xi_k(1+2it) \cdot \Lambda_k(2+2it, \chi_{\ell/k})$$

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4. *Degenerate Eisenstein series along anisotropic tori*

Let $[\ell : k] = n$ and let H be a copy of $\ell^\times$ in $G = GL_n(k)$. Let P be the standard $(n-1, 1)$ parabolic in G, namely, fixing the line spanned by the n^{th} standard basis vector e_n under right matrix multiplication. Let χ_s be the character on $P_\mathbb{A}$ defined by

$$\chi_s \begin{pmatrix} A & * \\ 0 & d \end{pmatrix} \;=\; \Big|\frac{\det A}{d^{n-1}}\Big|^s \qquad\qquad (\text{where } A \in GL_{n-1} \text{ and } d \in GL_1)$$

Define φ_s by

$$\varphi_s(p \cdot k) \;=\; \chi_s(p) \qquad\qquad (\text{with } k \text{ in maximal compact, } p \in P_\mathbb{A})$$

and define an Eisenstein series by

$$E_s(g) \;=\; \sum_{\gamma \in P_k \backslash G_k} \varphi_s(\gamma \cdot g)$$

We will compute the period of E_s along the copy of $\ell^\times$ in G. With Z the center of G, this is

$$\int_{Z_\mathbb{A} \ell^\times \backslash \ell^\times_\mathbb{A}} E_s \;=\; \int_{Z_\mathbb{A} \backslash \ell^\times_\mathbb{A}} \varphi_s$$

by unwinding, since

$$P_k \cdot \ell^\times \;=\; G_k$$

The function φ_s factors over primes, as does the unwound integral, giving v^{th} local factor

$$\int_{k_v^\times \backslash \ell_v^\times} \varphi_{s,v}(h)\, dh$$

To evaluate this, again use an integral representation of $\varphi_{s,v}$ as

$$\varphi_{s,v}(g) \;=\; \frac{1}{\zeta_{k,v}(ns)} |\det g|_v^s \int_{k_v^\times} |t|^{ns}\, \Phi(t \cdot e_n \cdot g)\, dt$$

where Φ is a suitable Schwartz function. Substituting this in the local integral, the local integral unwinds to

$$\int_{Z_\mathbb{A} \backslash \ell^\times_\mathbb{A}} \varphi_s \;=\; \frac{1}{\zeta_{k,v}(ns)} \int_{\ell^\times_\mathbb{A}} |\det h|_v^s\, \Phi(e_n \cdot g)\, dh \;=\; \frac{\zeta_\ell(s)}{\zeta_{k,v}(ns)}$$

Thus, apart from modifications at finitely-many places, the period of this degenerate Eisenstein series over a maximal anisotropic torus attached to the field extension ℓ of k is

$$\int_{Z_\mathbb{A} \ell^\times \backslash \ell^\times_\mathbb{A}} E_s \;=\; \frac{\xi_\ell(s)}{\xi_k(ns)}$$

5. *Appendix: zeta functions of quaternion algebras*

Let B be a quaternion algebra over a number field k. For a Schwartz function $\Phi = \bigotimes_v \Phi_v$ on $B_{\mathbb{A}}$, at almost all places v, we claim that

$$\int_{B_v^\times} |\det t|^s \; \Phi_v(t)\, dt \;=\; \zeta_{k_v}(s) \cdot \zeta_{k_v}(s-1)$$

Indeed, we can suppose that B_v is split, and use an Iwasawa decomposition NMK with

$$n \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad m \;=\; \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

In these coordinates, the Haar measure is

$$d(mnk) \;=\; dm\, dn\, dk$$

Then

$$\int_{B_v^\times} |\det t|^s \; \Phi_v(t)\, dt \;=\; \int_{M_v \cdot N_v} |\det t|^s \; \Phi_v(t)\, dt \;=\; \int_{M_v \cdot N_v} |ad|^s \; \Phi_v \begin{pmatrix} a & ax \\ 0 & d \end{pmatrix} dm\, dn$$

Almost everywhere, Φ_v is the characteristic function of the local integers. Replace x by x/a and integrate in x, leaving

$$\int_{M_v} |a|^{s-1}\, |d|^s \; \Phi_v \begin{pmatrix} a & a \\ 0 & d \end{pmatrix} dm$$

This is a product of two Iwasawa-Tate integrals, namely

$$\zeta_{k_v}(s-1)\, \zeta_{k_v}(s)$$

For ℓ a quadratic extension of k *not* splitting B, let $C = B \otimes_k \ell$ and

$$C^\natural \;=\; \{h \in C \;:\; h^\theta h \in k^\times\}$$

At a place v of k *splitting* in ℓ, the v^{th} local zeta integral for $\xi_{C^\natural}(s)$ becomes

$$\int \Phi_v \begin{pmatrix} a & ax \\ 0 & t/a \end{pmatrix} \Phi_v \begin{pmatrix} a' & a'x' \\ 0 & t/a' \end{pmatrix} \; |t|^{2s}$$

where Φ_v is the characteristic function of the integral-coordinate matrices. Replace x by x/a and x' by x'/a' and integrate x, x' away to obtain

$$\int \Phi_v \begin{pmatrix} a & a \\ 0 & t/a \end{pmatrix} \Phi_v \begin{pmatrix} a' & a' \\ 0 & t/a' \end{pmatrix} \; \frac{1}{|a|\,|a'|} \; |t|^{2s}$$

Thus, a and a' are integral, and t/a and t/a' are integral. Taking an outer integral over t, this becomes

$$\sum_{j \geq 0} (q^{-j})^{2s} \, \frac{1-q^{j+1}}{1-q} \, \frac{1-q^{j+1}}{1-q} \;=\; \frac{1}{(1-q)^2} \sum_{j \geq 0} \Big[(q^{-2s})^j - 2q \cdot (q^{1-2s})^j + q^2 \cdot (q^{2-2s})^j \Big]$$

$$= \; \frac{1}{(1-q)^2} \Big[\frac{1}{1-q^{-2s}} - \frac{2q}{1-q^{1-2s}} + \frac{q^2}{1-q^{2-2s}} \Big] \;=\; \frac{1+q^{1-2s}}{(1-q^{-2s})\,(1-q^{1-2s})\,(1-q^{2-2s})}$$

$$= \; \frac{1-q^{2-4s}}{(1-q^{-2s})\,(1-q^{1-2s})^2\,(1-q^{2-2s})} \qquad\qquad \text{(at split place)}$$

At a place v of k *inert* in ℓ, the v^{th} local zeta integral for $\xi_{C^\natural}(s)$ becomes

$$\int \Phi_v \begin{pmatrix} a & ax \\ 0 & t/a \end{pmatrix} \; |t|_\ell^s \;=\; \int \Phi_v \begin{pmatrix} a & ax \\ 0 & t/a \end{pmatrix} \; |t|^{2s}$$

since t is integrated just in $k_v^\times$. Replace x by x/a and integrate in x to obtain

$$\int \Phi_v \begin{pmatrix} a & a \\ 0 & t/a \end{pmatrix} \; \frac{1}{|a|_\ell} \; |t|^{2s}$$

This becomes

$$\sum_{j\geq 0} (q^{-j})^{2s} \, \frac{1-(q^2)^{j+1}}{1-q^2} \;=\; \frac{1}{1-q^2}\Big[\frac{1}{1-q^{-2s}} - \frac{q^2}{1-q^{2-2s}}\Big] \;=\; \frac{1}{(1-q^{-2s})\,(1-q^{2-2s})} \qquad \text{(at inert place)}$$

To compare the inert place computation with that for the split place, insert corresponding factors, giving

$$\frac{1-q^{2-4s}}{(1-q^{-2s})\,(1-q^{1-2s})\,(1+q^{2-2s})\,(1-q^{2-2s})} \qquad \text{(at inert place)}$$

The sign is $\chi_{\ell/k}$. Thus, up to finitely-many factors corresponding to ramification,

$$\xi_{C^\natural}(s) \;=\; \frac{\xi_k(2s)\,\xi_k(2s-2)\,\xi_k(2s-1)\,\Lambda_k(2s-1,\chi_{\ell/k})}{\xi_k(4s-2)} \;=\; \frac{\xi_k(2s)\,\xi_k(2s-2)\,\xi_\ell(2s-1)}{\xi_k(4s-2)}$$

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