(*January 16, 2008*)

Fujisaki's lemma (after Weil)

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[0.0.1] **Theorem:** *(Fujisaki's Lemma)* For a number field k let

$$\mathbb{J}^1 \;=\; \{\alpha \in \mathbb{J}_k \;:\; |\alpha| = 1\}$$

Then the quotient $k^\times \backslash \mathbb{J}^1$ is *compact*.

Proof: Give $\mathbb{A} = \mathbb{A}_k$ a Haar measure so that $k\backslash\mathbb{A}$ has measure 1. First, we have the Minkowski-like claim that a compact subset C of $\mathbb{A}$ with measure greater than 1 cannot *inject* to the quotient $k\backslash\mathbb{A}$. Indeed, suppose, to the contrary, that C injects to the quotient. Letting f be the characteristic function of C,

$$1 < \int_{\mathbb{A}} f(x)\,dx = \int_{k\backslash\mathbb{A}} \sum_{\gamma\in k} f(\gamma+x)\,dx \le \int_{k\backslash\mathbb{A}} 1\,dx = 1 \qquad \text{(last inequality by injectivity)}$$

contradiction, proving the claim.

Fix compact $C \subset \mathbb{A}$ with measure greater than 1. For idele α, the change-of-measure on $\mathbb{A}$ is

$$\frac{d(\alpha x)}{dx} = |\alpha|$$

Thus, neither αC nor $\alpha^{-1}C$ inject to the quotient $k\backslash\mathbb{A}$.

So there are $x \neq y$ in k so that $x + \alpha C = y + \alpha C$. Subtracting, $x - y \in \alpha(C - C) \cap k$. Since $x - y \neq 0$ and k is a field, $k^\times \cap \alpha(C - C) \neq \phi$. Likewise, $k^\times \cap \alpha^{-1}(C - C) \neq \phi$.

Thus, there are a, b in $k^\times$ such that

$$a \cdot \alpha \in C - C \qquad\qquad b \cdot \alpha^{-1} \in C - C$$

There is an obvious constraint

$$ab \;=\; (a\cdot\alpha)(b\cdot\alpha^{-1}) \;\in\; (C-C)^2 \cap k^\times \;=\; \text{compact} \cap \text{discrete} \;=\; \text{finite}$$

Let Ξ be the latter finite set. That is, given $|\alpha| = 1$, there is $a \in k^\times$ such that $a \cdot \alpha \in C - C$, and $\xi \in \Xi$ (ξ is ab just above) such that $(\xi a^{-1}) \cdot \alpha^{-1} \in C - C$. That is,

$$(a\cdot\alpha,\; (a\cdot\alpha)^{-1}) \;\in\; (C-C) \times \xi^{-1}(C-C)$$

The topology on $\mathbb{J}$ is obtained by imbedding $\mathbb{J} \to \mathbb{A} \times \mathbb{A}$ by $\alpha \to (\alpha, \alpha^{-1})$ and taking the subset topology. Thus, for each $\xi \in \Xi$,

$$\Big((C-C) \times \xi^{-1}(C-C)\Big) \;\cap\; \mathbb{J} \;=\; \text{compact in } \mathbb{J}$$

The continuous image in $k^\times\backslash\mathbb{J}$ of each of these finitely-many compacts is compact, and their union covers the closed subset $k^\times\backslash\mathbb{J}^1$, so the latter is compact. ///

Exercise: Adapt the proof to treat *division algebras* k: one must keep track of left and right more scrupulously than was done above.