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Intertwinings among principal series of $SL_2(\mathbb{C})$

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We compute natural integrals giving intertwining operators among principal series of $G = SL(2,\mathbb{C})$.

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1. *Principal series representations*

As usual, let

$$N = \{n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\} \qquad M = \{m_a = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{C}^\times\}$$

and

$$P \;=\; NM \;=\; MN$$

For $s \in \mathbb{C}$ and integer κ, the $(s,\kappa)^{th}$ *principal series* representation $I_{s,\kappa}$ is the space of smooth functions f on G with prescribed left equivariance

$$I_{s,\kappa} \;=\; \{f : f(pg) = \chi_{s,\kappa}(p)\, f(g) \text{ for all } p \in P,\ g \in G\} \qquad \text{(where } \chi_{s,\kappa}\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} = |a|^{4s}(\frac{a}{|a|})^\kappa)$$

with the normalization of the character to have the intertwining operator $T_{s,\kappa}$ below map from $I_{s,\kappa}$ to $I_{1-s,-\kappa}$ rather than have s transform in some other fashion. The group G acts on $I_{s,\kappa}$ by the *right regular* representation, that is, by right translation of functions:

$$(g \cdot f)(x) \;=\; f(xg) \qquad \text{(for } g, x \in G)$$

The *standard intertwining operator* $T = T_{s,\kappa} : I_{s,\kappa} \to I_{1-s,-\kappa}$ is defined, for $\mathrm{Re}(s)$ sufficiently large, by the integral

$$(T_{s,\kappa} f)(g) \;=\; \int_N f(w_o n \cdot g)\, dn$$

where the long Weyl element w_o is

$$w_o \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The integration is on the left, so does not disturb the right action of G. To verify that (assuming convergence) the image really does lie inside $I_{1-s,-\kappa}$, observe that $T_{s,\kappa} f$ is left N-invariant by construction, and that for $m \in M$

$$(T_{s,\kappa} f)(mg) = \int_N f(w_o n \cdot mg)\, dn = \int_N f(w_o m\, m^{-1} n m \cdot g)\, dn = \chi_1(m) \cdot \int_N f(w_o m\, n \cdot g)\, dn$$

by replacing n by mnm^{-1}, taking into account the change of measure $d(mnm^{-1}) = \chi_1(m) \cdot dn$ coming from

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \;=\; \begin{pmatrix} 1 & a^2 z \\ 0 & 1 \end{pmatrix}$$

Then this is

$$\chi_1(m)\cdot\int_N f(w_o m w_o^{-1}\cdot w_o\, n\cdot g)\,dn \;=\; \chi_1(m)\cdot\int_N f(m^{-1}\cdot w_o\, n\cdot g)\,dn$$
$$=\;\chi_1(m)\chi_{s,\kappa}(m^{-1})\cdot\int_N f(w_o\, n\cdot g)\,dn \;=\; \chi_{1-s,-\kappa}(m)\cdot(T_{s,\kappa}f)(g)$$

This verifies that $T_{s,\kappa}: I_{s,\kappa}\to I_{1-s,-\kappa}$.

The standard maximal compact is $K = SU(2)\subset G$, and G has Iwasawa decomposition $G = PK$. The overlap is

$$P\cap K \;=\; \{\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix} \;:\; |\mu|=1,\; \mu\in\mathbb{C}^\times\}$$

The restriction of $\chi_{s,\kappa}$ to $P\cap K$ does not depend on $s\in\mathbb{C}$, but only on κ. Write χ_κ for this restriction.

The complexified Lie algebra $\mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}\approx\mathfrak{sl}_2(\mathbb{C})$ has standard $\mathbb{C}$-basis

$$h \;=\; \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad x \;=\; \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad y \;=\; \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

The Cartan element h decomposes finite-dimensional complex representations σ of $SU(2)$ into eigenspaces, and there is a unique (up to scalars) non-zero h-eigenvector v_o annihilated by x, a *highest-weight vector* of σ. The h-eigenvalue of v_o is a non-negative integer ℓ, the *highest weight* of σ, and determines the isomorphism class of σ. Application of y to an h-eigenvector with eigenvalue λ shifts the eigenvalue to $\lambda-2$, or else annihilates the vector. The collection of *all* h-eigenvalues in the irreducible σ_ℓ with highest weight ℓ has eigenvalues exactly

$$-\ell,\; -\ell+2,\; -\ell+4,\; \ldots,\; \ell-4,\; \ell-2,\; \ell \qquad\qquad \text{(with (non-zero) multiplicities all 1)}$$

A convenient model for the irreducible σ_ℓ with highest weight ℓ is homogeneous polynomials of total degree ℓ on $\mathbb{C}^2$ treated as row vectors, with the action

$$(k\cdot f)(u,v) \;=\; \sigma_\ell(k)f(u,v) \;=\; f((u,v)\cdot k) \qquad\qquad \text{(with } k\in K \text{ and } f \text{ on } (u,v)\in\mathbb{C}^2\text{)}$$

The highest-weight vector is $(u,v)\to u^\ell$. The biregular representation of $K\times K$ on functions on K is

$$(k\times k')f(x) \;=\; f(k'^{-1}xk)$$

This decomposes the space of (for example) right K-finite functions as $\bigoplus_\sigma \sigma\otimes\check{\sigma}$, where $\check{\sigma}$ is the dual of σ, and σ runs through the irreducibles of K.

A function f in $I_{s,\kappa}$ is determined by its restriction to K, and must lie in

$$\mathrm{Ind}_{P\cap K}^K\, \chi_{s,\kappa}\Big|_{P\cap K} \;=\; \mathrm{Ind}_{P\cap K}^K\,\chi_\kappa$$

Conversely, for $s\in\mathbb{C}$, a smooth function f_o in $\mathrm{Ind}_{P\cap K}^K\,\chi_\kappa$ has a unique extension to $f\in I_{s,\kappa}$, by

$$f(pk) \;=\; \chi_s(p)\cdot f_o(k)$$

That is, for $f\in I_{s,\kappa}$ the restriction $f|_K$ is a κ-eigenvector for $h\in\mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}$ under the left action

$$(h\cdot f)(k) \;=\; \frac{\partial}{\partial t}\Big|_{t=0} f(e^{th}k)$$

This gives the *negative* of the eigenvalue under the left *regular* action.

The irreducible $\check{\sigma}_\ell$ has non-zero $-\kappa$ eigenspace $\check{\sigma}_\ell[-\kappa]$ for $\ell \in 2\mathbb{Z}$ for $\ell \geq |\kappa|$ and of the same parity. The eigenspace is one-dimensional. Thus, under the right regular representation of K on $\mathrm{Ind}_{P\cap K}^K\ \chi_\kappa$, each irreducible appearing appears with *multiplicity one*:

$$\mathrm{Ind}_{P\cap K}^K\ \chi_\kappa \;=\; \bigoplus_{|\kappa|\leq\ell\in\mathbb{Z},\ \ell=\kappa \bmod 2} \sigma_\ell\otimes\check{\sigma}_\ell[-\kappa] \;\approx\; \bigoplus_{|\kappa|\leq\ell\in\mathbb{Z},\ \ell=\kappa \bmod 2} \sigma_\ell \qquad \text{(right regular of } K=SU(2))$$

Let

$$R_{s,\kappa} \;:\; I_{s,\kappa} \longrightarrow \mathrm{Ind}_{P\cap K}^K\ \chi_\kappa \qquad (\text{by } R_{s,\kappa}f = f|_K)$$

and

$$E_{s,\kappa} \;:\; \mathrm{Ind}_{P\cap K}^K\ \chi_\kappa \longrightarrow I_{s,\kappa} \qquad (\text{by } (E_{s,\kappa}f_o)(pk) = \chi_{s,\kappa}(p)\cdot f_o(k))$$

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2. *The main computation*

We compute the effect of the intertwining operator $T_{s,\kappa}$ on a function f in $I_{s,\kappa}$ with a fixed K-type $\sigma=\sigma_\ell$.

The map $T_{s,\kappa}$ is a G-homomorphism, so does not disturb right K-types. However, as observed above, the composite

$$\tau_{s,\kappa} \;=\; R_{1-s,-\kappa}\circ T_{s,\kappa}\circ E_{s,\kappa}$$

has the effect

$$\tau_{s,\kappa} \;:\; \sigma\otimes\check{\sigma}[\kappa] \;\longrightarrow\; \sigma\otimes\check{\sigma}[-\kappa]$$

For $\kappa=0$, the two copies of $\sigma\otimes\check{\sigma}[\pm 0]$ are *identical*, not merely isomorphic, so by Schur's lemma $\tau_{s,0}$ is a scalar multiplication on $\sigma\otimes\check{\sigma}[0]$. For $\kappa\neq 0$, the copies of σ in the two induced representations require effort for comparison. We specify vectors $f_o\in\sigma\otimes\check{\sigma}[-\kappa]$ as *matrix coefficient functions*, as follows.

Use the model of σ_ℓ by homogeneous holomorphic polynomials of total degree ℓ in two complex variables with hermitian inner product

$$\langle\varphi_1,\varphi_2\rangle \;=\; \int_{\mathbb{C}^2} \varphi_1(u,v)\cdot\overline{\varphi}_2(u,v)\; e^{-\pi(|u|^2+|v|^2)}\; du\; dv$$

with the additive measure from $\mathbb{C}\approx\mathbb{R}^2$. Take $f_o\in\sigma\otimes\check{\sigma}$ to be a matrix coefficient function

$$f_o(k) \;=\; \langle k\cdot\varphi,\psi\rangle \qquad (\varphi,\psi\in\sigma)$$

using the hermitian inner product to identify σ with its dual. In that model, let

$$\varphi_{\ell,\kappa}(u,v) \;=\; u^{\frac{\ell+\kappa}{2}}\cdot v^{\frac{\ell-\kappa}{2}}$$

For $f_o(k)=\langle k\cdot\varphi,\psi\rangle$ to be a left $-\kappa$ eigenvector for h, ψ must be in $\check{\sigma}[-\kappa]$, so take

$$\psi(u,v) \;=\; \varphi_{\ell,-\kappa}(u,v) \;=\; u^{\frac{\ell-\kappa}{2}}\cdot v^{\frac{\ell+\kappa}{2}}$$

To make $(\tau_{s,\kappa}f_o)(1)\neq 0$, take

$$\varphi(u,v) \;=\; \varphi_{\ell,\kappa}(u,v) \;=\; u^{\frac{\ell+\kappa}{2}}\cdot v^{\frac{\ell-\kappa}{2}}$$

and

$$f_o(k) \;=\; \langle k\cdot\varphi,\psi\rangle \;=\; \langle k\cdot\varphi_{\ell,\kappa},\varphi_{\ell,-\kappa}\rangle$$

For all K-types σ_ℓ appearing, for $v\in\sigma_\ell$, $\tau_{s,\kappa}$ maps $v\otimes\varphi_{\ell,-\kappa}$ to a scalar multiple of $v\otimes\varphi_{\ell,\kappa}$, with scalar depending only on σ,s,κ, and the scalar can be computed as

$$(\tau_{s,\kappa}f_o)(1)\Big/\langle\varphi_{\ell,\kappa},\varphi_{\ell,\kappa}\rangle$$

First,

$$\langle \varphi_{\ell,\kappa},\, \varphi_{\ell,\kappa}\rangle \;=\; \int_{\mathbb{C}^2} |u^{\frac{\ell+\kappa}{2}} v^{\frac{\ell-\kappa}{2}}|^2\, e^{-\pi(|u|^2+|v|^2)}\, du\, dv \;=\; \int_{\mathbb{C}^2} |u|^{\ell+\kappa}\, |v|^{\ell-\kappa}\, e^{-\pi(|u|^2+|v|^2)}\, du\, dv$$

The latter will cancel, below, so we do not need further explication of this integral. To evaluate

$$(T_{s,\kappa}\circ E_{s,\kappa}\, f_o)(1) \;=\; \int_N f_o(w_o n)\, dn = \int_{\mathbb{C}} f_o(w_o n_z)\, dz \qquad (\text{with } n_z = \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix})$$

we need the Iwasawa decomposition $w_o n_z = pk$:

$$\begin{pmatrix} 0 & -1 \\ 1 & z\end{pmatrix} \;=\; w_o n_z \;=\; \begin{pmatrix} \frac{1}{\sqrt{1+|z|^2}} & \frac{-\overline{z}}{\sqrt{1+|z|^2}} \\ 0 & \sqrt{1+|z|^2}\end{pmatrix}\cdot\begin{pmatrix} \frac{\overline{z}}{\sqrt{1+|z|^2}} & \frac{-1}{\sqrt{1+|z|^2}} \\ \frac{1}{\sqrt{1+|z|^2}} & \frac{z}{\sqrt{1+|z|^2}}\end{pmatrix}$$

Thus,

$$(E_{s,\kappa} f_o)(w_o n_z) \;=\; (1+|z|^2)^{-2s}\cdot f_o(k_z) \qquad (\text{with } k_z = \begin{pmatrix} \frac{\overline{z}}{\sqrt{1+|z|^2}} & \frac{-1}{\sqrt{1+|z|^2}} \\ \frac{1}{\sqrt{1+|z|^2}} & \frac{z}{\sqrt{1+|z|^2}}\end{pmatrix})$$

Then

$$(k_z\cdot\varphi_{\ell,\kappa})(u,v) \;=\; \Big(\frac{u\overline{z}}{\sqrt{1+|z|^2}} + \frac{v}{\sqrt{1+|z|^2}}\Big)^{\frac{\ell+\kappa}{2}}\cdot\Big(\frac{-u}{\sqrt{1+|z|^2}} + \frac{vz}{\sqrt{1+|z|^2}}\Big)^{\frac{\ell-\kappa}{2}}$$

and

$$f_o(z) \;=\; \langle k_z\cdot\varphi_{\ell,\kappa},\, \varphi_{\ell,-\kappa}\rangle \;=\; \int_{\mathbb{C}^2} (k_z\cdot\varphi_{\ell,\kappa})(u,v)\cdot\overline{\varphi_{\ell,-\kappa}(u,v)}\; e^{-\pi(|u|^2+|v|^2)}\, du\, dv$$

$$= \int_{\mathbb{C}^2} \Big(\frac{u\overline{z}}{\sqrt{1+|z|^2}} + \frac{v}{\sqrt{1+|z|^2}}\Big)^{\frac{\ell+\kappa}{2}}\cdot\Big(\frac{-u}{\sqrt{1+|z|^2}} + \frac{vz}{\sqrt{1+|z|^2}}\Big)^{\frac{\ell-\kappa}{2}}\cdot\overline{u^{\frac{\ell-\kappa}{2}} v^{\frac{\ell+\kappa}{2}}}\; e^{-\pi(|u|^2+|v|^2)}\, du\, dv$$

$$= \; (1+|z|^2)^{-\ell/2}\int_{\mathbb{C}^2} (u\overline{z}+v)^{\frac{\ell+\kappa}{2}}\cdot(-u+vz)^{\frac{\ell-\kappa}{2}}\cdot\overline{u^{\frac{\ell-\kappa}{2}} v^{\frac{\ell+\kappa}{2}}}\; e^{-\pi(|u|^2+|v|^2)}\, du\, dv$$

$$= \; (1+|z|^2)^{-\ell/2}\sum_{j=0}^{\min(\frac{\ell+\kappa}{2},\frac{\ell-\kappa}{2})} \binom{\frac{\ell+\kappa}{2}}{j}\binom{\frac{\ell-\kappa}{2}}{j}\,(-1)^j\, |z|^{2j}\;\cdot\;\int_{\mathbb{C}^2} |u|^{\ell-\kappa}\, |v|^{\ell+\kappa}\, e^{-\pi(|u|^2+|v|^2)}\, du\, dv$$

The latter integral is $\langle\varphi_{\ell,\kappa},\varphi_{\ell,\kappa}\rangle$, with roles of u, v reversed. Thus, letting

$$\mu \;=\; \min(\frac{\ell+\kappa}{2},\frac{\ell-\kappa}{2}) \;=\; \frac{\ell-|\kappa|}{2}$$

the scalar by which the K,σ-isotype is multiplied under $T_{s,\kappa}: I_{s,\kappa}\longrightarrow I_{1-s,-\kappa}$ is

$$(\tau_{s,\kappa} f_o)(1)\Big/\langle\varphi_{\ell,\kappa},\varphi_{\ell,\kappa}\rangle \;=\; \int_{\mathbb{C}} (1+|z|^2)^{-2s-\frac{\ell}{2}}\sum_{j=0}^{\mu}\binom{\frac{\ell+\kappa}{2}}{j}\binom{\frac{\ell-\kappa}{2}}{j}\,(-1)^j |z|^{2j}\; dz$$

$$= \; 2\pi\int_0^\infty (1+r^2)^{-2s-\frac{\ell}{2}}\sum_{j=0}^{\mu}\binom{\frac{\ell+\kappa}{2}}{j}\binom{\frac{\ell-\kappa}{2}}{j}\,(-1)^j r^{2j}\; r\, dr \;=\; \pi\int_0^\infty (1+t)^{-2s-\frac{\ell}{2}}\sum_{j=0}^{\mu}\binom{\frac{\ell+\kappa}{2}}{j}\binom{\frac{\ell-\kappa}{2}}{j}\,(-t)^j\; dt$$

From this point, we follow a slightly simplified version of the device of [Duflo 1975] in the same computation, pp. 57-58. Use the identity

$$\sum_{j=0}^{c} \binom{a}{j}\binom{b}{j} (-t)^j \;=\; \frac{1}{c!}\Big(\frac{\partial}{\partial u}\Big)^c\Big|_{u=0}(1+u)^a\,(u-t)^b \qquad \text{(for } c\le \min(a,b))$$

With $a=\frac{\ell+\kappa}{2}$, $b=\frac{\ell-\kappa}{2}$, and $c=\mu$, the scalar is

$$\frac{\pi}{\mu!}\int_0^\infty (1+t)^{-2s-\frac{\ell}{2}}\Big(\frac{\partial}{\partial u}\Big)^\mu\Big|_{u=0}(1+u)^{\frac{\ell+\kappa}{2}}\,(u-t)^{\frac{\ell-\kappa}{2}}\;dt$$

We would like to have the differentiation be used in an integration by parts. To this end, *separate variables* in both $1+u$ and $u-t$, that is, for some function $p(t)$ and functions $q(\tau)$, $r(\tau)$ of a new variable τ, to have $1+u=p(t)\cdot q(\tau)$ and $u-t=p(t)\cdot r(\tau)$. Subtraction gives

$$1+t \;=\; (1+u)-(u-t) \;=\; p\cdot(q-r)$$

which suggests $p(t)=1+t$ and $q(\tau)-r(\tau)=1$. Take $q(\tau)=\tau$ and $r(\tau)=\tau-1$, so $u=(1+t)\tau-1$, and

$$1+u \;=\; (1+t)\tau \qquad\qquad u-t \;=\; (1+t)(\tau-1) \qquad\qquad \Big(\frac{\partial}{\partial u}\Big)^\mu \;=\; (1+t)^{-\mu}\Big(\frac{\partial}{\partial \tau}\Big)^\mu$$

The evaluation occurs at $\tau=\frac{1}{t+1}$. In these terms, the scalar is

$$\frac{\pi}{\mu!}\int_0^\infty (1+t)^{-2s-\frac{\ell}{2}}\cdot(1+t)^{-\mu}\Big(\frac{\partial}{\partial \tau}\Big)^\mu\Big|_{\tau=\frac{1}{t+1}}\Big((1+t)\tau\Big)^{\frac{\ell+\kappa}{2}}\Big((1+t)(\tau-1)\Big)^{\frac{\ell-\kappa}{2}}\;dt$$

$$=\;\frac{\pi}{\mu!}\int_0^\infty (1+t)^{-\mu+\frac{\ell}{2}-2s}\Big(\frac{\partial}{\partial \tau}\Big)^\mu\Big|_{\tau=\frac{1}{t+1}}\tau^{\frac{\ell+\kappa}{2}}(\tau-1)^{\frac{\ell-\kappa}{2}}\;dt$$

Let $v=\frac{1}{t+1}$, that is, $t=\frac{1}{v}-1$, so $dt=\frac{-dv}{v^2}$, and $1+t=\frac{1}{v}$, and the scalar is

$$\frac{\pi}{\mu!}\int_0^1 \Big(\frac{1}{v}\Big)^{-\mu+\frac{\ell}{2}-2s}\Big(\frac{\partial}{\partial \tau}\Big)^\mu\Big|_{\tau=v}\tau^{\frac{\ell+\kappa}{2}}(\tau-1)^{\frac{\ell-\kappa}{2}}\;\frac{dv}{v^2} \;=\; \frac{\pi}{\mu!}\int_0^1 v^{2s-\frac{\ell}{2}+\mu-2}\Big(\frac{\partial}{\partial v}\Big)^\mu\Big(v^{\frac{\ell+\kappa}{2}}(v-1)^{\frac{\ell-\kappa}{2}}\Big)\;dv$$

writing differentiation followed by evaluation in the more usual fashion. Integrate by parts μ times, with boundary terms vanishing, obtaining

$$\frac{\pi}{\mu!}(-1)^\mu\int_0^1\Big(\frac{\partial}{\partial v}\Big)^\mu v^{2s-\frac{\ell}{2}+\mu-2}\;\cdot\;v^{\frac{\ell+\kappa}{2}}(v-1)^{\frac{\ell-\kappa}{2}}\;dv$$

$$=\;\frac{\pi}{\mu!}(-1)^\mu\int_0^1\Big((2s-\tfrac{\ell}{2}+\mu-2)(2s-\tfrac{\ell}{2}+\mu-3)\ldots(2s-\tfrac{\ell}{2}-1)\Big)\cdot v^{2s-\frac{\ell}{2}-2}\cdot v^{\frac{\ell+\kappa}{2}}(v-1)^{\frac{\ell-\kappa}{2}}\;dv$$

$$=\;\frac{\pi}{\mu!}(-1)^\mu\frac{\Gamma(2s-\frac{\ell}{2}+\mu-1)}{\Gamma(2s-\frac{\ell}{2}-1)}\int_0^1 v^{2s-\frac{\ell}{2}-2}\cdot v^{\frac{\ell+\kappa}{2}}(v-1)^{\frac{\ell-\kappa}{2}}\;dv$$

$$=\;\frac{\pi\,(-1)^{\frac{\ell-|\kappa|}{2}}}{\Gamma(\frac{\ell-|\kappa|}{2}+1)}\,\frac{\Gamma(2s-\frac{|\kappa|}{2}-1)}{\Gamma(2s-\frac{\ell}{2}-1)}\int_0^1 v^{2s-\frac{\kappa}{2}-2}\,(v-1)^{\frac{\ell+\kappa}{2}}\;dv$$

Standard computational devices give

$$\int_0^1 v^a\,(1-v)^b\;dv \;=\; \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}$$

so the scalar is

$$\frac{\pi\,(-1)^{\frac{\ell-|\kappa|}{2}}}{\Gamma(\frac{\ell-|\kappa|}{2}+1)}\,\frac{\Gamma(2s-\frac{|\kappa|}{2}-1)}{\Gamma(2s-\frac{\ell}{2}-1)}\;\cdot\;(-1)^{\frac{\ell+\kappa}{2}}\,\frac{\Gamma(2s-\frac{\kappa}{2}-2+1)\,\Gamma(\frac{\ell+\kappa}{2}+1)}{\Gamma(2s-\frac{\kappa}{2}-2+\frac{\ell+\kappa}{2}+2)}$$

$$=\;\pi\,(-1)^{\frac{|\kappa|-\kappa}{2}}\;\frac{\Gamma(2s-\frac{|\kappa|}{2}-1)\,\Gamma(2s-\frac{\kappa}{2}-1)\,\Gamma(\frac{\ell+\kappa}{2}+1)}{\Gamma(\frac{\ell-|\kappa|}{2}+1)\,\Gamma(2s-\frac{\ell}{2}-1)\,\Gamma(2s+\frac{\ell}{2})}$$

For $\kappa=0$ and $\ell\in 2\mathbb{Z}$, this simplifies to

$$\pi\,\frac{\Gamma(2s-1)\,\Gamma(2s-1)}{\Gamma(2s-\frac{\ell}{2}-1)\,\Gamma(2s+\frac{\ell}{2})}\qquad\text{(with } \kappa=0 \text{ and } 0\le\ell\in 2\mathbb{Z})$$

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3. *Smooth vectors*

From the previous computation, and from

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}\;\sim\;z^{a-b}\qquad\text{(fixed } a,b)$$

and

$$\Gamma(1-z)\cdot\Gamma(z)\;=\;\frac{\pi}{\sin\pi z}$$

we see that, for fixed κ and $s\in C$, the scalar by which $T_{s,\kappa}$ maps $\sigma_\ell\otimes\check{\sigma}_\ell[-\kappa]$ to $\sigma_\ell\otimes\check{\sigma}_\ell[\kappa]$ is of *polynomial growth* in ℓ.

Thus, these intertwinings extend to the *smooth* vectors of the representation, since the L^2 norms of σ_ℓ^{th} Fourier components of smooth functions on $SU(2)$ decrease rapidly with ℓ, and have sup-norms bounded by a constant multiple of $\sqrt{\dim\sigma_\ell}$ times their L^2-norms.

For $SL(2,\mathbb{R})$, see

[Ehrenpreis-Mautner 1955] L. Ehrenpreis and F. Mautner, *Some properties of the Fourier transform on semi-simple Lie groups, I*, Annals of Math. **61** (1955), 406-439; *II*, Trans. AMS **84** (1957) 1-55; *III*, *ibid* **90** (1959), 431-484.

[Kunze-Stein 1960] R. Kunze and E. Stein, *Uniformly bounded representations and harmonic analysis of the 2-by-2 real unimodular group*, Amer. J. Math. **82** (1960), 1-62.

I thank V. Drinfeld for the following bibliographic notes: $SL_2(\mathbb{R})$ is treated in 7.17, and the outcome is stated for $SL_2(\mathbb{C})$ in 7.23, in

[Wallach 1977/79] N. Wallach, *Representations of reductive Lie groups*, in *Automorphic forms, representations, and L-functions*, Proc. Symp. AMS XXXIII, part 1 (1979), 71-86.

The case of $SL_2(\mathbb{C})$ is Prop 3.7 pp 57-58 in Chap III of

[Duflo 1975] M. Duflo, Représentations irréductibles des groupes semisimples complexes, in *Analyse harmonique sur les groupes de Lie*, Springer LNS **497**, 1975, 26-88.

An earlier treatment of this aspect of $SL_2(\mathbb{C})$ is

[Želobenko 1963] D. P. Želobenko, *Harmonic analysis of functions on semisimple Lie groups. I.* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1343-1394.
