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Factorization of unitary representations of adele groups

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The result sketched here is of fundamental importance in the 'modern' theory of automorphic forms and L-functions. What it amounts to is proof that *representation theory* is in principle relevant to study of automorphic forms and L-function.

The goal here is to get to a coherent statement of the basic *factorization theorem* for irreducible unitary representations of reductive linear adele groups, starting from very minimal prerequisites. Thus, the writing is discursive and explanatory. All necessary background definitions are given. There are no proofs. Along the way, many basic concepts of wider importance are illustrated, as well.

One disclaimer is necessary: while *in principle* this factorization result makes it clear that representation theory is relevant to study of automorphic forms and L-functions, *in practice* there are other things necessary. In effect, one needs to know that the representation theory of reductive linear p-adic groups is *tractable*, so that conversion of other issues into representation theory is a change for the better. Thus, beyond the material here, one will need to know (at least) the basic properties of spherical and unramified principal series representations of reductive linear groups over local fields. The fact that in some sense *there is just one irreducible representation of a reductive linear p-adic group* will have to be pursued later.

References and historical notes will be added later, maybe.

Many of the statements made here without proof are very difficult to prove! Just because no mention of proof is made it should not be presumed that it's 'just an exercise'!

1. Topological groups, Haar measures, Haar integrals

Let G be a **topological group**. Usually this tacitly requires also that the group is *locally compact* and *Hausdorff*, and probably also *separable*, in the sense that it has a countable dense subset. The latter hypothesis is necessary to apply a Baire-category style arguments to prove that any Hausdorff space X on which G acts transitively is homeomorphic to G/G_x, where $x \in X$ is fixed and G_x is the isotropy group of x in G.

We are interested only in *positive regular Borel measures* μ on G. That is, we are interested only in non-negative real-valued measures μ defined on σ-algebras containing the open sets so that

$$\mu(E) = \text{ inf of } \mu(U) \text{ where } U \text{ is open containing } E\text{: } \textit{outer regularity}$$

$$\mu(E) = \text{ sup of } \mu(K) \text{ where } K \text{ is compact contained in } E\text{: } \textit{inner regularity}$$

And a positive regular Borel measure μ is a **right Haar measure** if for all measurable sets E and $g \in G$

$$\mu(Eg) = \mu(E)$$

Similarly a **left Haar measure** has the property

$$\mu(gE) = \mu(E)$$

The theorem is that *a topological group has a unique right Haar measure, up to positive real multiples.* Similarly, up to positive real multiples, there is a unique *left* Haar measure.

A topological group is **unimodular** if a *right* Haar measure is also a *left* Haar measure.

2. Topologies on operators on Hilbert spaces

Let V be a (complex) Hilbert space with inner product $\langle , \rangle$ and norm $||$. Let $\mathcal{B}(V)$ be the algebra of continuous linear operators $T : V \to V$. There are at least 3 important topologies on $\mathcal{B}(V)$.

The **uniform** or **norm** topology is the strongest topology we will consider, and gives $\mathcal{B}(V)$ the structure of *Banach space*. This topology is defined via the **operator norm**

$$|T| = \sup_{|v|=1} |Tv|$$

where as indicated v ranges over unit vectors (in V).

The **strong topology** on $\mathcal{B}V$ is defined by a collection of *semi-norms*

$$\nu_v(T) = |Tv|$$

as v ranges over V. Note that it is unlikely that there is a *countable* collection of semi-norms giving this topology, so it is therefore *not* obviously metrizable.

The **weak topology** on $\mathcal{B}(V)$ is defined by a collection of *semi-norms*

$$\nu_{v,w}(T) = |\langle Tv, w\rangle|$$

as v, w range over V.

There are also ultra-strong and ultra-weak topologies, and others besides, but we don't need them here.

3. Unitary representations of topological groups

Let $\mathcal{B}(V)^\times$ be the group of continuous linear operators on a Hilbert space V having continuous *inverses*.

A (continuous) **representation** of a *topological* group G on a Hilbert space V is a group homomorphism

$$\pi : G \to \mathcal{B}(V)^\times$$

so that also π is *continuous* in the *strong* topology on $\mathcal{B}(V)$.

It is easy to arrange examples to show that we should not expect nor demand continuity in the *norm* topology, but only the *strong* topology.

A **unitary representation** π of a group G on a Hilbert space V is a representation so that for all $v, w \in V$

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$$

A unitary representation (π, V) of G often is denoted simply by 'π' or 'V' for reasons of brevity.

4. Representations of the convolution algebra

Let G be a unimodular topological group. Let $L^1(G)$, as usual, denote the collection of absolutely integrable complex-valued functions on G, and let $|\;|_1$ denote the usual norm on this Banach space. We have a **convolution** $*$ on $L^1(G)$ defined by

$$(f * \phi)(g) = \int_G f(gh^{-1})\; \phi(h)\; dh$$

where dh refers to a fixed choice of Haar measure on G. One fundamental result, which follows from Fubini's theorem, is

$$|f * \phi|_1 = |f|_1 \cdot |\phi|_1$$

Any unitary representation (π, V) of G gives rise to an **algebra representation** of $L^1(G)$ by 'defining'

$$\pi(f)v = \int_G f(g) \cdot \pi(g)v\; dg$$

(There are several ways to be sure that this integral makes sense). It is formal that

$$\pi(f * \phi) = \pi(f) \circ \pi(\phi)$$

which is to say that the convolution product fits into this story just right.

We can estimate the operator norm of operators $\pi(f)$, as follows. Then for $v \in V$

$$|\pi(f)v| \le \int |f(g)\pi(g)v|\; dg = \int |f(g)|\; |\pi(g)v|\; dg$$

$$\int |f(g)|\; |v|\; dg = |v| \int |f(g)|\; dg = |v|\; |f|_1$$

since π is *unitary*. Thus,

$$|\pi(f)| \le |f|_1$$

5. Commutants, Schur's lemma, central characters

Let A be a subalgebra of the algebra $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space V. Then the **commutant** A' of A is defined to be

$$A' = \{T \in \mathcal{B}(V) : T \circ \alpha = \alpha \circ T, \text{ for all } \alpha \in A\}$$

Schur's Lemma asserts that, if (π, V) is an irreducible unitary Hilbert space representation of a topological group G, then the commutant $\rho(G)'$ of $\rho(G)$ consists just of the scalar operators $\mathbf{C} \cdot 1$ on V. This is an immediate consequence of elementary spectral theory for bounded operators on Hilbert spaces.

One important and basic corollary is that on an irreducible representation (π, V) of a topological group G the center Z of G acts by scalars. Indeed, one can see that there is a (continuous) group homomorphism $\omega : Z \to \mathbf{C}^\times$ so that for $z \in Z$

$$\pi(z) = \omega(z) \cdot 1_V$$

where 1_V is the identity on V. This 'character' (meaning one-dimensional representation) ω is the **central character** of π. So, as corollary of Schur's lemma for unitary representations, we find that *irreducible unitary representations have central characters.*

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6. Banach algebras, C-star-algebras, representations

A (complex) **Banach algebra** is an associative **C**-algebra B with a norm $||$ with the **sub-multiplicative property**

$$|xy| \leq |x| \cdot |y|$$

The basic example of such is the collection $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space V.

An **involution** on an algebra B is a **C**-conjugate-linear map $B \to B$ so that

$$x^{**} = x \qquad (xy)^* = y^* x^*$$

A Banach algebra is a **Banach $*$-algebra** if there is an **involution** $x \to x^*$ on B so that also

$$|x^*| = |x|$$

A Banach $*$-algebra is a **C^*-algebra** if for all $x \in A$

$$|x^* x| = |x|^2$$

The fundamental example of a C^*-algebra is the collection $\mathcal{B}(V)$ of all continuous linear operators on a Hilbert space V, with $*$ being the usual *adjoint*.

A **representation π of a C^*-algebra** A on a Hilbert space V is a **C**-algebra homomorphism

$$\phi : A \to \mathcal{B}(V)$$

which also preserves the involution $*$, that is,

$$f(x^*) = f(x)^*$$

where x^* denotes the involution in A and $f(x)^*$ denotes the usual involution by adjoint in $\mathcal{B}(V)$.

An element p of a C^*-algebra is **self-adjoint** if $p^* = p$. This is in complete analogy with the terminology for operators on Hilbert spaces. An element p of a C^*-algebra is an **idempotent** if $p^2 = p$. This is also standard.

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7. Stellar algebras of topological groups

For unimodular topological group G we already have the convolution algebra $L^1(G)$ of G, which acts upon any unitary representation space (π, V) of G by

$$\pi(f)v = \int_G f(g) \cdot \pi(g)v \, dg$$

where dg denotes Haar measure. That is, this convolution algebra is the first analogue of the *group algebra* of a finite group.

We define the **stellar norm** on $L^1(G)$ by

$$|f|_* = \sup_\sigma |\sigma(f)|$$

where σ ranges over all unitary representations of G. Especially when G is *separable* we could just as well take the sup over *isomorphism classes* of *separable* Hilbert space representations, making clear that the sup is taken over a countable *set* rather than some dangerously big *class* which might lead to some set-theoretic fallacies.

The straightforward little computation done above shows that

$$|f|_* \le |f|_{L^1}$$

The completion $C^*(G)$ of $L^1(G)$ is the **stellar algebra** of G. The inequality comparing the two norms shows that $L^1(G)$ imbeds continuously into $C^*(G)$. An equivalent definition of the stellar algebra is that it is the completion of the convolution algebra C_c^o of compactly-supported continuous complex-valued functions on G.

By the definition of the stellar norm, any unitary representation of G induces a representation of $C^*(G)$.

This stellar algebra has the involution arising from

$$f^*(g) = \overline{f(g^{-1})}$$

With respect to this involution, *$C^*(G)$ is a C^*-algebra.*

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8. Tensor products of Hilbert spaces

Let V, W be two Hilbert spaces, with inner products $\langle , \rangle$ and $(,)$ respectively. Let $V \otimes W$ be the tensor product of the two complex vector spaces over $\mathbf{C}$. This tensor product has a positive-definite (hermitian) inner product $[,]$ defined by taking the sesquilinear extension of

$$[v \otimes w, v' \otimes w'] = \langle v, v' \rangle (w, w')$$

(As usual, *sesquilinear* means linear in the first argument and conjugate-linear in the second). The *completion* of this (pre-Hilbert) space with respect to this hermitian form is the **Hilbert tensor product** or **completed tensor product**

$$V \hat{\otimes} W$$

of the two Hilbert spaces.

One can check that without overtly completing the tensor product is *not* complete unless one of the two spaces is finite-dimensional.

If we replace W by the conjugate Hilbert space $\overline{W}$ (viewed as the topological dual of W), then the tensor product $V \otimes \overline{W}$ can be naturally identified with the collection of *finite-rank linear operators* $W \to V$ by taking

$$v \otimes \lambda \to T_{v \otimes \lambda}$$

where

$$T_{v\otimes\lambda}(w) = \lambda(w)\cdot v$$

The completion is (essentially by definition) the completion with respect to the **Hilbert-Schmidt norm** on finite-rank operators. (This norm does *not* extend to all of $\mathcal{B}(V)$).

The analogous tensor product construction for a countable collection $V_1, V_2, V_3, \ldots$ of Hilbert spaces V_i is a little more complicated. Indeed, it is necessary to specify a unit vector e_i in each Hilbert space V_i in the collection. Then the **restricted tensor product** of the V_i with respect to the unit vectors e_i is the vector space of all finite sums of tensors

$$v_1 \otimes v_2 \otimes v_3 \otimes \ldots$$

where for *almost all* (that is, all but finitely-many) indices i we have

$$v_i = e_i$$

The inner product on this pre-Hilbert space is given by (the hermitian extension of)

$$\langle v_1 \otimes v_2 \otimes v_3 \otimes \ldots, v_1' \otimes v_2' \otimes v_3' \otimes \ldots\rangle = \langle v_1, v_1'\rangle\langle v_2, v_2'\rangle\langle v_2, v_2'\rangle \ldots$$

Since all but finitely-many of the v_i and v_i' are actually just the corresponding unit vector e_i, all but finitely-many factors in the last product are just 1, so there is no issue of convergence.

Then the **completed restricted tensor product**

$$\widehat{\bigotimes}_i{}' \, V_i$$

of the spaces V_i with respect to choice of the unit vectors $e_i \in V_i$ is the completion of the restricted tensor product with respect to this inner product.

Note that there is usually no overt reference to the unit vectors in discussion of such a restricted tensor product. In practice this is harmless, since the choice of the unit vectors will be given us by some other mechanism.

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9. Type I groups

A unitary representation ρ of a topological group G is a **factor representation** or simply a **factor** if the double commutant is trivial, that is, if

$$\rho(G)'' \cap \rho(G)' = \mathbf{C}\cdot 1$$

Then, the algebra $\rho(G)''$ has center just $\mathbf{C}\cdot 1$ (In the terminology of von Neumann algebras, $\rho(G)''$ is *the von Neumann algebra generated by* $\rho(G)$).

A unitary Hilbert space representation (σ, V) of a topological group G is (π-) **isotypic** (where π is an irreducible unitary representation) if (σ, V) is a sum (not necessarily direct) of G-homomorphic images of copies of π. Sometimes it is said that σ is a **multiple** of π. Then from a good version of spectral theory it follows that (σ, V) is of the form

$$(\pi, V_\pi)\hat{\otimes}(1, W)$$

where V_π is the representation space of π, where $(1, W)$ denotes the trivial representation on a Hilbert space W, and where the $\hat{\otimes}$ denotes the completed tensor product. The converse, that such a completed tensor product is isotypic, is easy.

A *factor* representation of a topological group G is said to be **of Type I** if it is *isotypic*.

A locally compact Hausdorff topological group is defined to be of **Type I** if every *factor representation* of it is of Type I, i.e., is *isotypic*. It is fairly easy to show that an isotypic representation is a factor representation.

As an example of the manner in which factor representations occur, let π be an irreducible representation of the product $G \times H$ of two topological groups. Then the restriction of π to a representation of G is a *factor representation* of G. (Indeed, the same conclusion holds if π itself is merely a factor).

The most important result here is the **Factorization Lemma:** If G is a Type I group and if π is an irreducible unitary Hilbert space representation of $G \times H$, then π is of the form $\pi_1 \otimes \pi_2$ where π_1 is an irreducible unitary representation of G and π_2 is an irreducible unitary representation of H.

Corollary: A finite product of Type I groups is of Type I.

Unfortunately, the terminology about 'Type I' in reference to groups is *not* compatible with the analogous terminology in reference to von Neumann algebras, although it is compatible with the analogous terminology for C^*-algebras (discussed below).

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10. Criterion for Type I-ness: liminal and postliminal algebras

Of course it is not a trivial matter to prove that any given group is of Type I. And, in practice we want a stronger conclusion anyway.

One helpful criterion for a factor representation π of a topological group G to be of Type I-ness is the following: Suppose that $\pi(f)$ is a non-zero *compact operator*, for some $f \in C_c^o(G)$. Then π is *isotypic*, so is (by definition) of Type I.

Thus, as a corollary, *if $\pi(f)$ is a compact operator for every $f \in C_c^o(G)$ and for every irreducible unitary representation π of G, then every factor representation of G is of Type I, that is, is isotypic. That is, under this hypothesis, G is of Type I.*

A C^*-algebra A is called **liminal** or **CCR** if $\pi(f)$ is a *compact operator* for all irreducible representations π of A and for all $f \in A$.

A C^*-algebra A is called **postliminal** or **GCR** or **Type I** if $\pi(A)$ contains *some compact operator* for all irreducible representations π of A and for all $f \in A$. This is a weaker condition than the condition of liminality or CCR-ness.

By the criterion just above, for the *stellar algebra* $C^*(G)$ of a topological group,

$$C^*(G) \text{ liminal } (=\text{CCR}) \Rightarrow C^*(G) \text{ postliminal } (=\text{GCR}=\text{Type I}) \Rightarrow \; G \text{ Type I}$$

Thus, there is a slight distinction between the sense of 'Type I' for C^*-algebras and for topological groups.

Again, this terminology is *not* compatible with the analogous terminology for von Neumann algebras.

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11. Reductive linear groups

Rather than give general definitions, we give the simplest example, which already illustrates the important issues, without requiring sharper methods, broader background, and more mature viewpoint.

In characteristic zero, the prototypical local fields are the field of real numbers $\mathbf{R}$ and the fields $\mathbf{Q}_p$ of p-adic numbers. For the sake of a unifying notation, we may write

$$\mathbf{R} = \mathbf{Q}_\infty$$

and in the notation $\mathbf{Q}_p$ allow $p \leq \infty$, meaning that p is either a prime (in the usual sense) or is the symbol ∞.

As usual, $\mathbf{Q}_p$ is the completion of the rational numbers $\mathbf{Q}$ with respect to the **p-adic norm** defined by

$$|\frac{a}{b}p^n|_p = p^{-n}$$

where a, b are integers relatively prime to p and n is an integer. So

$$|p|_p = \frac{1}{p}$$

As usual, the ring $\mathbf{Z}_p$ of p-adic integers can be described as the completion of $\mathbf{Z}$ in $\mathbf{Q}_p$, or also simply as

$$\mathbf{Z}_p = \{\alpha \in \mathbf{Q}_p : |\alpha|_p \leq 1\}$$

Alternatively, we can describe $\mathbf{Z}_p$ as a *projective limit* of the finite rings $\mathbf{Z}/p^N$. Then $\mathbf{Q}_p$ is $\mathbf{Z}_p[\frac{1}{p}]$ (as a ring) or is a *direct limit* of the $\mathbf{Z}_p$-modules $p^{-n}\mathbf{Z}_p$ as $n \to +\infty$ (with inclusion maps).

The (rational) **adeles A** or **adele ring** consists of elements

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \alpha_5, \alpha_7, \ldots) \in \prod_p \mathbf{Q}_p$$

(where the indexing is by primes, apart from the first component which is a real number) with the further condition that

$$\alpha_p \in \mathbf{Z}_p \text{ for all but finitely-many primes } p$$

Alternatively, it is the direct limit (with inclusion maps) of the rings

$$\mathbf{A}_S = \mathbf{R} \times \left(\prod_{p \in S} \mathbf{Q}_p\right) \times \left(\prod_{p \notin S} \mathbf{Z}_p\right)$$

as S ranges over larger and larger *finite* subsets of the collection of all primes.

Let R be a commutative ring. The prototypical **reductive group** is the **general linear group**

$$GL(n, R) = \text{ invertible } n\text{-by-}n \text{ matrices with entries in } R$$

In particular, an n-by-n matrix is in $GL(n, R)$ if and only if its *determinant* is in the group of units $R^\times$ of R.

Thus, we have prototypical **reductive linear groups over local fields**

$$GL(n, \mathbf{R}) \qquad GL(n, \mathbf{Q}_p)$$

Like the adeles themselves, the **adele group** $GL(n, \mathbf{A})$ is *not* simply the product of the groups $GL(n, \mathbf{Q}_p)$. Rather, this adele group is the collection of

$$\alpha = (\alpha_\infty, \alpha_2, \alpha_3, \alpha_5, \alpha_7, \ldots) \in \prod_{p \leq \infty} GL(n, Q_p)$$

so that

$$\alpha_p \in GL(n, \mathbf{Z}_p) \text{ for all but finitely-many primes } p$$

Alternatively, it is the direct limit (with inclusion maps) of the groups

$$GL(n, \mathbf{A}_S) = GL(n, \mathbf{R}) \times \left(\prod_{p \in S} GL(n, \mathbf{Q}_p)\right) \times \left(\prod_{p \notin S} GL(n, \mathbf{Z}_p)\right)$$

as S ranges over larger and larger *finite* subsets of the collection of all primes.

In general, 'prime' is used as synonym for 'completion', so is meant to include the completion $\mathbf{R}$ of $\mathbf{Q}$, for example. Up to isomorphism, the real numbers $\mathbf{R}$ and the p-adic numbers $\mathbf{Q}_p$ (as p varies over primes) exhaust the list of all completions of $\mathbf{Q}$.

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12. Reductive groups over local fields are Type I

The very important basic result here is that *stellar algebras of reductive linear groups over local fields are liminal.*

Therefore, *reductive linear groups over local fields are of Type I.*

This *local result* then has *global application* (to representations of adele groups), via the general factorization property that irreducible unitary representations of products of Type I groups have. This mechanism is what makes representation theory important to the theory of automorphic forms.

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13. Factorization of representations of adele groups

The idea of the result here is that then *irreducible unitary representations of adele groups of reductive linear groups factor over primes.*

This factorization can be described a more precisely, as follows. Let G be a reductive linear group defined over a number field k. For a completion k_v of k, let G_v be the group of k_v-valued points of G. (Now, in contrast to the previous section, we are using the letter v instead of p as index).

Then the assertion is that for any irreducible unitary representation π of the adelization $G_{\mathbf{A}}$ of G, there are irreducible unitary representations π_v of the groups $G_v = G(k_v)$ so that π is a *completed restricted direct product*

$$\pi \approx \widehat{\bigotimes}' \pi_v$$

Further, for almost all (meaning for all but finitely-many) completions k_v, the 'local' representation π_v is **spherical**, meaning that it is *irreducible* and has a one-dimensional subspace of K_v-fixed vectors, where K_v is a maximal compact subgroup of G_v.

In the case of p-adic groups $GL(n, \mathbf{Q}_p)$, the standard maximal compact subgroups are

$$K_v = K_p = GL(n, \mathbf{Z}_p)$$

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14. Admissibility of representations

Again, to prove that reductive groups over local fields are Type I, one proves the stronger result that their stellar algebras are *liminal.* Here we only consider the p-adic case.

A unitary representation (π, V) of a p-adic group G (such as $GL(n, \mathbf{Q}_p)$) is **admissible** if, for every compact open subgroup K of G the subspace π^K of K-fixed vectors is *finite-dimensional.*

It is a fundamental but difficult result that *every irreducible unitary representation of a reductive linear p-adic group is admissible.*

It is elementary that the collection of finite linear combinations of translates of the characteristic functions of compact open subgroups is *dense* in the stellar algebra $C^*(G)$, for G a p-adic group. Therefore, if (π, V) is admissible, then the stellar algebra acts by compact operators on V.

Thus, *since every irreducible unitary representation of a p-adic group is admissible, the stellar algebra is liminal, and thus the group is of Type I.* This is the way the Type I property is proven, and, thus, the way that the factorization over primes is obtained.