(*February 19, 2005*)

Gelfand-Kazhdan criterion

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

- Gelfand pairs (Euler subgroups)
- The Gelfand-Kazhdan criterion

It is important to know that induced representations are *multiplicity-free*, meaning contain at most one copy of a given irreducible representation, if and when this is the case.

A typical example of interest is an induced representation

$$\text{c-Ind}_H^G\,\mathbf{C}$$

of the trivial representation $\mathbf{C}$ of a closed unimodular subgroup H of a unimodular totally disconnected subgroup G. As usual, the induced representation space consists of left H-invariant, locally constant $\mathbf{C}$-valued functions which are compactly supported left modulo H.

The basic idea (which is useful already in the representation theory of finite groups) is that if no irreducible occurs twice inside a representation, then the endomorphism algebra should be *commutative*, and vice-versa (by Schur's Lemma). Unfortunately, this principle is not quite valid in general. After the necessary adaptations are made, instead we have the **Gelfand-Kazhdan criterion**.

Proofs that endomorphism (convolution) algebras are commutative most often depend upon identifying an anti-involution to interchange the order of factors, but which nevertheless acts as the identity on the algebra or suitable subalgebras. Silberger gave such an argument for the spherical Hecke algebras of p-adic reductive groups, for example. Apparently the first occurrence of the Gelfand-Kazhdan criterion idea is in I.M. Gelfand, 'Spherical functions on symmetric spaces', *Dokl. Akad. Nauk SSSR 70* (1950), pp. 5-8. An extension of that idea appeared in I.M. Gelfand and D. Kazhdan, 'Representations of the group $GL(n,k)$ where k is a local field', in *Lie Groups and their Representations*, Halsted, New York, 1975, pp. 95-118. A relatively recent survey of some of this is given by B. Gross, 'Some applications of Gelfand pairs to number theory', Bull. A.M.S. 24, no. 2 (1991), pp. 277-301.

Another typical example, in which the representation to be induced from the subgroup is non-trivial, is the proof that the **Whittaker space** is multiplicity-free. That is, for suitable 'generic' character ψ on the unipotent radical N of a minimal parabolic in a p-adic reductive group G, consider the Whittaker space

$$\mathrm{Wh}_\psi = \mathrm{Ind}_N^G\,\psi$$

(Consideration of this object was motivated in part by abstracting the classical use of Fourier coefficients of modular forms.) The catch-phrase for this multiplicity-freeness property is *uniqueness of Whittaker models*, although what is really meant is

$$\dim_C \mathrm{Hom}_G(\pi, \mathrm{Wh}_\psi) \le 1$$

for irreducibles π, since not every irreducible π imbeds into the Whittaker space.

---

1. Gelfand pairs (Euler subgroups)

We assume that the (locally-compact, Hausdorff) topological group G has a *countable basis*, and is *totally disconnected*. The space $C_c^\infty(G)$ of **test functions** on G is the space of a compactly-supported locally constant complex-valued function on G. As a colimit (that is, direct limit) of finite-dimensional complex vector spaces, this space has a uniquely determined topology. The space $C_c^\infty(G)^*$ of **distributions** on G is the complex-linear dual to $C_c^\infty(G)$. (Every linear functional is continuous.)

Let H be a closed subgroup of G, and let σ be a smooth representation of H_1 on a complex vectorspace V. Define the *compactly-induced* representation

$$\text{c-Ind}_H^G\sigma = \left\{ \begin{array}{l} \sigma\text{-valued functions } f \text{ on } G, \text{ locally constant, compactly-supported left modulo } H, \\ \text{so that } f(hg) = \sigma(h)f(g) \text{ for all } g \in G \text{ and } h \in H \end{array} \right\}$$

Also define the (not-necessarily-compactly-supported) induced representation

$$\mathrm{Ind}_H^G \sigma = \left\{ \begin{array}{l} \sigma\text{-valued functions } f \text{ on } G, \textit{ uniformly} \text{ locally constant, so that } f(hg) = \sigma(h) f(g) \\ \text{for all } g \in G \text{ and } h \in H \end{array} \right\}$$

Note that the quotient $H\backslash G$ has a right G-invariant measure, since H and G are unimodular.

Let $\mathbf{C}$ denote the trivial representation of G (or of H). Say that (G, H) is a **Gelfand pair** or equivalently that H is an **Euler subgroup** of G if, for all irreducible *admissible* representations π of G, we have

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \mathbf{C}) \times \dim \mathrm{Hom}_G(\check{\pi}, \mathrm{Ind}_H^G \mathbf{C}) \le 1$$

where $\check{\pi}$ is the contragredient of π. By Frobenius Reciprocity, this condition is equivalent to

$$\dim \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \mathbf{C}) \times \dim \mathrm{Hom}_H(\mathrm{Res}_H^G \check{\pi}, \mathbf{C}) \le 1$$

The separate questions about multiplicities seem more difficult to address.

More generally, we will be interested in *trying* to show that

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \psi) \le 1$$

for a one-dimensional representation ψ of H. In fact, again, instead we will prove a symmetrized assertion involving the contragredient $\check{\pi}$ of π as well.

---

2. Gelfand-Kazhdan criterion

We treat $\mathrm{Ind}_H^G \psi$ with ψ one-dimensional. The case that ψ is the trivial one-dimensional representation $\mathbf{C}$ is already of considerable interest, and the following argument can be meaningfully read assuming that ψ is trivial.

An **anti-involution** σ on a group G is a bijection $G \to G$ so that

$$(gh)^\sigma = h^\sigma g^\sigma$$

Theorem: (Gelfand-Kazhdan) Let ψ and ψ^σ be one-dimensional representations of a closed unimodular subgroup H of G. Suppose that there is an anti-involution σ of G so that σ stabilizes H, $\psi(h^\sigma) = \psi^\sigma(h)$, and σ acts trivially on *all* distributions u so that

$$u(L_h \eta) = \psi(h) \cdot u(\eta)$$

$$u(R_h \eta) = \psi^\sigma(h)^{-1} \cdot u(\eta)$$

for $\eta \in C_c^\infty(G)$. Then

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \psi) \times \dim \mathrm{Hom}_H(\mathrm{Res}_H^G \check{\pi}, \psi^\sigma) \le 1$$

Proof: By Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \psi) \approx \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \psi)$$

Thus, supposing that there are non-trivial intertwinings from π and $\check{\pi}$ to $\mathrm{Ind}_H^G \psi_i$ (with $i = 1, 2$, respectively) amounts to supposing that we have non-zero H-homomorphisms

$$s : \pi \to \psi$$

$$t : \check{\pi} \to \psi^\sigma$$

We obtain G-homomorphisms

$$S : \mathrm{C}_c^\infty \to \check{\pi}$$

$$T : \mathrm{C}_c^\infty \to \check{\check{\pi}}$$

by taking (for $\eta \in \mathrm{C}_c^\infty(G)$, $v \in \pi$, $\lambda \in \check{\pi}$)

$$(S\eta)(v) = \int_G \eta(x)\, s(x \cdot v)\, dx$$

$$(T\eta)(\lambda) = \int_G \eta(x)\, t(x \cdot \lambda)\, dx$$

The admissibility of π implies that π is *reflexive*, that is, that

$$\check{\check{\pi}} \approx \pi$$

By direct computation, right translation R_g by $g \in G$, and left translation L_h by $h \in H$ interact with S and T by

$$S(R_g\eta) = g \cdot (S\eta)$$

$$T(R_g\eta) = g \cdot (T\eta)$$

$$S(L_h\eta) = \psi(h) \cdot S\eta$$

$$T(L_H\eta) = \psi^\sigma(h) \cdot T\eta$$

The first assertion, for example, is verified as follows: for $v \in \pi$,

$$S(R_g\eta)(v) = \int_G \eta(xg)\, s(x \cdot v)\, dx = \int_G \eta(x)\, s(xg^{-1} \cdot v)\, dx = S\eta(g^{-1}v)$$

by replacing x by xg^{-1}. The last expression is simply the contragredient action of g, that is, on $\check{\pi}$. The left H-invariance follows by

$$S(L_h\eta)(v) = \int_G \eta(h^{-1}x)\, s(x \cdot v)\, dx = \int_G \eta(x)\, s(hx \cdot v)\, dx = \int_G \eta(x)\, \psi(h)\, s(x \cdot v)\, dx = S(\eta)(v)$$

where we replace x by hx, and then invoke the H-equivariance of s. The corresponding assertions for T are proven similarly. That is, both S and T are left H-equivariant as indicated, and are right G-equivariant. In particular, they do give G-homomorphisms from $\mathrm{C}_c^\infty(G)$ (with right regular representation) to π and $\check{\pi}$, respectively.

Let $\langle , \rangle$ be the usual canonical bilinear map

$$\langle , \rangle : \pi \times \check{\pi} \to \mathbf{C}$$

by

$$\langle v, \lambda \rangle = \lambda(v)$$

and denote the induced linear map

$$\langle , \rangle : \pi \otimes \check{\pi} \to \mathbf{C}$$

by the same symbol. Define

$$B = \langle , \rangle \circ (T \otimes S) : \mathrm{C}_c^\infty(G) \otimes \mathrm{C}_c^\infty(G) \to \pi \otimes \check{\pi} \to \mathbf{C}$$

(Note the reversal of S and T.) The functional B is in the space of distributions $C_c^{\infty *}(G \times G)$, is left $(H, \psi^\sigma) \times (H, \psi)$-equivariant and right G^Δ-invariant, where G^Δ is the diagonal copy of G in $G \times G$. Note the reversal of ψ and ψ^σ due to the reversal of S and T.

Lemma: With B, t, S as above, for α, β in $C_c^\infty(G)$, we have

$$B(\alpha \otimes \beta) = t(S(\beta * \alpha))$$

Proof: Apart from an issue of interchange of integration and application of linear operators, this is a purely formal computation:

$$B(\alpha \otimes \beta) = T\alpha(S\beta) = \int_G \alpha(x)\, t(x \cdot S\beta)\, dx = \int_G \alpha(x)\, t(S(R_x^{-1} \cdot \beta))\, dx$$

by the G-equivariance of S. Moving the integral inside $t \circ S$, this becomes

$$(t \circ S)\left(\int_G \alpha(x)\, R_x^{-1} \cdot \beta\, dx\right) = (t \circ S)(\beta * \alpha)$$

by definition of convolution.

To justify the exchange of integration and application of the operator $t \circ S$, note that the indicated integral is actually a finite sum. Or, as a more general approach, start from the observation that $C_c^\infty(G)$ is a countable colimit of finite-dimensional vectorspaces, so is an LF-space, and thus is quasi-complete. This implies that the Gelfand-Pettis integral of any compactly-supported continuous $C_c^\infty(G)$-valued function f exists. (!) Thus, further, for any continuous linear operator L on $C_c^\infty(G)$,

$$L\left(\int_G f(x)\, dx\right) = \int_G L(f(x))\, dx$$

The desired exchange is a special case of this. ///

Corollary: The distribution u on G defined by $u(\eta) = t(S(\eta))$ is left H-equivariant by ψ and right H-equivariant by $(\psi^\sigma)^{-1}$, meaning that

$$u(L_h\eta) = \psi(h) \cdot u(\eta)$$
$$u(R_h\eta) = \psi^\sigma(h)^{-1} \cdot u(\eta)$$

Proof: Given $\eta \in C_c^\infty(G)$ and given $h \in H$, let $\beta \in C_c^\infty(G)$ be such that

$$R_h\eta * \beta = R_h\eta$$

For example, if K is a small-enough compact open subgroup of G so that $R_h\eta$ is left K-invariant, take β to be $\text{meas}\,(K)^{-1}$ on K and 0 off K. Then

$$u(R_h\eta) = (t \circ S)(R_h\eta) = (t \circ S)((R_h\eta) * \beta) = (t \circ S)(\eta * L_h^{-1}\beta) = B(L_h^{-1}\beta, \eta)$$

by the way that convolution and translations interact. Then this is

$$B(L_h^{-1}\beta \otimes \eta) = \psi^\sigma(h)^{-1} \cdot B(\beta \otimes \eta)$$

by the left H-equivariance of B by ψ^σ in its first argument. Going back by the same procedure, we conclude that

$$u(R_h\eta) = \psi^\sigma(h)^{-1} \cdot u(\eta)$$

Even more simply, for β so that

$$(L_h\eta) * \beta = L_h\eta$$

we compute that

$$u(L_h\eta) = B(\beta \otimes L_h\eta) = \psi(h) \cdot B(\beta \otimes \eta) = \psi(h) \cdot u(\eta)$$

This proves the equivariance. ///

For anti-involution σ define an action on $\eta \in \mathrm{C}_c^\infty(G)$ by

$$\eta^\sigma(g) = \eta(g^\sigma)$$

As usual, we have

Lemma: For α, β in $\mathrm{C}_c^\infty(G)$,

$$(\alpha * \beta)^\sigma = \beta^\sigma * \alpha^\sigma$$

Proof: This is by computation: for $g \in G$

$$(\alpha * \beta)^\sigma(g) = (\alpha * \beta)(g^\sigma) = \int_G \alpha(g^\sigma x^{-1})\,\beta(x)\,dx = \int_G \alpha(x^{-1})\,\beta(xg^\sigma)\,dx$$

$$= \int_G \alpha(x)\,\beta(x^{-1}g^\sigma)\,dx = \int_G \alpha(x^\sigma)\,\beta((gx^{-1})^\sigma)\,dx$$

replacing x successively by xg, x^{-1}, and x^σ. This is

$$\int_G \alpha^\sigma(x)\,\beta^\sigma(gx^{-1})\,dx = (\beta^\sigma * \alpha^\sigma)(g)$$

as claimed. ///

Corollary:

$$B(\alpha \otimes \beta) = B(\beta^\sigma \otimes \alpha^\sigma)$$

Proof: The hypothesis of σ-invariance of u is (by definition) that

$$u(\eta^\sigma) = u(\eta)$$

for all $\eta \in \mathrm{C}_c^\infty(G)$. Therefore,

$$B(\alpha \otimes \beta) = u(\beta * \alpha) = u((\beta * \alpha)^\sigma) = u(\alpha^\sigma * \beta^\sigma) = B(\beta^\sigma \otimes \alpha^\sigma)$$

///

Corollary: For η in $\mathrm{C}_c^\infty(G)$, $T\eta = 0$ implies $S(\eta^\sigma) = 0$, and similarly $S\eta = 0$ implies $T(\eta^\sigma) = 0$.

Proof: Suppose that $T\alpha = 0$. Then for all β in $\mathrm{C}_c^\infty(G)$

$$0 = \langle T\eta, S\beta\rangle = B(\eta \otimes \beta) = B(\beta^\sigma \otimes \eta^\sigma) = \langle T\beta^\sigma, S\eta^\sigma\rangle$$

by the identity of the previous corollary. That is, $S\eta^\sigma$ gives the trivial linear functional on π, so must be 0 in $\check{\pi}$. The other assertion is similarly proven. ///

That is, $\ker T$ determines $\ker S$ and vice-versa.

Now π is irreducible, so by Schur's lemma the kernel of $S : \mathrm{C}_c^\infty(G) \to \pi$ determines S uniquely up to a constant multiple. Since π is irreducible admissible, the same assertion holds for T. And we can certainly recover $s : \pi \to \mathbf{C}$ unambiguously from S (and t from T), as follows. Given $v \in \pi$, let η be $\mathrm{meas}\,(K)^{-1}$ times the characteristic function of K, where K is any sufficiently small compact open subgroup of G. Then

$$(S\eta)(v) = \int_G \eta(x)\,s(x \cdot v)\,dx = s(v)$$

That is, from $\ker S$ we recover S uniquely up to a constant, and then recover s uniquely up to a constant. The analogous assertion holds for $\ker T$, T, and t.

Then t certainly determines T, which determines $\ker T$. From above, $\ker T$ determines $\ker S$, which (by the previous paragraph) determines s up to a constant. We could have fixed t and let s be arbitrary, which would show that *if* the space of t's were positive-dimensional then the space of s's would be *at most* one-dimensional. The symmetrical argument reversing the role of s and t goes through in the same manner, wherein we use the assumed admissibility of π (and, thus, $\check{\pi}$). This proves the theorem. ///