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Extended automorphic forms

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Many computations in the harmonic analysis of automorphic forms, especially concerning Eisenstein series, or, even worse, trace formulas, expose one to the danger that a naive formal approach leads to incorrect manipulation of expressions whose convergence is fragile or even volatile. At the same time it is often clear that some improved formalism can be correct, and may be more intelligible than a classical treatment. Indeed, the extreme technicality of strictly classical versions of many of these computations gives considerable impetus to consideration of alternatives.

In [Casselman 1993] it is observed that an approach reminiscent of Hadamard's *partie finie* [Hadamard 1932] is helpful in this regard. Casselman notes that [Zagier 1982] raises similar issues.

In the spirit of [Gelfand-Shilov 1958], in effect following [M. Riesz 1938/40] and [M. Riesz 1949], one can present *partie finie* functionals as being obtained by meromorphic continuation with respect to a natural auxiliary parameter, rather than as results of an *ad hoc* classical construction as in [Hadamard 1932]. Nevertheless, one may view meromorphic continuation as *ad hoc* itself.

Thus, we present an elementary uniqueness result (from [Casselman 1993]) for extensions of functionals satisfying differential equations or similar conditions. Some version of this uniqueness result underlies the discussion of 'regularization' in [Gelfand-Shilov 1958] although it is not made explicit.

As in [Casselman 1993], we set up a formalism sufficient to prove that the volume of $SL(2, \mathbb{Z})\backslash SL(2, \mathbb{R})$ is determined by the constant term of a spherical Eisenstein series, and prove the simplest case of the Maaß-Selberg formula for the inner product of truncated Eisenstein series.

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1. *Hadamard's example*

In [Hadamard 1932], Hadamard considered the behavior of functionals of the form

$$\int_\varepsilon^1 \frac{f(x)}{x^{3/2}}\,dx$$

as $\varepsilon \to 0^+$. If f is continuous and $f(0) \neq 0$, then this expression blows up as $\varepsilon \to 0^+$. Nevertheless, Hadamard attached meaning to the integral as follows.

Before letting $\varepsilon \to 0^+$, integrate by parts:

$$\int_\varepsilon^1 \frac{f(x)}{x^{3/2}}\,dx = [\frac{-2f(x)}{x^{1/2}}]_\varepsilon^1 + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx$$

$$= -2f(1) + \frac{-2f(\varepsilon)}{\varepsilon^{1/2}} + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx = -2f(1) + \frac{(-2)(f(\varepsilon) - f(0))}{\varepsilon^{1/2}} + \frac{-2f(0)}{\varepsilon^{1/2}} + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx$$

Of the four summands, only $-2f(0)/\varepsilon^{1/2}$ blows up as $\varepsilon \to 0^+$. In fact, assuming that f is at least once continuously differentiable, the term $(-2)(f(\varepsilon) - f(0))/\varepsilon^{1/2}$ goes to 0.

Hadamard's surprising insight was to simply *drop* the term $-2f(0)/\varepsilon^{1/2}$ entirely, calling what remained the *partie finie* ('finite part') of the integral, denoted

$$\text{p.f.} \int_0^1 \frac{f(x)}{x^{3/2}}\,dx = -2f(1) + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx$$

This should rightfully appear to be a scandalous lapse, not obviously justifiable or purposeful. (Nevertheless, Hadamard developed this idea sufficiently to apply to hyperbolic partial differential equations.)

A few years later [M.Riesz 1938/40] showed that such *partie finie* functionals are the *meromorphic continuations* of convergent integrals, as developed later at length in [Gelfand-Shilov 1958]. In the example above, consider

$$u_s(f) = \int_0^1 f(x)\,x^s\,dx$$

for f at least once continuously differentiable, and for $\mathrm{Re}(s) > -1$. Integration by parts gives

$$u_s(f) = [\frac{f(x)x^{s+1}}{s+1}]_0^1 - \frac{1}{s+1}\int_0^1 f'(x)\,x^{s+1}\,dx = \frac{f(1)}{s+1} - \frac{1}{s+1}\,u_{s+1}(f')$$

Iteration of this gives a meromorphic continuation of u_s to $\mathbb{C}$ with $-1,-2,-3,\ldots$ removed. In particular, there is no pole at $s=-3/2$, and the latter equation gives

$$u_{-3/2}(f) = \frac{f(1)}{(-3/2)+1} - \frac{1}{(-3/2)+1}\int_0^1 f'(x)\,x^{(-3/2)+1}\,dx = -2f(1) + 2\int_0^1 \frac{f'(x)}{x^{1/2}}\,dx$$

It is striking that meromorphic continuation recovers Hadamard's formula. While on one hand this makes Hadamard's *partie finie* less suspect, on the other hand it simultaneously illustrates that the extensions of functionals obtained by meromorphic continuation may have counter-intuitive aspects.

It is important to realize that there is the additional technical issue of understanding various notions of holomorphy or meromorphy of distribution-valued functions, but this is not a serious obstacle.

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2. *Schwartz functions*

A test function on a smooth manifold is is simply a compactly-supported (complex-valued) smooth function. The set $C_c^\infty(X)$ of all test functions on a smooth manifold X (without boundary) is a diffeomorphism invariant of X. In general, spaces of test functions are not Fréchetunless X is compact, but in general are LF-spaces (strict colimits of Fréchetspaces). Let $C_c^\infty(X)^*$ be the dual of $C_c^\infty(X)$, that is, the distributions on X.

By contrast, Schwartz spaces of functions on non-compact smooth manifolds M (without boundary) are *not* diffeomorphism invariants. The definition depends upon a choice of 'compactification', that is, depends upon an open imbedding

$$i : M \to \tilde{M}$$

of M into a *compact* manifold $\tilde{M}$ (of the same dimension as M). In the sequel, we will simply take M to be an open subset of $\tilde{M}$. Then the space $\mathscr{S}(M)$ of Schwartz functions on M (with dependence upon i implicit) is a closed subspace of $C_c^\infty(\tilde{M})$:

$$\mathscr{S}(M) = \text{ closure in } C_c^\infty(\tilde{M}) \text{ of} C_c^\infty(M)$$

Note that $C_c^\infty(M)$ lies inside $C_c^\infty(\tilde{M})$, but need *not* inherit its own LF topology from the topology of $C_c^\infty(\tilde{M})$, since $\tilde{M}$ is compact but M need not be. On the other hand, the compactness of $\tilde{M}$ implies that $C_c^\infty(\tilde{M})$ is a Fréchetspace, from which it follows that the closed subspace $\mathscr{S}(M)$ is a Fréchetspace, not merely an LF space.

Recall that a distribution u is said to *vanish on* an open set U if $uf = 0$ for all test functions with support inside U. The *support* of u is the complement (in the compact manifold $\tilde{M}$) of the union of all open sets on which u vanishes. Recall that a distribution vanishes on the complement of its support, which is seen as follows. Since there exists a locally finite cover $\{\psi_i\}$ subordinate to the collection of open sets U on which u vanishes. Then a test function f with support in the union of the sets U is expressible as $f = \sum_i \psi_i f$, and this sum is finite. Since $\psi_i f$ has support in some U on which u vanishes, $uf = \sum_i u\psi_i f = 0$.

Also, for an open subset M of a compact smooth manifold $\tilde{M}$, the Schwartz space $\mathscr{S}(M)$ is expressible as

$$\mathscr{S}(M) = \{f \in C_c^\infty(\tilde{M}) : uf = 0 \text{ for all distributions } u \in C_c^\infty(\tilde{M})^* \text{ with } \operatorname{spt} u \subset \tilde{M} - M\}$$

Indeed, the set of distributions with support in $\tilde{M} - M$ is simply the 'orthogonal complement' $C_c^\infty(M)^\perp$ of $C_c^\infty(M)$ in $C_c^\infty(\tilde{M})$. Since the latter is locally convex, by the Hahn-Banach theorem

$$\text{closure of subspace } X \;= (X^\perp)^\perp$$

Thus, one might take the viewpoint that Schwartz functions vanish to infinite order at the boundary of M in its compactification $\tilde{M}$.

Either characterization of Schwartz spaces makes it clear that for M' another open subset of $\tilde{M}$ with $M' \subset M$ there is a natural inclusion

$$\mathscr{S}(M') \subset \mathscr{S}(M)$$

with $\mathscr{S}(M')$ a closed subspace of $\mathscr{S}(M)$. Further, for $M'' \subset M'$ a yet smaller open subset, the obvious triangle commutes:

$$\begin{array}{ccc} \mathscr{S}(M') & \to & \mathscr{S}(M) \\ \uparrow & \nearrow & \\ \mathscr{S}(M'') & & \end{array}$$

And note that for any test function F on $\tilde{M}$, for $f \in \mathscr{S}(M)$, $f \cdot F$ is in $\mathscr{S}(M)$. Indeed, with f_i in $C_c^\infty(M)$ so that $f_i \to f$ in the Fréchettopology on $C_c^\infty(\tilde{M})$, $F \cdot f_i \to F \cdot f$, because multiplication by test functions is a continuous map from $C_c^\infty(\tilde{M})$ to itself. Certainly the supports of all the functions $F \cdot f_i$ are inside M, since multiplication can only shrink supports, so the limit $F \cdot f$ is in the closure $\mathscr{S}(M)$ of $C_c^\infty(M)$ in $C_c^\infty(\tilde{M})$.

Following L. Schwartz, we may define the Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$ to be test functions on a *smooth* one-point compactification $\mathbb{R}^n \cup \{\infty\} \approx S^n$ (as an n-sphere) which vanish to infinite order at the point at infinity, that is, which are annihilated by all distributions supported on $\{\infty\}$. This depends upon the imbedding (or, equivalently, upon the metric): the (stereographic) imbedding $i : \mathbb{R}^n \to S^n$ appropriate to this example is

$$i(x) = 2(|x|^2+1)^{-1} \cdot x \oplus (|x|^2-1)(|x|^2-1)^{-1} \in \mathbb{R}^n \oplus \mathbb{R}$$

Define Fréchetspaces $\mathscr{S}(0,\infty)$, and restrictions

$$\mathscr{S}[0,\infty) = \operatorname{Res}_{[0,\infty)} \mathscr{S}(-\infty,\infty) = \operatorname{Res}_{[0,\infty)} \mathscr{S}(S^1) = \operatorname{Res}_{[0,\infty)} C_c^\infty(S^1)$$

$$\mathscr{S}(0,\infty] = \operatorname{Res}_{(0,\infty]} \mathscr{S}(-\infty,\infty)$$

$$\mathscr{S}[0,\infty] = C_c^\infty(S^1) \,/\, \mathscr{S}(-\infty,0)$$

[2.0.1] Proposition: For any $T_1 < T_2$,

$$\mathscr{S}(0,\infty) = \mathscr{S}(0,T_2) + \mathscr{S}(T_1,\infty)$$

$$\mathscr{S}[0,\infty] = \mathscr{S}[0,T_2) + \mathscr{S}(T_1,\infty]$$
$$\mathscr{S}(T_1,T_2) = \mathscr{S}(0,T_2) \cap \mathscr{S}(T_1,\infty)$$
$$\mathscr{S}(T_1,T_2) = \mathscr{S}[0,T_2) \cap \mathscr{S}(T_1,\infty]$$

The inversion map

$$\mathrm{inv} : x \to \frac{1}{x}$$

stabilizes $\mathscr{S}(0,\infty)$ and $\mathscr{S}[0,\infty]$ and gives a natural isomorphism

$$\mathscr{S}[0,\infty) \to \mathscr{S}(0,\infty]$$

simply by sending $x \to f(x)$ to $x \to f(1/x)$.

Define a space of locally integrable functions of moderate growth by

$$\mathrm{Mod}(0,\infty) = \{f \in L^1_{\mathrm{loc}}(0,\infty) : \sup_{x>0} (x+\frac{1}{x})^{-N}|f(x)| < +\infty \text{ for some } N\}$$

Any $u \in \mathrm{Mod}(0,\infty)$ gives a continuous functional on $\mathscr{S}(0,\infty)$ by integration, with positive-dilation-invariant measure dx/x: for $f \in \mathscr{S}(0,\infty)$,

$$u(f) = \int_0^\infty u(x)\, f(x)\, \frac{dx}{x}$$

Also, let

$$C^\infty_{\mathrm{mod}}(0,\infty) = C^\infty(0,\infty) \cap \mathrm{Mod}(0,\infty)$$

be the smooth functions on $(0,\infty)$ of moderate growth at 0 and infinity. The indicated integral formula imbeds $\mathrm{Mod}(0,\infty)$ (and, hence, $\mathscr{S}(0,\infty)$ and $C^\infty_{\mathrm{mod}}(0,\infty)$) in the dual:

$$\mathscr{S}(0,\infty) \subset C^\infty_{\mathrm{mod}}(0,\infty) \subset \mathrm{Mod}(0,\infty) \subset \mathscr{S}(0,\infty)^*$$

The inversion map $x \to 1/x$ acts on the duals by

$$\mathrm{inv}(u)(f) = u(\mathrm{inv}\, f)$$

for $f \in \mathscr{S}(0,\infty)$. Since *inv* preserves the measure dx/x, this action of *inv* on elements in the dual, when restricted to $\mathrm{Mod}(0,\infty)$, agrees with the natural action on functions.

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3. *A formal lemma on extensions*

[3.0.1] Proposition: Let A, B, C be modules over a (not necessarily commutative) $\mathbb{C}$-algebra R. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{s} C \to 0$$

be a short exact sequence, and let T be an R-endomorphism of B which stabilizes A (as subobject of B), so induces an R-endomorphism on $C \approx B/A$ by

$$T(b+A) = Tb + A$$

Then we have a natural exact sequence

$$0 \to \ker_A T \to \ker_B T \to \ker_C T \to A/TA \to B/TB \to C/TC \to 0$$

Proof: This is the long exact homology sequence attached to the short exact sequence

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & A & \xrightarrow{i} & B & \xrightarrow{s} & C & & \to 0 \\ & & T\downarrow & & T\downarrow & & T\downarrow & & \\ 0 \to & & A & \xrightarrow{i} & B & \xrightarrow{s} & C & & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

of the complexes

$$\tilde{A}:\ 0 \to A \xrightarrow{T} A \to 0, \qquad \tilde{B}:\ 0 \to B \xrightarrow{T} B \to 0, \qquad \tilde{C}:\ 0 \to C \xrightarrow{T} C \to 0$$

That is, $H_o(\tilde{A}) = \ker_A T$, $H_1(\tilde{A}) = A/TA$, and similarly for B and C. ///

[3.0.2] Corollary: In the situation of the previous proposition, if T gives a bijection of A to itself, then the natural map

$$\ker_B T \to \ker_C T$$

is an isomorphism. ///

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4. *Extension of integration by parts*

Let

$$\langle , \rangle : A \times S \to \mathbb{C}$$

be a complex bilinear pairing of $\mathbb{C}[D]$-modules. Assume that

$$\langle Du, f \rangle = -\langle u, Df \rangle$$

for all $u \in A$ and $f \in S$. Let A_o be a $\mathbb{C}[D]$-subspace of A possessing a $\mathbb{C}$-algebra structure, and on which D acts as a derivation in the sense that

$$D(u \cdot v) = Du \cdot v + u \cdot Dv$$

Suppose that there is an element 1 in S with the property that $D1 = 0$. The following is an extension of the usual integration by parts formula (without boundary terms):

[4.0.1] Proposition: For $u, v \in A_o$,

$$\langle Du \cdot v, 1 \rangle = \langle -u \cdot Dv, 1 \rangle$$

Proof: This is a direct computation:

$$0 = \langle u \cdot v, 0 \rangle = \langle u \cdot v, D1 \rangle = -\langle D(u \cdot v), 1 \rangle = -\langle Du \cdot v + u \cdot Dv, 1 \rangle$$

from which we have

$$\langle Du \cdot v, 1 \rangle = -\langle u \cdot Dv, 1 \rangle$$

as desired. ///

Let A and S be $U = \mathbb{C}[X_1, \ldots, X_n]$-modules, with a U-subspace A_o which has an algebra structure, where the X_i do not necessarily commute with each other. Assume that for any X_i

$$\langle X_i u, f \rangle = -\varepsilon_i \langle u, X_i f \rangle$$

where the sign $\varepsilon_i = \pm 1$ may depend upon i. Assume that each X_i acts as a derivation on A_o. Let

$$\Delta = \varepsilon_1 X_1^2 + \ldots + \varepsilon_n X_n^2$$

Suppose that there is an element $1 \in S$ so that $X_i 1 = 0$ for all indices i. Then we have an extension of the formula often known as *Green's formula:*

[4.0.2] Proposition: For $u, v \in A_o$,

$$\langle \Delta u \cdot v, 1\rangle = \langle u \cdot \Delta v, 1\rangle$$

Proof: It certainly suffices to proves that for each $X = X_i$ we have the corresponding identity

$$\langle X^2 u \cdot v, 1\rangle = \langle u \cdot X^2 v, 1\rangle$$

Note that this would make the coefficients ε_i irrelevant. Observe that

$$X(Xu \cdot v - u \cdot Xv) = (X^2 u \cdot v + Xu \cdot Xv) - (Xu \cdot Xv + u \cdot X^2 v) = X^2 u \cdot v - u \cdot X^2 v$$

Therefore, with $X = X_i$ and $\varepsilon = \varepsilon_i$,

$$0 = \langle Xu \cdot v - u \cdot Xv, 0\rangle = \langle Xu \cdot v - u \cdot Xv, X1\rangle = -\varepsilon\langle X(Xu \cdot v - u \cdot Xv), 1\rangle = -\varepsilon\langle X^2 u \cdot v - u \cdot X^2 v, 1\rangle$$

which yields the desired identity. ///

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5. *Maaß-Selberg relation for* $SL(2, \mathbb{Z})$

Granting the legitimacy of computations with extended versions of integrals as above, we can give an elegant proof of the Maaß-Selberg relations for $SL(2, \mathbb{Z})$.

For $z = x + iy$ in the complex upper half-plane $\mathfrak{H}$, following Maaßdefine the simplest spherical Eisenstein series as usual by

$$E_s(x + iy) = \sum_{m,n \text{ coprime}} \frac{y^s}{|mz + n|^{2s}}$$

This converges nicely for $\Re(s) > 1$, and has a Fourier expansion

$$E_s(x + iy) = y^s + c(s)\, y^{1-s} + (\text{higher-order terms})$$

where the higher-order terms do not immediately concern us, apart from the fact that they are of rapid decay as $y \to +\infty$. Direct computation shows that

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)}$$

where

$$\xi(s) = \pi^{-s/2}\, \Gamma(\frac{s}{2})\, \zeta_{\mathbb{Q}}(s)$$

with Riemann's zeta function $\zeta_{\mathbb{Q}}$. The expression $y^s + c(s) y^{1-s}$ is the **constant term** of the Eisenstein series. It is well known that E_s has a meromorphic continuation in s. The explicit details of the situation would allow us to conclude directly that the only pole in the half-plane $\Re(s) > \frac{1}{2}$ is at $s = 1$, but we want an approach to study of poles that will apply in situations where much less explicit information is available.

Because of the nature of the constant term, away from poles and zeros of $c(s)$ the Eisenstein series E_s cannot be square-integrable on the usual fundamental domain

$$\mathcal{F} = \{x+iy \in \mathfrak{H} : |x| \le \frac{1}{2},\ |x+iy| \ge 1\}$$

with regard to the $SL(2,\mathbb{R})$-invariant measure $dx\,dy/y^2$. However, for large positive T define the **truncation** $\Lambda^T E_s = E_s^T$ of the Eisenstein series by

$$E_s^T(x+iy) = \begin{cases} y^s + c(s)y^{1-s} & + & \text{(higher-order terms)} & \text{(for } y \le T) \\ 0 & + & \text{(higher-order terms)} & \text{(for } y > T) \end{cases}$$

Since the higher-order terms are rapidly decreasing on the fundamental domain, the truncated Eisenstein series *are* square-integrable on the fundamental domain.

Note that, from the definition, the Eisenstein series and the function $c(s)$ behave nicely with regard to complex conjugation:

$$E_{\bar{s}} = \overline{E_s} \qquad c(\bar{s}) = \overline{c(s)}$$

[5.0.1] Theorem: *(Maaß-Selberg)* For two complex r, s, with $r(r-1) \ne s(s-1)$ (so that the denominators in the following formula are not zero),

$$\int_{SL_2(\mathbb{Z})\backslash\mathfrak{H}} E_r^T(z)\, E_s^T(z)\, \frac{dx\,dy}{y^2}$$

$$= \frac{T^{r+s-1}}{r+s-1} + c(r)\,\frac{T^{(1-r)+s-1}}{(1-r)+s-1} + c(s)\,\frac{T^{r+(1-s)-1}}{r+(1-s)-1} + c(r)\,c(s)\,\frac{T^{(1-r)+(1-s)-1}}{(1-r)+(1-s)-1}$$

Proof: Although the whole Eisenstein series are not square-integrable, we have extended the pairing on functions (given initially by absolutely convergent integrals), and we have an extension of Green's identity

$$\langle (\Delta E_r)\cdot E_s, 1\rangle = \langle E_r \cdot \Delta(E_s), 1\rangle$$

for products paired against 1. In particular, since $\Delta E_s = s(s-1)E_s$, this gives

$$r(r-1)\,\langle E_r \cdot E_s, 1\rangle = \langle (\Delta E_r)\cdot E_s, 1\rangle = \langle E_r \cdot \Delta(E_s), 1\rangle = s(s-1)\,\langle E_r\cdot E_s, 1\rangle$$

For $r(r-1) \ne s(s-1)$, this implies that

$$\langle E_r \cdot E_s, 1\rangle = 0$$

Thus, letting $Y_s^T = E_s - E_s^T$ denote the 'tail',

$$0 = \langle E_r\cdot E_s, 1\rangle = \langle E_r^T \cdot E_s^T, 1\rangle + \langle E_r^T\cdot Y_s^T, 1\rangle + \langle Y_r^T \cdot E_s^T, 1\rangle + \langle Y_r^T\cdot Y_s^T, 1\rangle = 0 + 0 + \langle Y_r^T \cdot Y_s^T, 1\rangle$$

The first 0 is because $\langle E_r \cdot E_s, 1\rangle = 0$. The second and third 0's are because the 'tails' Y_s^T are 'orthogonal' to the truncated Eisenstein series in the sense that the integral is absolutely convergent and can be evaluated as

$$\langle Y_r^T \cdot E_s^T, 1\rangle = \int_{SL_2(\mathbb{Z})\backslash\mathfrak{H}} Y_r^T(z)\, E_s^T(z)\, \frac{dx\,dy}{y^2} = \int_{|x|\le 1/2}\int_{y\ge T} Y_r^T(z)\, E_s^T(z)\,\frac{dx\,dy}{y^2}$$

$$= \int_{y\ge T} (y^r + c(r)y^{1-r})\,(\text{constant Fourier coefficient of } E_s^T(z))\, y^{-1}\,\frac{dy}{y}$$

$$= \int_{y \ge T} (y^r + c(r)y^{1-r})\, 0\, y^{-1}\,\frac{dy}{y} = 0$$

since, by construction,the constant term of E_s^T vanishes for $y > T$. Thus, again,

$$0 = \langle E_r \cdot E_s, 1\rangle = \langle Y_r^T \cdot Y_s^T, 1\rangle = \int_T^\infty (y^r + c(r)y^{1-r}) \cdot (y^s + c(s)y^{1-s})\, y^{-1}\, \frac{dy}{y}$$

where the last 'integral' is understood as being in the extended sense (since, after all, for no values of r, s does it converge). Thus, rearranging the equation $0 = \langle E_r \cdot E_s, 1\rangle$, we have

$$\langle E_r^T \cdot E_s^T, 1\rangle = -\int_T^\infty (y^r + c(r)y^{1-r}) \cdot (y^s + c(s)y^{1-s})\, y^{-1}\, \frac{dy}{y}$$

$$= \frac{T^{r+s-1}}{r+s-1} + c(r)\, \frac{T^{(1-r)+s-1}}{(1-r)+s-1} + c(s)\, \frac{T^{r+(1-s)-1}}{r+(1-s)-1} + c(r)\, c(s)\, \frac{T^{(1-r)+(1-s)-1}}{(1-r)+(1-s)-1}$$

of course evaluating the 'integrals' in an extended sense. ///

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