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Universality of Holomorphic Discrete Series

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The goal here is to recover an apocryphal result on the structure of **holomorphic discrete series** representations of symplectic groups $Sp_n(\mathbf{R})$ and unitary groups $U(p,q)$ for *sufficiently high highest weight of the lowest K-type*. The same sort of argument applies to other groups of hermitian type, for example the classical groups $O(n,2)$ and $O^*(2n)$.

For $Sp(n)$, the maximal compact is isomorphic to $U(n)$. For ρ with highest weight $(m_1,\ldots,m_n)$ it is sufficient to assume that

$$m_1 \geq \ldots \geq m_n \geq n$$

to reach the conclusions below. For $U(p,q)$, the maximal compact is $U(p)\times U(q)$, and for ρ with highest weight $(m_1,\ldots,m_p)\times(m'_1,\ldots,m'_q)$ it is sufficient to assume that

$$m_1 \geq \ldots \geq m_p \geq \frac{p+q-1}{2} \quad \text{and} \quad m'_1 \geq \ldots \geq m'_q \geq \frac{p+q-1}{2}$$

We prove that the *universal* $(\mathbf{g},K)$-module with a K-type ρ annihilated by the (lowering) operators $\mathbf{p}^-$ is irreducible for such ρ. Thus, for such ρ, the K-isomorphism class of ρ determines the $(\mathbf{g},K)$-isomorphism class of the holomorphic discrete series with lowest K-type ρ.

In particular, as a $\mathcal{U}(\mathbf{p}^+)$-module and K-module, the whole $(\mathbf{g},K)$ module is

$$\mathcal{U}(\mathbf{p}^+)\otimes_{\mathbf{C}} \rho$$

These structural results are essential in study of Maaß-Shimura differential operators on automorphic forms.

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1. Finite-dimensional representations of $U(n)$

Let $\mathbf{n}$ be the upper-triangular matrices in the complexification of the Lie algebra of $U(n)$ (identified with all n-by-n complex matrices), and $\mathbf{m}$ the diagonal matrices. Let e_{ii} be the element of $\mathbf{m}$ with a 1 at the i^{th} place on the diagonal and zeros elsewhere. An eigenvector v for the action of $\mathbf{m}$ in a finite-dimensional representation ρ of $U(n)$ is a **weight vector**. There are integers m_i such that

$$e_{ii}\cdot v = m_i \cdot v$$

The n-tuple $(m_1,\ldots,m_n)$ is the **weight** of the vector v. A **highest weight vector** in ρ is a (weight) vector $v\in\rho$ such that also

$$\mathbf{n}\cdot v = 0$$

Recall that any n-tuple $(m_1,\ldots,m_n)$ of integers with

$$m_1 \geq m_2 \geq \ldots \geq m_n$$

can occur as the highest weight of an irreducible, and uniquely determines the isomorphism class of the representation.

The **standard** representation σ of $U(n)$ is left matrix multiplication on $\mathbf{C}^n$, and has highest weight vector $(1,0,\ldots,0)$. The tensor product $\sigma\otimes\rho$ of the standard representation σ with an irreducible ρ with highest weight $(m_1,\ldots,m_n)$ is the direct sum of the irreducibles with highest weights

$$(m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_n)$$

where we *omit* an alleged highest weight if by mischance

$$m_{i-1} < m_i + 1$$

which would violate the inequality required of highest weights. In the same vein, the tensor product of the second exterior power $\wedge^2\sigma$ with ρ is the sum of irreducibles with highest weights

$$(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_{j-1}, m_j + 1, m_{j+1}, \ldots, m_n)$$

where the two indices $i < j$ vary over all indices so that the indicated expression still satisfies $m_1 \geq \ldots \geq m_n$. The second symmetric power $\mathrm{Sym}^2\sigma$ has highest weight $(2, 0, \ldots, 0)$. It arises as

$$\sigma^{\otimes 2} \approx \mathrm{Sym}^2\sigma \oplus \wedge^2\sigma$$

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2. Eigenvalues of the Casimir operator for $Sp(n, \mathbf{R})$

The point here is direct computation of the eigenvalues of (essentially) the Casimir operator on certain representation spaces.

Let $\mathbf{g}$ be the Lie algebra of $G = Sp(n, \mathbf{R})$, where the latter is the isometry group of the standard alternating form given by the $2n$-by-$2n$ matrix

$$J = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

Let c be the **Cayley element** in the complexification

$$c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

The copy K of $U(n)$ in G is

$$K = \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + iB \in U(n) \}$$

and

$$c^{-1}Kc = \{ \begin{pmatrix} A + iB & 0 \\ 0 & (A+iB)^{\top -1} \end{pmatrix} : A + iB \in U(n) \}$$

The center

$$Z = \{ \begin{pmatrix} a \cdot 1_n & b \cdot 1_n \\ -b \cdot 1_n & a \cdot 1_n \end{pmatrix} : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in SO(2) \}$$

of K decomposes $\mathbf{g_C}$ into three pieces $\mathbf{p}^+$, $\mathbf{p}^-$, and $\mathbf{k_C}$, best described by their images under conjugation by the Cayley element:

$$(Ad\, c^{-1})(\mathbf{p}^+) = c^{-1}\mathbf{p}^+c = \{ \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \quad \text{(with } S \text{ symmetric)} \}$$

$$(Ad\, c^{-1})(\mathbf{p}^-) = c^{-1}\mathbf{p}^-c = \{ \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \quad \text{(with } S \text{ symmetric)} \}$$

and the complexified Lie algebra of K

$$(Ad\, c^{-1})(\mathbf{k_C}) = c^{-1}\mathbf{k}_{\mathbf{C}}^{-}c = \{ \begin{pmatrix} \theta & 0 \\ 0 & -\theta^\top \end{pmatrix} \}$$

The elements of $\mathbf{p}^+$ are the **raising operators** and the elements of $\mathbf{p}^-$ are the **lowering operators**. The presentation just above makes clear that as a K representation space $\mathbf{p}^+$ is $\mathrm{Sym}^2\sigma$, having highest weight $(2, 0, \ldots, 0)$, and $\mathbf{p}^-$ has highest weight $(0, \ldots, 0, -2)$.

Let

$$\langle \alpha, \beta \rangle = \mathrm{tr}\,(\alpha\beta)$$

be the (pseudo-) Killing form on $\mathbf{g} \times \mathbf{g}$. For a basis $\{E_i\}$ of $\mathbf{g}$ or $\mathbf{g}_{\mathbf{C}}$, let E_i^* be a dual basis with respect to the (pseudo-) Killing form. Then (up to a constant) the **Casimir operator** in the center of the universal enveloping algebra $\mathcal{U}(\mathbf{g})$ is

$$C = \sum_i E_i \circ E_i^*$$

(This convenient normalization is not quite the canonical one, but we use it throughout.)

Let V be a $(\mathbf{g}, K)$-module generated by an irreducible K-type ρ, and suppose $\mathbf{p}^- \cdot \rho = \{0\}$. Call V a **holomorphic discrete series** module with **lowest K-type** ρ.

Lemma: Let V be a holomorphic discrete series module with lowest K-type ρ. Then the Casimir operator acts on V with eigenvalue

$$\lambda(\rho) = \frac{1}{2} \sum_{1 \le i \le n} m_i(m_i/2 - i)$$

where ρ has highest weight $(m_1, \ldots, m_n)$.

Proof: Let e_{ij} be the n-by-n matrix with 0 everywhere but for a 1 at the ij^{th} place. We choose a basis for $\mathbf{g}_{\mathbf{C}}$ consisting of

$$\begin{aligned}
X_{ii} &= \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix} && \in \mathbf{p}^+ \quad (1 \le i \le n) \\
X_{ij} &= \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix} && \in \mathbf{p}^+ \quad (1 \le i < j \le n) \\
Y_{ii} &= \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix} && \in \mathbf{p}^- \quad (1 \le i \le n) \\
Y_{ij} &= \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix} && \in \mathbf{p}^- \quad (1 \le i < j \le n) \\
\theta_{ij} &= \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix} && \in \mathbf{k}_{\mathbf{C}} \quad (1 \le i \le n,\ 1 \le j \le n)
\end{aligned}$$

Note that the corresponding dual basis with respect to the Killing form is

$$\begin{aligned}
X_{ii}^* &= Y_{ii} \\
X_{ij}^* &= Y_{ij}/2 \quad (i \ne j) \\
Y_{ii}^* &= X_{ii} \\
Y_{ij}^* &= X_{ij}/2 \quad (i \ne j) \\
\theta_{ij}^* &= \theta_{ji}/2
\end{aligned}$$

Then the Casimir operator is

$$C = \sum_{ij} \theta_{ij}\theta_{ij}^* + \sum_{i \le j} X_{ij}X_{ij}^* + \sum_{i \le j} Y_{ij}Y_{ij}^*$$

$$= \sum_{ij} \theta_{ij}\theta_{ji}/2 + \sum_i X_{ii}Y_{ii} + \sum_{i<j} X_{ij}Y_{ji}/2 + \sum_i Y_{ii}X_{ii} + \sum_{i<j} Y_{ij}X_{ij}/2$$

On any vector annihilated by $\mathbf{p}^-$, the operators Y_{ij} are all 0, so on such a vector this simplifies to

$$\sum_{ij} \theta_{ij}\theta_{ji}/2 + \sum_i Y_{ii}X_{ii} + \sum_{i<j} Y_{ij}X_{ij}/2$$

And since

$$Y_{ii}X_{ii} = [Y_{ii}, X_{ii}] + X_{ii}Y_{ii} = -\theta_{ii} + X_{ii}Y_{ii}$$

and

$$Y_{ij}X_{ij} = [Y_{ij}, X_{ij}] + X_{ij}Y_{ij} = -\theta_{ii} - \theta_{jj} + X_{ij}Y_{ij}$$

on vectors annihilated by $\mathbf{p}^-$ this simplifies further to

$$\sum_{ij} \theta_{ij}\theta_{ji}/2 - \sum_i \theta_{ii} - \sum_{i<j} (\theta_{ii} + \theta_{jj})/2$$

Next, consider the action of this on the highest weight vector v_o annihilated by all θ_{ij} with $i < j$. First, about half the quadratic terms disappear, giving

$$\sum_i \theta_{ii}^2/2 + \sum_{i<j} \theta_{ij}\theta_{ji}/2 - \sum_i \theta_{ii} - \sum_{i<j} (\theta_{ii} + \theta_{jj})/2$$

Using the relations

$$\theta_{ij}\theta_{ji} = [\theta_{ij}, \theta_{ji}] + \theta_{ji}\theta_{ij} = \theta_{ii} - \theta_{jj} + \theta_{ji}\theta_{ij}$$

on the highest weight vector the expression simplifies further to

$$\sum_i \theta_{ii}^2/2 + \sum_{i<j} (\theta_{ii} - \theta_{jj})/2 - \sum_i \theta_{ii} - \sum_{i<j} (\theta_{ii} + \theta_{jj})/2$$

There is a bit of obvious cancellation, giving

$$\sum_{ii} \theta_{ii}^2/2 - \sum_{i<j} \theta_{jj} - \sum_i \theta_{ii} = \sum_i \left[\theta_{ii}^2/2 - (i-1+1)\theta_{ii}\right] = \sum_i \theta_{ii}(\theta_{ii}/2 - i)$$

which gives the claimed formula. ///

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3. Eigenvalues of the Casimir operator for $U(p, q)$

Now we do a computation for $U(p, q)$ entirely analogous to the that done just above for $Sp(n, \mathbf{R})$. The point here is direct computation of the eigenvalues of (essentially) the Casimir operator on certain representation spaces. In some regards the unitary group computation is even simpler than that for the symplectic group.

Let $\mathbf{g}$ be the Lie algebra of $G = U(p, q)$, where the latter is the isometry group of the standard hermitian form given by the $(p + q)$-by-$(p + q)$ matrix

$$H = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$$

The copy K of $U(p) \times U(q)$ in G is

$$K = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in U(p), B \in U(q)\}$$

The center

$$Z = \{\begin{pmatrix} a \cdot 1_p & 0 \\ 0 & b \cdot 1_q \end{pmatrix} : a, b \in U(1)\}$$

of K decomposes $\mathbf{g_C}$ into three pieces $\mathbf{p}^+$, $\mathbf{p}^-$, and $\mathbf{k_C}$

$$\mathbf{p}^+ = \{\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \quad \text{(with } S \text{ } p\text{-by-}q)\}$$

$$\mathbf{p}^- = \{\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \quad \text{(with } S \text{ } q\text{-by-}p)\}$$

and the complexified Lie algebra $\mathbf{k_C}$ of K. The elements of $\mathbf{p}^+$ are the **raising operators** and the elements of $\mathbf{p}^-$ are the **lowering operators**. The presentation just above makes clear that as a K representation space $\mathbf{p}^+$ is $\sigma \otimes \sigma'$ where σ is the standard representation of $U(p)$ and σ' is the standard representation of $U(q)$.

As in the case of $Sp(n, \mathbf{R})$ we choose a convenient normalization of a Killing form and Casimir operator, as follows. Let

$$\langle \alpha, \beta \rangle = \mathrm{tr}\,(\alpha\beta)$$

be the Killing form on $\mathbf{g} \times \mathbf{g}$. For a basis $\{E_i\}$ of $\mathbf{g_C}$, let E_i^* be a dual basis with respect to the Killing form. Then (up to a normalizing constant) the **Casimir operator** in the center of the universal enveloping algebra $\mathcal{U}(\mathbf{g})$ is

$$C = \sum_i E_i \circ E_i^*$$

(As before, the normalization is not the canonical one.)

Let V be a $(\mathbf{g}, K)$-module generated by an irreducible K-type ρ, and suppose $\mathbf{p}^- \cdot \rho = \{0\}$. Call V a **holomorphic discrete series** module with **lowest** K**-type** ρ.

Lemma: Let V be a holomorphic discrete series module with lowest K-type ρ. Then the Casimir operator acts on V with eigenvalue

$$\lambda(\rho) = \sum_{1 \le i \le p} m_i(m_i - (p - 2i + 1 - q)) + \sum_{1 \le j \le q} m'_j(m'_j - (q - 2j + 1 - p))$$

where ρ has highest weight $(m_1, \ldots, m_p) \times (m'_1, \ldots, m'_q)$.

Proof: Let e_{ij} be the n-by-n matrix with 0 everywhere but for a 1 at the ij^{th} place. We choose a basis for $\mathbf{g}_C$ consisting of

$$\begin{array}{rclll} X_{ij} & = & \begin{pmatrix} 0 & e_{ij} \\ 0 & 0 \end{pmatrix} & \in & \mathbf{p}^+ \quad (1 \le i \le p, 1 \le j \le q) \\ Y_{ij} & = & \begin{pmatrix} 0 & e_{ij} \\ 0 & 0 \end{pmatrix} & \in & \mathbf{p}^- \quad (1 \le i \le q, 1 \le j \le p) \\ \theta_{ij} & = & \begin{pmatrix} e_{ij} & 0 \\ 0 & 0 \end{pmatrix} & \in & \mathbf{k_C} \quad (1 \le i \le p,\ 1 \le j \le p) \\ \theta'_{ij} & = & \begin{pmatrix} 0 & 0 \\ 0 & -e_{ji} \end{pmatrix} & \in & \mathbf{k_C} \quad (1 \le i \le q,\ 1 \le j \le q) \end{array}$$

Note that the corresponding dual basis with respect to the Killing form is

$$\begin{array}{rcl} X_{ij}^* & = & Y_{ji} \\ Y_{ij}^* & = & X_{ji} \\ \theta_{ij}^* & = & \theta_{ji} \\ {\theta'_{ij}}^* & = & \theta'_{ji} \end{array}$$

Then the Casimir operator is

$$C = \sum_{ij} \theta_{ij}\theta_{ij}^* + \sum_{ij} \theta'_{ij}{\theta'_{ij}}^* + \sum_{i \le j} X_{ij}X_{ij}^* + \sum_{i \le j} Y_{ij}Y_{ij}^*$$

$$= \sum_{ij} \theta_{ij}\theta_{ji} + \sum_{ij} \theta'_{ij}\theta'_{ji} + \sum_{i,j} X_{ij}Y_{ji} + \sum_{i,j} Y_{ij}X_{ji}$$

On any vector annihilated by $\mathbf{p}^-$, the operators Y_{ij} are all 0, so on such a vector this simplifies to

$$\sum_{ij} \theta_{ij}\theta_{ji} + \sum_{ij} \theta'_{ij}\theta'_{ji} + \sum_{i,j} Y_{ij}X_{ji}$$

And since

$$Y_{ij}X_{ji} = [Y_{ij}, X_{ji}] + X_{ji}Y_{ij} = -\theta_{jj} - \theta'_{ii} + X_{ij}Y_{ij}$$

on vectors annihilated by $\mathbf{p}^-$ this simplifies further to

$$\sum_{ij} \theta_{ij}\theta_{ji} + \sum_{ij} \theta'_{ij}\theta'_{ji} - \sum_{ij} (\theta_{jj} + \theta'_{ii})$$

Next, consider the action of this on the highest weight vector v_o annihilated by all θ_{ij} and θ'_{ij} with $i < j$. First, about half the quadratic terms disappear, giving

$$\sum_{i<j} \theta_{ij}\theta_{ji} + \sum_{i<j} \theta'_{ij}\theta'_{ji} - \sum_{ij} (\theta_{jj} + \theta'_{ii})$$

Using the relations

$$\theta_{ij}\theta_{ji} = [\theta_{ij}, \theta_{ji}] + \theta_{ji}\theta_{ij} = m_i - m_j + \theta_{ji}\theta_{ij}$$

and similarly for θ'_{ij}, on the highest weight vector the expression simplifies further to

$$\sum_i m_i^2 + \sum_{i<j} (m_i - m_j) + \sum_i {m'_i}^2 + \sum_{i<j} (m'_i - m'_j) - \sum_{i<j} (m_j + m'_i)$$

This simplifies to

$$\sum_i m_i^2 - \sum_i m_i(p - 2i - 1 - q) + \sum_i {m'_i}^2 - \sum_i m'_i(q - 2i - 1 - p)$$

which gives the claimed formula. ///

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4. Irreducibility of universal modules

We can treat simultaneously the cases

$$(G, K) = (Sp(n, \mathbf{R}), U(n))$$

and

$$(G, K) = (U(p, q), U(p) \times U(q))$$

Let $\mathbf{g}$ be the Lie algebra of G. Let ρ, W be an irreducible representation of K. We can construct a **universal holomorphic discrete series** $(\mathbf{g}, K)$-module V with lowest K-type ρ, in the sense that for any other $(\mathbf{g}, K)$-module V' generated by $\rho' \approx \rho$ and with $\mathbf{p}^- \cdot \rho' = \{0\}$, there is a surjective $(\mathbf{g}, K)$-module morphism $V \to V'$. We construct the universal module V as follows.

Let I_ρ be the left ideal in $\mathcal{U}(\mathbf{k_C})$ annihilating the highest-weight vector v_o in W. Let $\pi = \pi_\rho$ be the left $(\mathbf{g}, K)$-module expressed as a quotient of the universal enveloping algebra $\mathcal{U}(\mathbf{g_C})$ by

$$V = \mathcal{U}(\mathbf{g_C})/\mathcal{U}(\mathbf{p}^-)I_\rho$$

Note that

$$W = \mathcal{U}(\mathbf{k_C}) \cdot v_o \approx \mathcal{U}(\mathbf{k_C})/I_\rho$$

Thus, by construction, $\mathbf{p}^-$ annihilates this copy of W inside V. And, by construction, this V is the *universal* $(\mathbf{g}, K)$-module generated by an irreducible of K isomorphic to ρ, all of whose vectors are annihilated by $\mathbf{p}^-$. It is the **universal holomorphic discrete series** representation with **lowest K-type** ρ. An important aspect of the universality is that, from the Poincaré-Birkhoff-Witt theorem, as a left $\mathcal{U}(\mathbf{p}^+) \times K$-module

$$\mathcal{U}(\mathbf{p}^+) \otimes_{\mathbf{C}} W \approx V$$

Corollary: The *universal* holomorphic discrete series

$$V = \mathcal{U}(\mathbf{g_C})/\mathcal{U}(\mathbf{p}^-)I_\rho \approx \mathcal{U}(\mathbf{p}^+) \otimes_{\mathbf{C}} W$$

with lowest K-type ρ is (irreducible) if the highest weight of ρ is sufficiently high. Specifically, it suffices that that the highest weight $(m_1, \ldots, m_n)$ of ρ satisfies

$$m_1 \geq m_2 \geq \ldots \geq m_n \geq n$$

in the case of $Sp(n, \mathbf{R})$, and

$$m_1 \geq \ldots \geq m_p \geq \frac{p+q-1}{2} \quad \text{and} \quad m'_1 \geq \ldots \geq m'_q \geq \frac{p+q-1}{2}$$

in the case of $U(p, q)$.

Remark: Again, the fact that the map $\mathcal{U}(\mathbf{p}^+) \otimes_{\mathbf{C}} W \approx V$ is an isomorphism follows from Poincaré-Birkhoff-Witt and the construction. Thus, rather than begin with an irreducible $(\mathbf{g}, K)$-module and attempt to prove something about the structure, we take the universal (largest possible) $(\mathbf{g}, K)$-module and prove that (by a miracle) it is irreducible.

Remark: The key point in the proof of the corollary is a monotonicity property of the eigenvalues of the Casimir operator for highest weight in a certain range.

Proof: Order the highest weights of $U(n)$ representations by

$$(m'_1, \ldots, m'_n) \leq (m''_1, \ldots, m''_n)$$

if $m'_i \leq m''_i$ for all indices. All K-types ρ' occurring are summands of $\mathcal{U}(\mathbf{p}^+) \otimes \rho$, and

$$\begin{array}{ll} \mathbf{p}^+ \approx \mathrm{Sym}^2\sigma & (\text{for } Sp(n, \mathbf{R}) \text{ and } U(n)) \\ \mathbf{p}^+ \approx \sigma \otimes \sigma' & (\text{for } U(p,q) \text{ and } U(p) \times U(q)) \end{array}$$

(with standard representations σ). Thus, there are no infinite descending sequences among highest weights occuring in V. Suppose X were a proper $(\mathbf{g}, K)$-submodule of V. Let ρ' be a K-type with least highest weight among all K-types occurring in X. Then $\mathbf{p}^- \cdot \rho' = \{0\}$, since otherwise $\mathbf{p}^- \cdot \rho' \subset X$ would have a lower highest weight than the supposed minimum. That is, $\mathcal{U}(\mathbf{g}) \cdot v_1$ would be a holomorphic discrete series with lowest K-type ρ'.

The expression for the eigenvalue of Casimir is strictly increasing (as some weight increases, and no weight decreases) in the ranges indicated. Since ρ' has a higher highest weight than ρ (or else X is all of V), the eigenvalue $\lambda(\rho')$ of the (pseudo-) Casimir on X would satisfy

$$\lambda(\rho') > \lambda(\rho)$$

But the Casimir operator is central, so *every* vector in V has the same eigenvalue $\lambda(\rho)$. Thus, there can be no non-zero holomorphic vectors in V other than the original lowest K-type ρ. This proves the irreducibility of the universal module. ///

Corollary: For ρ having highest weight satisfying the given inequalites, any holomorphic discrete series representation V with lowest K-type ρ is isomorphic to the universal one constructed above, and ρ determines the $(\mathbf{g}, K)$-isomorphism class of V completely.

Proof: Again, let I_ρ be the left ideal in $\mathcal{U}(\mathbf{k}_\mathbf{C})$ annihilating a highest weight vector v_o in ρ. The structure of $\mathcal{U}(\mathbf{g}_\mathbf{C})/\mathcal{U}(\mathbf{p}^-)I$ is completely determined by I. And any holomorphic discrete series with lowest K-type ρ in that range is isomorphic to this universal one, since the universal one is irreducible. ///
