(*July 3, 2006*)

Poles of half-degenerate Eisenstein series

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

- Review of a well-known example
- Simplest non-self-associate functional equation
- Half-cuspidal data
- Associate Eisenstein series

We study the meromorphic continuation and functional equations for a useful special class of half-degenerate Eisenstein series for maximal proper parabolics

$$P = P^{r-q,q} = \{\begin{pmatrix} (r-q)\text{-by-}(r-q) & * \\ 0 & q\text{-by-}q \end{pmatrix}\}$$

on GL_r over a number field k, involving *cuspidal data* on the lower-right GL_q factor of the (standard) Levi component and *non-cuspidal* data on the upper-left GL_{r-q} factor. In particular, with a normalizing factor of a (single) Godement-Jacquet zeta integral attached to the cuspidal data, *there are no poles.*

As a simple example, let f be a cuspform on GL_q for $1 < q < r$, and define

$$\varphi_{s,f}(p) = \left|\frac{(\det a)^q}{(\det x)^{r-q}}\right|^s \cdot f(x) \qquad \left(\text{for } p = \begin{pmatrix} a & b \\ 0 & x \end{pmatrix} \in P^{r-q,q}\right)$$

and extend φ to all of $GL_r(\mathbb{A})$ by right invariance with respect to the standard maximal compact subgroups K_ν at all places ν of k. Form the Eisenstein series

$$E_{s,f}(g) = \sum_{\gamma \in P_k \backslash GL_r(k)} \varphi_{s,f}(g) \qquad (\text{where } P = P^{r-q,q})$$

To normalize this Eisenstein series, let Φ be a Schwartz function on $\mathbb{A}^{q\times q}$, and suppose that Φ is right and left K_ν-invariant for all places ν. Let $\xi_f(s)$ be the Godement-Jacquet global zeta integral

$$\xi_f(s) = \int_{GL_q(\mathbb{A})} f(t^{-1})\, |\det t|^s\, \Phi(t)\, dt$$

We will show that $\xi_f(rs) \cdot E_{s,f}$ is *entire* in s.

The argument is a straightforward combination of ideas of [Godement-Jacquet 1972] (using Poisson summation, extending the Tate-Iwasawa argument for GL_1) with the Ingham-Rankin-Selberg integral Mellin representation of Eisenstein series (see [Godement 1966] for GL_2).

These Eisenstein series are *partly degenerate* in the sense that (except for extreme cases) while including some cuspidal data they do also use non-cuspidal data (determinants) on one of the simple factors of the Levi component. Thus, these Eisenstein series play no direct role in the L^2 spectral theory on the arithmetic quotients, but, rather, are *residues* of the non-degenerate Eisenstein series which *do* enter the spectral theory (as in [Langlands 1976] and [Moeglin Waldspurger 1989], [Moeglin Waldspurger 1995]).

The present special approach gives better results on possible poles of these special semi-degenerate Eisenstein series than an iterated residues viewpoint. This is useful in integral representations of L-functions.

Both the classical treatment of GL_2 and the present integral representation can be viewed as *theta correspondences*, for groups GL_n viewed as globally split forms of unitary groups.

1. *Review of a well-known example*

First we review some extremely degenerate Eisenstein series (really a *pair* of Eisenstein series), on GL_r over a number field k, with continuation properties established by Poisson summation in one step. This material is standard. The (vacuously) *cuspidal data* here is a Hecke character χ. The Eisenstein series of this section *may* have poles.

View $\mathbb{A}^r$ and k^r as *row* vectors. Let $e_1, \ldots, e_r$ the standard basis for k^r. We use the $(r-1,1)$ parabolic

$$P = \{p \in GL_r \ : \ (k \cdot e_r) \cdot p = k \cdot e_r\} \ = \ \{\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \ : \ a \in GL_{r-1}, \ d \in GL_1\} \ = \ \text{stabilizer of the line } ke_r$$

We make vectors [1] φ constructed from a Hecke character [2] χ and a Schwartz function Φ on $\mathbb{A}^r$, by

$$\varphi(g) = \chi(\det g) \int_{\mathbb{J}} \chi(t)^r \, \Phi(t \cdot e_r \cdot g) \, dt$$

The r^{th} power of χ in the integrand and the leading factor of $\chi(\det g)$ combine to give the invariance $\varphi(zg) = \varphi(g)$ for all z in the center $Z_{\mathbb{A}}$ of $GL_r(\mathbb{A})$.

By changing variables in the integral, and by the product formula, we observe the left equivariance

$$\varphi(pg) = \chi(\det pg) \int_{\mathbb{J}} \chi(t)^r \, \Phi(t \cdot e_r \cdot pg) \, dt$$

$$= \chi(\det a)\chi(d)^{1-r} \cdot \varphi(g) \qquad\qquad (\text{for } p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P_{\mathbb{A}})$$

with respect to $P_{\mathbb{A}}$. [3]

Note that the normalization is *not* $\varphi(1) = 1$, but, rather,

$$\varphi(1) = \int_{\mathbb{J}} \chi(t)^r \, \Phi(t \cdot e_r) \, dt \qquad\qquad (\text{Tate-Iwasawa zeta integral at } \chi^r)$$

Denote this zeta integral by $\xi = \xi(\chi^r, \Phi(0, *))$, indicating that it only depends upon the values of Φ along the last coordinate axis. Thus, we should construe the Eisenstein series, associated to φ in the usual fashion, as having a factor of $\xi(\chi^r, \Phi(0, *))$ included, namely

$$\xi(\chi^r, \Phi(0, *)) \cdot E(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g) \qquad\qquad (\text{convergent for } \mathrm{Re}(\chi) >> 0)$$

---

[1] These vectors lie in a parametrized family of induced representations, specifically, degenerate principal series, but this fact is not immediately necessary.

[2] As usual, a *Hecke character* is a continuous group homomorphism of $GL_1(k)\backslash GL_1(\mathbb{A})$ to $\mathbb{C}^\times$. The idele class group quotient $\mathbb{J}/k^\times = GL_1(\mathbb{A})/GL_1(k)$ can be factored as $(\mathbb{J}^1/k^\times) \times \mathbb{R}^+$, where $\mathbb{J}^1$ is ideles with idele norm 1, and $\mathbb{R}^+$ is the group of positive real numbers, imbedded in $\mathbb{J}$ by $t \longrightarrow (t^{1/N}, \ldots, t^{1/N}, 1, 1, \ldots)$, with $N = [k : \mathbb{Q}]$, and $t^{1/N}$ appears at the archimedean places. A Hecke character χ has a corresponding decomposiion $\chi(t) = \chi_1(t) \cdot |t|^s$, where χ_1 is a Hecke character *trivial* on the copy of $\mathbb{R}^+$, and $s \in \mathbb{C}$. As usual, write $\mathrm{Re}(\chi) = \mathrm{Re}(s)$.

[3] This left equivariance is exactly the left equivariance required of a vector in *degenerate principal series* representations of $GL_r(\mathbb{A})$ induced from that character $p \longrightarrow \chi(\det a)\chi(d)^{1-r}$ on $P_{\mathbb{A}}$. For now, we have no need of any properties of this representation, but it is convenient that for *generic* χ (in the sense of *typical*, rather than in the technical sense of *having a Whittaker model*) these degenerate principal series are *irreducible*. Specifically, these representations are irreducible for the complex parameter s *off a discrete set of points* (with no limit point in $\mathbb{C}$). Thus, these integral expressions inevitably produce *all* vectors in the representation. This irreducibility is non-trivial, and in the immediate sequel we will not use either it or the implied surjectivity.

Poisson summation proves the meromorphic continuation of this Eisenstein series, just as the Tate-Iwasawa version of Riemann's argument for the Euler-Riemann zeta. Let

$$\mathbb{J}^+ = \{t \in \mathbb{J} \;:\; |t| \geq 1\} \qquad \mathbb{J}^- = \{t \in \mathbb{J} \;:\; |t| \leq 1\}$$

and $g^\iota = (g^\top)^{-1}$ (transpose inverse)

$$\xi \cdot E(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g) = \chi(\det g) \sum_{\gamma \in P_k \backslash G_k} \int_{\mathbb{J}} \chi(t)^r \, \Phi(t \cdot x \cdot \gamma g)\, dt$$

$$= \chi(\det g) \sum_{\gamma \in P_k \backslash G_k} \int_{k^\times \backslash \mathbb{J}} \chi(t)^r \sum_{\lambda \in k^\times} \Phi(t \cdot \lambda e_r \cdot \gamma g)\, dt$$

$$= \chi(\det g) \int_{k^\times \backslash \mathbb{J}} \chi(t)^r \sum_{x \in k^r - 0} \Phi(t \cdot x \cdot g)\, dt$$

Let [4]

$$\Theta(g) = \sum_{x \in k^r} \Phi(t \cdot x \cdot g)$$

Then

$$\xi \cdot E(g) = \chi(\det g) \int_{k^\times \backslash \mathbb{J}^+} \chi(t)^r \, [\Theta(g) - \Phi(0)]\, dt + \chi(\det g) \int_{k^\times \backslash \mathbb{J}^-} \chi(t)^r \, [\Theta(g) - \Phi(0)]\, dt = (\text{entire}) + (?)$$

since the usual sort of estimate shows that the integral over $k^\times \backslash \mathbb{J}^+$ converges absolutely for all χ. Following Riemann *et alia*, via Poisson summation we rewrite the second part of the integral as an analogous integral over $k^\times \backslash \mathbb{J}^+$. Poisson summation asserts that

$$\sum_{x \in k^r - 0} \Phi(t \cdot x \cdot g) + \Phi(0) = |t|^{-r} |\det g|^{-1} \sum_{x \in k^r - 0} \widehat{\Phi}(t^{-1} \cdot x \cdot g^\iota) + |t|^{-r} |\det g|^{-1} \widehat{\Phi}(0)$$

using standard change-of-variables properties of the Fourier transform. [5] Let

$$\Theta'(g^\iota) = \sum_{x \in k^r} \widehat{\Phi}(t \cdot x \cdot g^\iota) \cdot$$

Then, removing the $\Phi(0)$ and $\widehat{\Phi}(0)$ terms, and replacing t by t^{-1} in the integral over $k^\times \backslash \mathbb{J}^-$ turns this integral into

$$|\det g|^{-1} \chi(\det g) \int_{k^\times \backslash \mathbb{J}^+} \left(|t| \chi(t)^{-1}\right)^r \, [\Theta'(g^\iota) - \widehat{\Phi}(0)]\, dt$$

$$-\chi(\det g) \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} \chi(t)^r \, dt + |\det g|^{-1} \chi(\det g) \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} \chi(t)^r \, |t|^{-r}\, dt$$

The integral over $k^\times \backslash \mathbb{J}^+$ is entire. Thus, the non-elementary part of the integral is converted into two *entire* integrals over $k^\times \backslash \mathbb{J}^+$ together with two elementary integrals that give the only possible poles:

$$\xi \cdot E(g) = (\text{entire}) - \chi(\det g) \Phi(0) \int_{k^\times \backslash \mathbb{J}^-} \chi(t)^r \, dt + |\det g|^{-1} \chi(\det g) \widehat{\Phi}(0) \int_{k^\times \backslash \mathbb{J}^-} \chi(t)^r \, |t|^{-r}\, dt$$

[4] As in Iwasawa-Tate theory, this function θ has a role analogous to the Jacobi theta function in the classical argument.

[5] Any non-trivial additive character ψ on $\mathbb{A}/k$, and any non-degenerate pairing $\langle , \rangle$ on $\mathbb{A}^r$ (k-valued on $k^r \times k^r$) can be used to define a Fourier transform. The specific choice does not matter, but of course the notion of *transpose* is defined via the pairing.

As in Tate-Iwasawa theory, with χ decomposed as

$$\chi(t) = \chi_1(t) \cdot |t|^s \qquad \text{(with } \chi_1 \text{ trivial on the copy of } \mathbb{R}^+ \text{ in } \mathbb{J})$$

the relatively elementary integrals can be evaluated

$$\int_{k^\times\backslash\mathbb{J}^-} |t|^{rs}\,\chi_1(t)^r\,dt = \int_{k^\times\backslash\mathbb{J}^1} \chi_1(t)^r\,dt \cdot \int_0^1 t^{rs}\,dt = \frac{1}{rs}\int_{k^\times\backslash\mathbb{J}^1} \chi_1(t)^r\,dt = \begin{cases} \frac{\mathrm{vol}\,(k^\times\backslash\mathbb{J}^1}{rs} & (\text{for } \chi_1^r = 1) \\ 0 & (\text{for } \chi_1^r \neq 1) \end{cases}$$

Similarly,

$$\int_{k^\times\backslash\mathbb{J}^-} |t|^{r(s-1)}\,\chi_1(t)^r\,dt = \begin{cases} \frac{\mathrm{vol}\,(k^\times\backslash\mathbb{J}^1}{r(s-1)} & (\text{for } \chi_1^r = 1) \\ 0 & (\text{for } \chi_1^r \neq 1) \end{cases}$$

That is, the only possible poles of $\xi \cdot E(g)$ are at $s = 0, 1$, and these occur only when $\chi_1^r = 1$, and the residues are $\Phi(0)$ at $s = 0$ and $\widehat{\Phi}(0)$ at $s = 1$ (times the volume constant):

$$\text{[1.1]} \qquad \begin{aligned} \xi(\chi^r, \Phi(0,*)) \cdot E(g) \quad = \quad & \chi(\det g) \textstyle\int_{k^\times\backslash\mathbb{J}^+} \chi(t)^r\,[\Theta(g) - \Phi(0)]\,dt \\ & + |*|\chi^{-1}(\det g^\iota) \textstyle\int_{k^\times\backslash\mathbb{J}^+} |*|\chi^{-1}(t)^r\,[\Theta'(g^\iota) - \widehat{\Phi}(0)]\,dt \\ & + \begin{cases} -\frac{\chi(\det g)\,\mathrm{vol}\,(k^\times\backslash\mathbb{J}^1)}{rs} + \frac{|*|\chi^{-1}(\det g^\iota)\,\mathrm{vol}\,(k^\times\backslash\mathbb{J}^1)}{r(s-1)} & \text{for } \chi^r = 1 \\ 0 & \text{for } \chi^r \neq 1 \end{cases} \end{aligned}$$

Thus, the possible poles of $E(g)$ itself, *without* the zeta factor $\xi(\chi^r, \Phi(0,*))$, may be at $s = 0, 1$ *and* at the *zeros* of ξ. For reasonable choices of Φ, these zeros occur only in the strip $0 < \mathrm{Re}\,(s) < \frac{1}{r}$.

[1.2] **Remark:** The estimate here on possible poles is stronger than the otherwise-natural estimates that arise when these Eisenstein series are analyzed as residues of non-degenerate Eisenstein series.

[1.3] **Remark:** Although it is a good indicator of analytic properties, the above expression for $E(g)$ requires further interpretation to give a functional equation connecting it to another Eisenstein series. This is explicated in the next section.

---

2. *Simplest non-self-associate functional equation*

Except for the case $r = 2$, the functional equation of the degenerate Eisenstein series (above) attached to the $(r-1, 1)$ parabolic does *not* relate it to itself. We will see that there is a functional equation relating it to an Eisenstein series for the $(1, r-1)$ parabolic [6]

$$Q = \{\begin{pmatrix} \text{1-by-1} & * \\ 0 & (r-1)\text{-by-}(r-1) \end{pmatrix}\}$$

Indeed, the discussion of meromorphic continuation presents g^ι inside $\widehat{\Phi}$. In the special case $r = 2$, up to a scalar, g^ι is *conjugate* to g.

[6] The fact that the diagonal block sizes in P and Q are merely permuted is no coincidence. In a broader context, we would say that these two parabolics are *associate*.

Let $E^P = E^P(\chi, \Phi)$ be the Eisenstein series of the previous section. We want to understand the Fourier-transformed part of the integral expression for E^P as being made from a left-equivariant function φ' which should be roughly like φ. An obvious candidate,

$$\varphi^?(g) = \chi(g^\iota) \int_{\mathbb{J}} \chi(t)^r \, \widehat{\Phi}(t \cdot e_r \cdot g^\iota) \, dt \qquad (?)$$

is left equivariant by the image P^ι of P under the involution ι, but P^ι is not a standard parabolic subgroup. Further, except for $r = 2$, the subgroup P^ι is not even *conjugate* to P, but is conjugate to the (standard) parabolic Q via

$$w_\circ P^\iota w_\circ = Q \qquad \left(\text{where } w_\circ = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}, \text{ anti-diagonal}\right)$$

Thus, unsurprisingly, noting that $e_1 = e_r \cdot w_\circ$, we should take

$$\varphi'(g) = \chi(g^\sigma) \int_{\mathbb{J}} \chi(t)^r \, \widehat{\Phi}(t \cdot e_1 \cdot g^\iota) \, dt = \text{left equivariant by } Q_{\mathbb{A}} \qquad (\text{note occurrence of } e_1, \text{ not } e_r)$$

As before, the normalization involves a zeta integral

$$\xi(\chi^r, \widehat{\Phi}(*, 0)) = \varphi'(1) = \int_{\mathbb{J}} \chi(t)^r \, \widehat{\Phi}(t \cdot e_1) \, dt$$

Thus, we should take the viewpoint that this zeta integral occurs as an implied factor when we form the Eisenstein series $E^Q(\chi^r, \widehat{\Phi})$ attached to φ', namely

$$\xi(\chi^r, \widehat{\Phi}(*, 0)) \cdot E^Q(g) = \sum_{\gamma \in Q_k \backslash GL_r(k)} \varphi'(\gamma \, g)$$

The visible symmetry in the integral expression for E^P from above gives the functional equation

$$\xi(\chi^r, \Phi(*, 0)) \;\cdot\; E^P(\chi, \Phi) \;=\; \xi((|*|\chi^{-1})^{-1}, \widehat{\Phi}(0, *)) \;\cdot\; E^Q(|*|\chi^{-1}, \widehat{\Phi})$$

---

3. *Half-cuspidal data*

Take $1 < q < r$, and consider the standard maximal proper parabolic

$$P = P^{r-q,q} = \{ \begin{pmatrix} (r-q)\text{-by-}(r-q) & * \\ 0 & q\text{-by-}q \end{pmatrix} \}$$

We will demonstrate the meromorphic continuation of an Eisenstein series attached to *degenerate* data on the copy of GL_{r-q}, and to *cuspidal* data on the copy of GL_r. The general form of the discussion is parallel to the previous.

Let f be a cuspform on $GL_q(\mathbb{A})$, in the strong sense that f is in $L^2(GL_q(k)\backslash GL_q(\mathbb{A})^1)$, f meets the Gelfand-Fomin condition [7]

$$\int_{N_k \backslash N_{\mathbb{A}}} f(ng) \, dn = 0 \qquad (\text{for almost all } g)$$

[7] In words, this condition is *integrating to* 0 *over horocycles.*

and f generates an irreducible representation of $GL_q(k_\nu)$ locally at all places ν of k. For a Schwartz function Φ on $\mathbb{A}^{q\times r}$ and Hecke character χ, let

$$\varphi(g) = \varphi_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(\mathbb{A})} f(h^{-1})\,\chi(\det h)^r\,\Phi(h\cdot[0_{q\times(r-q)}\ 1_q]\cdot g)\,dh$$

This function φ has the same central character as f. It is left invariant by the adele points of the unipotent radical

$$N = \{\begin{pmatrix} 1_{r-q} & * \\ 0 & 1_r \end{pmatrix}\} \qquad \text{(unipotent radical of } P = P^{r-q,q})$$

The function φ is left invariant under the k-*rational* points of the standard Levi component of P,

$$M = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a\in GL_{r-q},\ d\in GL_r\}$$

To understand the normalization, observe that

$$\xi(\chi^r, f, \Phi(0,*)) = \varphi(1) = \int_{GL_q(\mathbb{A})} f(h^{-1})\,\chi(\det h)^r\,\Phi(h\cdot[0_{q\times(r-q)}\ 1_q])\,dh$$

is a Godement-Jacquet zeta integral for the standard L-function attached to the cuspform f (or perhaps a contragredient). Thus, the corresponding Eisenstein series includes this zeta integral as a factor, so write [8]

$$\xi(\chi^r, f, \Phi(0,*))\ \cdot\ E^P_{\chi,f,\Phi}(g)\ =\ \sum_{\gamma\in P_k\backslash GL_r(k)} \varphi(\gamma\, g) \qquad \text{(convergent for } \mathrm{Re}(\chi) >> 0)$$

Now prove the meromorphic continuation via Poisson summation.

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g)$$

$$= \chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{\alpha\in GL_q(k)} \Phi(h^{-1}\cdot[0\ \alpha]\cdot g)\,dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{full rank}} \Phi(h^{-1}\cdot y\cdot g)\,dh$$

---

[8] To understand φ, observe that for $p = \begin{pmatrix} a & b \\ 0 & x \end{pmatrix}$ in $P_{\mathbb{A}}$,

$$\varphi(p) = \chi(\det p)^q \int_{GL_q(\mathbb{A})} f(h^{-1})\,\chi(\det h)^r\,\Phi(h\cdot[0_{q\times(r-q)}\ x])\,dh$$

$$= \chi(\det p)^q\,\chi(\det x)^{-r} \int_{GL_q(\mathbb{A})} f(xh)\,\chi(\det h^{-1})^r\,\Phi(h^{-1}\cdot[0_{q\times(r-q)}\ 1_q])\,dh$$

by replacing h by h^{-1} then replacing h by xh. For simplicity, suppose that f is a *spherical* vector everywhere locally,

$$\int_{GL_q(\mathbb{A})} f(xh)\,\chi(\det h^{-1})^r\,\Phi(h^{-1}\cdot[0_{q\times(r-q)}\ 1_q])\,dh = f(x)\cdot \int_{GL_q(\mathbb{A})} f(h)\,\chi(\det h^{-1})^r\,\Phi(h^{-1}\cdot[0_{q\times(r-q)}\ 1_q])\,dh$$

and the last integral is a Godement-Jacquet zeta integral $\xi(\chi^r, f, \Phi(0,*))$.

The Gelfand-Fomin condition on f will compensate for the otherwise-irksome full-rank constraint. Anticipating that we can drop the rank condition suggests that we define

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

Now we show (as in Godement-Jacquet) that the non-full-rank terms integrate to 0. [9]

[3.1] Proposition: For f a cuspform, less-than-full-rank terms integrate to 0, that is,

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{rank}\ <q} \Phi(h^{-1}\cdot y\cdot g)\,dh = 0$$

Proof: Since this is asserted for arbitrary Schwartz functions Φ, we can take $g=1$. By linear algebra, given $y_o \in k^{q\times r}$ of rank ℓ, there is $\alpha\in GL_q(k)$ such that

$$\alpha\cdot y_0 = \begin{pmatrix} \text{rank-}\ell\ \ell\text{-by-}r\ \text{block} \\ 0_{(q-\ell)\times r}\end{pmatrix}$$

Thus, without loss of generality fix y_0 of the latter shape. Let Y be the orbit of y_0 under left multiplication by the rational points of the parabolic

$$P^{\ell,q-\ell} = \begin{pmatrix} \ell\text{-by-}\ell & * \\ 0 & (q-\ell)\text{-by-}(q-\ell)\end{pmatrix} \subset GL_q$$

This is some set of matrices of the same shape as y_0. Then the subsum over $GL_q(k)\cdot y_0$ is

$$\int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in GL_q(k)\cdot y_0} \Phi(h^{-1}\cdot y)\,dh = \int_{P_k^{\ell,q-\ell}\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\sum_{y\in Y}\Phi(h^{-1}\cdot y)\,dh$$

Let N and M be the unipotent radical and standard Levi component of $P^{\ell,q-\ell}$, namely, usual,

$$N = \begin{pmatrix} 1_\ell & * \\ 0 & 1_{q-\ell}\end{pmatrix} \qquad M = \begin{pmatrix} \ell\text{-by-}\ell & 0 \\ 0 & (q-\ell)\text{-by-}(q-\ell)\end{pmatrix}$$

Then the integral can be rewritten as an iterated integral

[9] There are issues of convergence. First, for $\mathrm{Re}(\chi)$ sufficiently large, the integral

$$\chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\,\Theta(h,g)\,dh$$

is absolutely convergent. Also, we have the integrals analogous to integrals over $k^\times\backslash\mathbb{J}^+$. That is, let $GL_q^+ = \{h\in GL_q(\mathbb{A}) : |\det h|\ge 1\}$. Then, for *arbitrary* χ, using the fact that cuspforms f are of rapid decay in Siegel sets,

$$\chi(\det g)^q \sum_{\gamma\in P_k\backslash GL_r(k)} \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r} \sum_{y\in k^{q\times r},\ \text{full rank}} \Phi(h^{-1}\cdot y\cdot g)\,dh$$

is absolutely convergent.

$$\int_{N_k M_k \backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r} \sum_{y\in Y} \Phi(h^{-1}\cdot y)\,dh$$

$$= \int_{N_\mathbb{A} M_k \backslash GL_q(\mathbb{A})} \sum_{y\in Y} \int_{N_k\backslash N_\mathbb{A}} f(nh)\,\chi(\det nh)^{-r}\,\Phi((nh)^{-1}\cdot y)\,dn\,dh$$

$$= \int_{N_\mathbb{A} M_k \backslash GL_q(\mathbb{A})} \sum_{y\in Y} \chi(\det h)^{-r}\,\Phi(h^{-1}\cdot y) \left(\int_{N_k\backslash N_\mathbb{A}} f(nh)\,dn\right) dh$$

since all fragments but $f(nh)$ in the integrand are left invariant by $N_\mathbb{A}$. But the inner integral of $f(nh)$ is 0, by the Gelfand-Fomin condition, so the whole is 0. ///

Let ι denote the transpose-inverse involution(s). Poisson summation gives

$$\Theta_\Phi(h,g) = \sum_{y\in k^{q\times r}} \Phi(h^{-1}\cdot y\cdot g)$$

$$= |\det(h^{-1})^\iota|^r\,|\det g^\iota|^q \sum_{y\in k^{q\times r}} \widehat{\Phi}((h^\iota)^{-1}\cdot y\cdot g^\iota) \;=\; |\det(h^{-1})^\iota|^r\,|\det g^\iota|^q\,\Theta_{\widehat{\Phi}}(h^\iota,g^\iota)$$

As with Θ_Φ, the not-full-rank summands in $\Theta_{\widehat{\Phi}}$ integrate to 0 against cuspforms. Thus, letting

$$GL_q^+ = \{h\in GL_q(\mathbb{A}) \;:\; |\det h|\geq 1\} \qquad\qquad GL_q^- = \{h\in GL_q(\mathbb{A}) \;:\; |\det h|\leq 1\}$$

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g) = \chi(\det g)^q \int_{GL_q(k)\backslash GL_q(\mathbb{A})} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh + \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$= \quad \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$+\;\chi(\det g)^q \int_{GL_q(k)\backslash GL_q^-} |\det(h^{-1})^\iota|^r\,|\det g^\iota|^q\,f(h)\,\chi(\det h)^{-r}\,\Theta_{\widehat{\Phi}}(h^\iota,g^\iota)\,dh$$

By replacing h by h^ι in the second integral we convert it to an integral over $GL_q(k)\backslash GL_q^+$, and the whole is

$$\xi(\chi^r, f, \Phi(0,*))\cdot E^P_{\chi,f,\Phi}(g) \;=\; \chi(\det g)^q \int_{GL_q(k)\backslash GL_q^+} f(h)\,\chi(\det h)^{-r}\,\Theta_\Phi(h,g)\,dh$$

$$+\;|*|\chi^{-1}(\det g^\iota)^q \int_{GL_q(k)\backslash GL_q^+} f(h^\iota)\;\;|*|\chi^{-1}(\det h^\iota)^{-r}\,\Theta_{\widehat{\Phi}}(h,g^\iota)\,dh$$

Since $f\circ\iota$ is a cuspform, the second integral is entire in χ. Thus, we have proven

$$\xi(\chi^r, f, \Phi(0,*))\;\cdot E^P_{\chi,f,\Phi} \quad \text{is entire}$$

[3.2] Remark: Except for the extreme case $q = r-1$, these Eisenstein series are degenerate, so occur only as residues of (purely) cuspidal-data Eisenstein series. Assessing poles of residues seems less effective in the present special circumstances than the above argument.

4. *Associate Eisenstein series*

The functional equation of the Eisenstein series $E^P_{\chi,f,\Phi}$ relates it not to itself but to an Eisenstein series for the parabolic Q associate to P, namely [10]

$$Q = P^{q,r-q} = \{\begin{pmatrix} q\text{-by-}q & * \\ 0 & (r-q)\text{-by-}(r-q) \end{pmatrix}\}$$

The functions

$$\varphi'(g) = \chi(\det g^\iota)^q \int_{GL_q(\mathbb{A})} f(h^\iota)\,\chi(\det h)^{-r}\,\widehat{\Phi}(h^{-1}\cdot[1_q\;0_{q\times(r-q)}]\cdot g^\iota)\,dh$$

arising in the meromorphic continuation for $E^P_{\chi,f,\Phi}$ are vectors in induced representations for Q. The values $\varphi'(1)$ are Godement-Jacquet zeta integrals

$$\xi(\chi^r, f\circ\iota, \widehat{\Phi}(*\,0)) = \int_{GL_q(\mathbb{A})} f(h^\iota)\,\chi(\det h)^{-r}\,\widehat{\Phi}(h^{-1}\cdot[1_q\;0_{q\times(r-q)}])\,dh$$

Then form an Eisenstein series by

$$\xi(\chi^r, f\circ\iota, \widehat{\Phi}(*\,0))\cdot E^Q_{\chi,f\circ\iota,\widehat{\Phi}} = \sum_{\gamma\in Q_k\backslash GL_r(k)} \varphi'(\gamma g)$$

The expression above that proves the meromorphic continuation also has a visible symmetry, giving the functional equation

$$\xi(\chi^r, f, \Phi(0\,*))\;\cdot\;E^P_{\chi,f,\Phi} \;=\; \xi((|*|\chi^{-1})^r, f\circ\iota, \widehat{\Phi}(*\,0))\;\cdot\;E^Q_{|*|\chi^{-1},f\circ\iota,\widehat{\Phi}}$$

[Godement 1966] R. Godement, *Decomposition of $L^2(\Gamma\backslash G)$ for $\Gamma = SL(2,Z)$*, in Proc. Symp. Pure Math. **9** (1966), AMS, 211-24.

[Godement Jacquet 1972] R. Godement, H. Jacquet, *Zeta functions of Simple Algebras*, Lecture Notes in Math. **260**, Springer-Verlag, 1972.

[Jacquet 1983] H. Jacquet, *On the residual spectrum of $GL(n)$*, in Lie Group Representations, II, Lecture Notes in Math. **1041**, Springer-Verlag, 1983.

[Langlands 1976] R. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, **544**, Springer-Verlag, Berlin and New York, 1976.

[Moeglin Waldspurger 1989] C. Moeglin, J.-L. Waldspurger, *Le spectre résiduel de $GL(n)$*, Ann. Sci. Éc. Norm. Sup. **22** (1989), 605-674.

[Moeglin Waldspurger 1995] C. Moeglin, J.-L. Waldspurger, *Spectral Decompositions and Eisenstein series*, Cambridge University Press, 1995.

[10] This applies even in the symmetrical case $r-q=q$, since the roles of the cuspidal data and degenerate data are reversed in the functional equation.