INTEGRAL MOMENTS OF AUTOMORPHIC L–FUNCTIONS

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Abstract. This paper exposes the underlying mechanism for obtaining second integral moments of GL_2 automorphic L–functions over an arbitrary number field. Here, moments for GL_2 are presented in a form enabling application of the structure of adele groups and their representation theory. To the best of our knowledge, this is the first formulation of integral moments in adele-group-theoretic terms, distinguishing global and local issues, and allowing uniform application to number fields. When specialized to the field of rational numbers $\mathbb{Q}$, we recover the classical results.

§1. Introduction

For ninety years, the study of mean values of families of automorphic L–functions has played a central role in analytic number theory for their applications to classical problems. In the absence of the Riemann Hypothesis, or the Grand Riemann Hypothesis, rather, when referring to general L–functions, suitable mean value results often served as a substitute. In particular, obtaining asymptotics or sharp bounds for integral moments of automorphic L–functions is of considerable interest. The study of integral moments was initiated in 1918 by Hardy and Littlewood (see [Ha-Li]) who obtained the second moment of the Riemann zeta-function

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \;\sim\; T\log T \tag{1.1}$$

About 8 years later, Ingham in [I] obtained the fourth moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4 dt \;\sim\; \frac{1}{2\pi^2}\cdot T(\log T)^4 \tag{1.2}$$

Subsequently, many papers by various authors were devoted to this subject. For instance see [At], [H-B], [G1], [M1], [J1]. Most of the existing results concern integral moments of automorphic L–functions defined over $\mathbb{Q}$. No analog of (1.1) or (1.2) is known over an arbitrary number field. The only previously known results, for fields other than $\mathbb{Q}$, are in [M4], [S1], [BM1], [BM2] and [DG2], all over quadratic fields.

Here we expose the underlying mechanism to obtain second integral moments of GL_2 automorphic L–functions over an arbitrary number field. Integral moments for GL_2 are presented in a form

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amenable to application of the representation theory of adele groups. To the best of our knowledge, this is the first formulation of integral moments on adele-groups, distinguishing global and local questions, and allowing uniform application to number fields. Precisely, for f an automorphic form on GL_2 and χ an idele class character of the number field, let $L(s, f\otimes\chi)$ denote the twisted L–function attached to f. We obtain asymptotics for averages

$$\sum_{\chi} \int_{-\infty}^{\infty} \left|L\left(\tfrac{1}{2}+it,\, f\otimes\chi\right)\right|^2 M_\chi(t)\, dt \tag{1.3}$$

for suitable smooth weights $M_\chi(t)$. The sum in (1.1) is over a certain (in general infinite) set of idele class characters. For general number fields, it seems that (1.1) is the correct structure of the second integral moment of GL_2 automorphic L–functions. This was first pointed out by Sarnak in [S1], where an average of the above type was studied over the Gaussian field $\mathbb{Q}(i)$; see also [DG2]. From the analysis of Section 2, it will become apparent that this comes from the Fourier transform on the idele class group of the field.

The course of the argument makes several points clear. First, the sum (over twists) of moments of L–functions is presented by an integral representation. Second, the kernel arise from a collection of *local* data, wound up into an automorphic form, and the proof proceeds by unwinding. Third, the local data at *finite* primes is of a mundane sort, already familiar from other constructions of Poincaré series. Fourth, the only subtlety resides in choices of archimedean data. Once this is understood, it is clear that Good's original idea in [G2], seemingly limited to $GL_2(\mathbb{Q})$, exhibits a good choice of local data for *real* primes. (See also [DG1].) Similarly, while [DG2] explicitly addresses only $GL_2(\mathbb{Z}[i])$, the discussion there does exhibit a good choice of local data for *complex* primes. That is, these two examples suffice to illustrate the (non-obvious) choices of local data for all archimedean places.

For applications, we will combine careful choices of archimedean data with extensions of the estimates in [Ho-Lo] and [S2] (or [BR]) to number fields. However, for now, we are content with a formulation suitable for both applications and extensions. In subsequent papers we will address convexity breaking in the t-aspect, and extend this approach to $GL(n)$.

§2. Unwinding to Euler product

Let k be a number field, $G = GL_2$ over k, and define the standard subgroups:

$$P = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \qquad N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \qquad M = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\} \qquad Z = \text{center of } G$$

Also, for any place ν of k, let $K_\nu^{\max}$ be the standard maximal compact subgroup. That is, for finite ν, we take $K_\nu^{\max} = GL_2(\mathfrak{o}_\nu)$, at real places $K_\nu^{\max} = O(2)$, and at complex places $K_\nu^{\max} = U(2)$.

We form a suitable Poincaré series, and unwind a corresponding global integral to express it as an inverse Mellin transform of an Euler product. The Poincaré series of the form

$$\text{Pé}(g) \;=\; \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g) \qquad\qquad (g\in G_{\mathbb{A}}) \tag{2.1}$$

for suitable function φ on $G_{\mathbb{A}}$. Let φ be a *monomial* vector

$$\varphi = \bigotimes_{\nu} \varphi_\nu$$

where for finite primes ν, the local component φ_ν is defined as follows. For the usual maximal compact open subgroup $K_\nu^{\max}$ of G_ν, and for a character $\chi_{0,\nu}$ of M_ν, put

$$\varphi_\nu(g) = \begin{cases} \chi_{0,\nu}(m) & \text{for } g = mk,\ m \in M_\nu \text{ and } k \in K_\nu^{\max} \\ 0 & \text{for } g \notin M_\nu \cdot K_\nu^{\max} \end{cases} \qquad (\nu \text{ finite}) \tag{2.2}$$

For ν infinite, for now we do not entirely specify φ_ν, only requiring left equivariance

$$\varphi_\nu(mnk) = \chi_{0,\nu}(m) \cdot \varphi_\nu(n) \qquad (\text{for } \nu \text{ infinite},\ m \in M_\nu,\ n \in N_\nu \text{ and } k \in K_\nu^{\max}) \tag{2.3}$$

Require that $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ be M_k–invariant. For clarity, and not needing anything more general, assume that

$$\chi_{0,\nu}(m) = \left|\frac{y_1}{y_2}\right|_\nu^v \qquad \left(m = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \in M_\nu,\ v \in \mathbb{C}\right) \tag{2.4}$$

Thus, φ has trivial central character. Then φ is left $M_{\mathbb{A}}$–equivariant by χ_0. For ν infinite, our assumptions imply that

$$x \to \varphi_\nu \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a function of $|x|$ only.

Let f_1 and f_2 be cuspforms on $G_{\mathbb{A}}$. Eventually we will take $f_1 = f_2$, but for now merely require the following. At all ν, require (without loss of generality) that f_1 and f_2 have the same right K_ν–type, that this K_ν–type is *irreducible*, and that f_1 and f_2 correspond to the same vector in the K-type (up to scalar multiples). (Schur's lemma assures that this makes sense, insofar as there are no non-scalar automorphisms.) Suppose that the representations of $G_{\mathbb{A}}$ generated by f_1 and f_2 are *irreducible*, with the same central character. Last, require that each f_i is a special vector locally everywhere in the representation it generates, in the following sense. Let

$$f_i(g) = \sum_{\xi \in Z_k \backslash M_k} W_i(\xi g) \tag{2.5}$$

be the Fourier expansion of f_i, and let

$$W_i = \bigotimes_{\nu \le \infty} W_{i,\nu}$$

be the factorization of the Whittaker function W_i into local data. By [JL], we may require that for all $\nu < \infty$ the Hecke type local integrals

$$\int_{y \in k_\nu^\times} W_{i,\nu} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|_\nu^{s-\frac{1}{2}}\, dy$$

differ by at most an exponential function from the local L–factors for the representation generated by f_i. Eventually we will take $f_1 = f_2$, compatible with these requirements.

The integral under consideration is (with notation suppressing details)

$$I(\chi_0) = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}(g)\, f_1(g)\, \bar{f}_2(g)\, dg \tag{2.6}$$

For χ_0 (and archimedean data) in the range of absolute convergence, the integral unwinds (via the definition of the Poincaré series) to

$$\int_{Z_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \varphi(g)\, f_1(g)\, \bar{f}_2(g)\, dg$$

Using the Fourier expansion

$$f_1(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_1(\xi\, g)$$

this further unwinds to

$$\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g)\, W_1(g)\, \bar{f}_2(g)\, dg \tag{2.7}$$

Let C be the idele class group $GL_1(\mathbb{A})/GL_1(k)$, and $\widehat{C}$ its dual. More explicitly, by Fujisaki's Lemma (see Weil [W2]), the idele class group C is a product of a copy of $\mathbb{R}^+$ and a compact group C_0. By Pontryagin duality, $\widehat{C} \approx \mathbb{R} \times \widehat{C}_0$ with $\widehat{C}_0$ discrete. It is well-known that, for any compact open subgroup $U_{\rm fin}$ of the finite-prime part in C_0, the dual of $C_0/U_{\rm fin}$ is finitely generated with rank $[k:\mathbb{Q}] - 1$. The general Mellin transform and inversion are

$$\begin{aligned} f(x) \;&=\; \int_{\widehat{C}} \int_C f(y) \chi(y)\, dy\, \chi^{-1}(x)\, d\chi \\ &=\; \sum_{\chi' \in \widehat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \int_C f(y) \chi'(y) |y|^s\, dy\, {\chi'}^{-1}(x) |x|^{-s}\, ds \end{aligned} \tag{2.8}$$

for suitable Haar measure on C.

To formulate the main result of this section, we need one more piece of notation. For ν infinite and $s \in \mathbb{C}$, let

$$\begin{aligned} \mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu) \;&=\; \int_{Z_\nu \backslash M_\nu N_\nu} \int_{Z_\nu \backslash M_\nu} \varphi_\nu(m_\nu n_\nu) W_{1,\nu}(m_\nu n_\nu) \\ &\quad \cdot \overline{W}_{2,\nu}(m'_\nu n_\nu)\, \chi_\nu(m'_\nu)\, |m'_\nu|_\nu^{s-\frac{1}{2}}\, \chi_\nu(m_\nu)^{-1}\, |m_\nu|_\nu^{\frac{1}{2}-s}\, dm'_\nu\, dn_\nu\, dm_\nu \end{aligned} \tag{2.9}$$

and

$$\mathcal{K}_\infty(s,\, \chi_0,\, \chi) \;=\; \prod_{\nu | \infty} \mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu) \tag{2.10}$$

Here $\chi_0 = \bigotimes_\nu \chi_{0,\nu}$ is the character defining the monomial vector φ, and $\chi = \bigotimes_\nu \chi_\nu \in \widehat{C}_0$. Call the monomial vector φ *admissible*, if the series (2.1) defining Pé(g) converges absolutely and uniformly on compact sets, and furthermore, there exist distinct positive numbers a and b such that the integral defining $\mathcal{K}_\nu(s,\, \chi_{0,\nu},\, \chi_\nu)$ converges absolutely for all $\nu | \infty$, $\chi \in \widehat{C}_0$ and $a < \Re(s) < b$. We do not need a complete determination of the class of admissible archimedean data, since we will use a limited range of explicit choices. For instance, take

(2.11)

$$\varphi_\nu(g) \;=\; \begin{cases} |y|^v \left(\frac{|y|}{\sqrt{x^2+y^2}}\right)^w & \text{for } \nu|\infty \text{ real, and } g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G_\nu \\ |y|^{2v} \left(\frac{|y|}{\sqrt{|x|^2+|y|^2}}\right)^{2w} & \text{for } \nu|\infty \text{ complex, and } g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G_\nu \end{cases} \qquad (v,\, w \in \mathbb{C})$$

Then the monomial vector φ generated by (2.4) and (2.11) is admissible for $\Re(v)$ and $\Re(w)$ sufficiently large. This choice of φ will be used in Section 4 to derive an asymptotic formula for the GL_2 integral moment over the number field k. The main result of this section is

Theorem 2.12. *For φ an admissible monomial vector as above, for suitable $\sigma > 0$,*

$$I(\chi_0) \;=\; \sum_{\chi \in \widehat{C}_0} \frac{1}{2\pi i} \int\limits_{\Re(s)=\sigma} L(\chi_0 \cdot \chi^{-1} |\cdot|^{1-s},\, f_1) \cdot L(\chi|\cdot|^s,\, \bar{f}_2)\, \mathcal{K}_\infty(s,\, \chi_0,\, \chi)\, ds$$

Let S be a finite set of places including archimedean places, all absolutely ramified primes, and all finite bad places for f_1 and f_2. Then the sum is over a set $\widehat{C}_{0,S}$ of characters unramified outside S, with bounded ramification at finite places, depending only upon f_1 and f_2.

Proof: Applying (2.8) to $\bar{f}_2$ via the identification

$$\left\{ \begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} : y' \in C \right\} \approx C$$

and using the Fourier expansion

$$f_2(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W_2(\xi\, g)$$

the integral (2.7) is

$$\int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi(g)\, W_1(g) \left(\int_{\widehat{C}} \int_C \bar{f}_2(m'g)\, \chi(m')\, dm'\, d\chi \right) dg$$

$$= \int_{\widehat{C}} \left(\int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi(g)\, W_1(g) \int_C \sum_{\xi \in Z_k \backslash M_k} \overline{W}_2(\xi m' g)\, \chi(m')\, dm'\, dg \right) d\chi$$

$$= \int_{\widehat{C}} \left(\int_{Z_\mathbb{A} \backslash G_\mathbb{A}} \varphi(g)\, W_1(g) \int_{\mathbb{J}} \overline{W}_2(m' g)\, \chi(m')\, dm'\, dg \right) d\chi$$

where $\mathbb{J}$ is the ideles.

For fixed f_1 and f_2, the finite-prime ramification of the characters $\chi \in \widehat{C}$ is bounded, so there are only finitely many bad finite primes for all the χ which appear. In particular, all the characters χ which appear are unramified outside S and with bounded ramification, depending only on f_1 and f_2, at finite places in S. Thus, for $\nu \in S$ finite, there exists a compact open subgroup U_ν of $\mathfrak{o}_\nu^\times$ such that the kernel of the ν^{th} component χ_ν of χ contains U_ν for all characters χ which appear.

Since f_1 and f_2 generate irreducibles locally everywhere, the Whittaker functions W_i factor

$$W_i(\{g_\nu : \nu \leq \infty\}) = \Pi_\nu W_{i,\nu}(g_\nu)$$

Therefore, the inner integral over $Z_{\mathbb{A}}\backslash G_{\mathbb{A}}$ and $\mathbb{J}$ factors over primes, and

$$I(\chi_0) = \int_{\widehat{C}} \Pi_\nu \left(\int_{Z_\nu\backslash G_\nu} \int_{k_\nu^\times} \varphi_\nu(g_\nu)\, W_{1,\nu}(g_\nu)\, \overline{W}_{2,\nu}(m'_\nu g_\nu)\, \chi_\nu(m'_\nu)\, dm'_\nu\, dg_\nu \right) d\chi$$

Let ω_ν be the ν^{th} component of the central character ω of f_2. Define a character of M_ν by

$$\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \to \omega_\nu \begin{pmatrix} y_2 & 0 \\ 0 & y_2 \end{pmatrix} \chi_\nu \begin{pmatrix} y_1/y_2 & 0 \\ 0 & 1 \end{pmatrix}$$

Still denote this character by χ_ν, without danger of confusion. In this notation, the last expression of $I(\chi_0)$ is

$$I(\chi_0) = \int_{\widehat{C}} \Pi_\nu \left(\int_{Z_\nu\backslash G_\nu} \int_{Z_\nu\backslash M_\nu} \varphi_\nu(g_\nu)\, W_{1,\nu}(g_\nu)\, \overline{W}_{2,\nu}(m'_\nu g_\nu)\, \chi_\nu(m'_\nu)\, dm'_\nu\, dg_\nu \right) d\chi$$

Suppressing the index ν, the ν^{th} local integral is

$$\int_{Z\backslash G} \int_{Z\backslash M} \varphi(g)\, W_1(g)\, \overline{W}_2(m'g)\, \chi(m')\, dm'\, dg$$

Take ν *finite* such that both f_1 and f_2 are right $K_\nu^{\max}$–invariant. Use a ν–adic Iwasawa decomposition $g = mnk$ with $m \in M$, $n \in N$, and $k \in K_\nu^{\max}$. The Haar measure is $d(mnk) = dm\, dn\, dk$ with Haar measures on the factors. The integral becomes

$$\int_{Z\backslash MN} \int_{Z\backslash M} \varphi(mn)\, W_1(mn)\, \overline{W}_2(m'mn)\, \chi(m')\, dm'\, dn\, dm$$

To symmetrize the integral, replace m' by $m'm^{-1}$ to obtain

$$\int_{Z\backslash MN} \int_{Z\backslash M} \varphi(mn)\, W_1(mn)\, \overline{W}_2(m'n)\, \chi(m')\, \chi(m)^{-1}\, dm'\, dn\, dm$$

The Whittaker functions W_i have left N–equivariance

$$W_i(ng) = \psi(n)\, W_i(g) \quad \text{(fixed non-trivial } \psi)$$

so

$$W_1(mn) = W_1(mnm^{-1}\, m) = \psi(mnm^{-1})\, W_1(m)$$

and similarly for W_2. Thus, letting

$$X(m, m') = \int_N \varphi(n)\, \psi(mnm^{-1})\, \overline{\psi}(m'nm'^{-1})\, dn$$

the local integral is

$$\int_{Z\backslash M}\int_{Z\backslash M} \chi_0(m)\, W_1(m)\, \overline{W}_2(m')\, \chi(m')\, \chi^{-1}(m)\, X(m,m')\, dm'\, dm$$

We claim that for m and m' in the supports of the Whittaker functions, the inner integral $X(m,m')$ is constant, independent of m,m'. (And is 1 for almost all finite primes.) First, $\varphi(mn)$ is 0, unless $n \in M\cdot K^{\max}\cap N$, that is, unless $n\in N\cap K^{\max}$. On the other hand,

$$\psi(mnm^{-1})\cdot W_1(mk) = \psi(mnm^{-1})\cdot W_1(m) = W_1(mn) = W_1(m) \qquad (\text{for } n\in N\cap K)$$

Thus, for $W_1(m)\neq 0$, necessarily $\psi(mnm^{-1})=1$. A similar discussion applies to W_2. So (up to normalization) the inner integral is 1 for m,m' in the supports of W_1 and W_2. Then

$$\begin{aligned}&\int_{Z\backslash M}\int_{Z\backslash M} \chi_0(m)\, W_1(m)\, \overline{W}_2(m')\, \chi(m')\, \chi^{-1}(m)\, dm\, dm' \\ &= \int_{Z\backslash M} (\chi_0\cdot\chi^{-1})(m)\, W_1(m)\, dm \cdot \int_{Z\backslash M} \chi(m')\, \overline{W}_2(m')\, dm' \\ &= L_\nu(\chi_{0,\nu}\cdot\chi_\nu^{-1}|\cdot|_\nu^{1/2},\, f_1)\cdot L_\nu(\chi_\nu|\cdot|_\nu^{1/2},\, \bar{f}_2)\end{aligned}$$

i.e., the product of local factors of the standard L–functions in the theorem (up to exponential functions at finitely-many finite primes) by our assumptions on f_1 and f_2.

For non-trivial right K–type σ, the argument is similar but a little more complicated. The key point is that the inner integral over N (as above) should not depend on mk and $m'k$ for mk and $m'k$ in the support of the Whittaker functions. Changing conventions for a moment, look at V_σ–valued Whittaker functions, and consider any W in the ν^{th} Whittaker space for f_i having right K–isotype σ. Thus,

$$W(gk) = \sigma(k)\cdot W(g) \qquad (\text{for } g\in G \text{ and } k\in K)$$

For $\varphi(mn)\neq 0$, again $n\in N\cap K$. Then

$$\sigma(k)\cdot\psi(mnm^{-1})\cdot W(m) = W(mnk) = \sigma(k)\cdot W(mn) = \sigma(k)\cdot\sigma(n)\cdot W(m)$$

where in the last expression n comes out on the right by the right σ–equivariance of W. For m in the support of W, $\sigma(n)$ acts by the scalar $\psi(mnm^{-1})$ on $W(mk)$, for all $k\in K$. Thus, $\sigma(n)$ is scalar on that copy of V_σ. At the same time, this scalar is $\sigma(n)$, so is independent of m if $W(m)\neq 0$. Thus, except for a common integral over K, the local integral falls into two pieces, each yielding the local factor of the L–function. The common integral over K is a constant (from Schur orthogonality), non-zero since the two vectors are collinear in the K-type. □

At this point the archimedean local factors of the Euler product are not specified. The option to vary the choices is *essential* for applications.

§3. Spectral decomposition of Poincaré series

The objective now is to spectrally decompose the Poincaré series defined in (2.1). As we shall see, in general Pé(g) is not square-integrable. However, choosing the archimedean part of the monomial vector φ to have enough decay, and after an obvious Eisenstein series is subtracted, the Poincaré series is not only in L^2 but also has sufficient decay so that its integrals against Eisenstein series converge absolutely. In particular, if the archimedean data is specialized to (2.11), the Poincaré series Pé(g) has meromorphic continuation in the variables v and w. This is achieved via spectral decomposition and meromorphic continuation of the spectral fragments, with estimates on the decomposition coefficients. See [DG1], [DG2] when $k = \mathbb{Q}$, $\mathbb{Q}(i)$.

Let k be a number field, $G = GL_2$ over k, and ω a unitary character of $Z_k \backslash Z_{\mathbb{A}}$. Recall the decomposition

$$L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}, \omega) \;=\; L^2_{\text{cusp}}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}, \omega) \;\oplus\; L^2_{\text{cusp}}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}, \omega)^{\perp}$$

The orthogonal complement

$$L^2_{\text{cusp}}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}, \omega)^{\perp} \;\approx\; \{1-\text{dimensional representations}\} \oplus \int^{\oplus}_{(GL_1(k)\backslash GL_1(\mathbb{A}))^{\widehat{}}} \bigotimes_{\nu} \text{Ind}_{P_\nu}^{G_\nu}(\chi_\nu\, \delta_\nu^{1/2})\, d\chi$$

where δ the modular function on $P_{\mathbb{A}}$, and the isomorphism is via Eisenstein series.

The projection to cuspforms is straightforward componentwise. We have

Proposition 3.1. *Let f be a cuspform on $G_{\mathbb{A}}$ generating a spherical representation locally everywhere, and suppose f corresponds to a spherical vector everywhere locally. In the region of absolute convergence of the Poincaré series Pé(g), the integral*

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \bar{f}(g)\, \text{Pé}(g)\, dg$$

is an Euler product. At finite ν, the corresponding local factors are $L_\nu(\chi_{0,\nu}\,|\cdot|_\nu^{1/2}, \bar{f}\,)$ (up to a constant depending on the set of absolutely ramified primes in k).

Proof: The computation uses the same facts as the Euler factorization in the previous section. Using the Fourier expansion

$$f(g) \;=\; \sum_{\xi \in Z_k \backslash M_k} W(\xi g)$$

unwind

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \bar{f}(g)\, \text{Pé}(g)\, dg \;=\; \int_{Z_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}} \sum_{\xi} \overline{W}(\xi g)\, \varphi(g)\, dg \;=\; \int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \overline{W}(g)\, \varphi(g)\, dg$$

$$= \prod_{\nu} \left(\int_{Z_\nu \backslash G_\nu} \overline{W}_\nu(g_\nu)\, \varphi_\nu(g_\nu)\, dg_\nu \right)$$

At finite ν, suppressing the subscript ν, the integrand in the ν^{th} local integral is right $K_\nu^{\max}$–invariant, so we can integrate over MN with left Haar measure. The ν^{th} Euler factor is

$$\int_{Z\backslash M} \int_N \overline{W}(mn)\, \varphi(mn)\; dn\, dm \backslash = \int_{Z\backslash M} \int_N \overline{\psi}(mnm^{-1})\, \overline{W}(m)\, \chi_0(m)\, \varphi(n)\, dn\, dm$$

for all finite primes ν. The integral over n is

$$\int_N \overline{\psi}(mnm^{-1})\,\varphi(n)\,dn$$

For $\varphi(n)$ to be non-zero requires n to lie in $M \cdot K$, which further requires, as before, that $n \in N \cap K$. And, again, $W(m) = 0$ unless

$$m(N \cap K)m^{-1} \subset N \cap K$$

The character ψ is trivial on $N \cap K$. Thus, the integral over N is really the integral of 1 over $N \cap K$. Thus, at finite primes ν, the local factor is

$$\int_{Z\backslash M} \overline{W}(m)\,\chi_0(m)\,dm \;=\; L_\nu(\chi_{0,\nu}\,|\cdot|_\nu^{1/2},\,\bar{f}\,) \qquad \square$$

Of course, the spectral decomposition of a right $K_{\mathbb{A}}$–invariant automorphic form only can involve everywhere locally spherical cuspforms.

Assume that φ is given by (2.11). Taking $\Re(v)$ and $\Re(w)$ sufficiently large to ensure absolute convergence of Pé(g), the local integral in Proposition 3.1 at infinite ν is

$$\int_{Z_\nu\backslash G_\nu} \overline{W}_\nu(g_\nu)\,\varphi_\nu(g_\nu)\,dg_\nu \;=\; \mathcal{G}_\nu(\tfrac{1}{2} + i\bar{\mu}_{f,\nu}; v, w)$$

where, up to a constant,

$$\mathcal{G}_\nu(s; v, w) = \pi^{-v}\,\frac{\Gamma\big(\frac{v+1-s}{2}\big)\Gamma\big(\frac{v+w-s}{2}\big)\Gamma\big(\frac{v+s}{2}\big)\Gamma\big(\frac{v+w+s-1}{2}\big)}{\Gamma\big(\frac{w}{2}\big)\Gamma\big(v+\frac{w}{2}\big)} \tag{3.2}$$

for $\nu|\infty$ real, and

$$\mathcal{G}_\nu(s; v, w) = (2\pi)^{-2v}\,\frac{\Gamma(v+1-s)\Gamma(v+w-s)\Gamma(v+s)\Gamma(v+w+s-1)}{\Gamma(w)\Gamma(2v+w)} \tag{3.3}$$

for $\nu|\infty$ complex. In the above expression $\mu_{f,\nu}$ denotes the local parameter of f at ν. Then, the sum corresponding to discrete spectrum in the spectral decomposition of Pé(g) converges absolutely for $(v, w) \in \mathbb{C}^2$, apart from the poles of $\mathcal{G}_\nu(\frac{1}{2} + i\bar{\mu}_{f,\nu}; v, w)$.

For the remaining decomposition, subtract (as in [DG1], [DG2]) a finite linear combination of Eisenstein series from the Poincaré series, leaving a function in L^2 with sufficient decay to be integrated against Eisenstein series. The correct Eisenstein series to subtract becomes visible from the dominant part of the constant term of the Poincaré series (below).

Write the Poincaré series as

$$\text{Pé}(g) \;=\; \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g) \;=\; \sum_{\gamma\in P_k\backslash G_k}\;\sum_{\beta\in N_k} \varphi(\beta\gamma g)$$

By Poisson summation

$$\text{Pé}(g) \;=\; \sum_{\gamma\in P_k\backslash G_k}\;\sum_{\psi\in(N_k\backslash N_{\mathbb{A}})^\frown} \widehat{\varphi}_{\gamma g}(\psi) \tag{3.4}$$

where, $\varphi_g(n) = \varphi(ng)$, and $\widehat{\varphi}$ is Fourier transform along $N_{\mathbb{A}}$. The trivial–ψ (that is, with $\psi = 1$) Fourier term

$$\sum_{\gamma\in P_k\backslash G_k} \widehat{\varphi}_{\gamma g}(1) \tag{3.5}$$

is an Eisenstein series, since the function

$$g \to \widehat{\varphi}_g(1) = \int_{N_{\mathbb{A}}} \varphi(ng)\, dn$$

is left $M_{\mathbb{A}}$–equivariant by the character $\delta\chi_0$, and left $N_{\mathbb{A}}$–invariant.

For $\xi \in M_k$,

$$\begin{aligned}\widehat{\varphi}_{\xi g}(\psi) \;&=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(n\xi g)\, dn \\ &=\; \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \varphi(\xi\cdot\xi^{-1}n\xi\cdot g)\, dn \;=\; \int_{N_{\mathbb{A}}} \overline{\psi}(\xi n\xi^{-1})\, \varphi(n\cdot g)\, dn \;=\; \widehat{\varphi}_g(\psi^{\xi})\end{aligned}$$

where $\psi^{\xi}(n) = \psi(\xi n \xi^{-1})$, by replacing n by $\xi n\xi^{-1}$, using the left M_k–invariance of φ, and invoking the product formula to see that the change-of-measure is trivial. Since this action of $Z_k\backslash M_k$ is transitive on non-trivial characters on $N_k\backslash N_{\mathbb{A}}$, for a fixed choice of non-trivial character ψ, the sum over non-trivial characters can be rewritten as a more familiar sort of Poincaré series

$$\begin{aligned}\sum_{\gamma\in P_k\backslash G_k}\ \sum_{\psi'\in (N_k\backslash N_{\mathbb{A}})^{\frown}} \widehat{\varphi}_{\gamma g}(\psi') \;&=\; \sum_{\gamma\in P_k\backslash G_k}\ \sum_{\xi\in Z_k\backslash M_k} \widehat{\varphi}_{\gamma g}(\psi^{\xi}) \\ &=\; \sum_{\gamma\in P_k\backslash G_k}\ \sum_{\xi\in Z_k\backslash M_k} \widehat{\varphi}_{\xi\gamma g}(\psi) \;=\; \sum_{\gamma\in Z_k N_k\backslash G_k} \widehat{\varphi}_{\gamma g}(\psi)\end{aligned}$$

Denote this version of the original Poincaré series, with the Eisenstein series subtracted, by

$$\text{Pé}^*(g) \;=\; \sum_{\gamma\in Z_kN_k\backslash G_k} \widehat{\varphi}_{\gamma g}(\psi) \;=\; \text{Pé}(g) \;-\; \sum_{\gamma\in P_k\backslash G_k} \widehat{\varphi}_{\gamma g}(1) \tag{3.6}$$

Granting that Pé* is not only in $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ but also has sufficient decay so that its integrals against Eisenstein series (with parameter in a bounded vertical strip containing the critical line) converge absolutely, the continuous-spectrum components of Pé* are computed by integrals against Eisenstein series

$$E(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \eta(\gamma g)$$

for η left P_k–invariant, left $M_{\mathbb{A}}$–equivariant, and left $N_{\mathbb{A}}$–invariant. The Fourier expansion of this Eisenstein series is

$$E(g) \;=\; \sum_{\psi'\in (N_k\backslash N_{\mathbb{A}})^{\frown}} \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi'}(n)\, E(ng)\, dn$$

For a fixed non-trivial character ψ, the ψ^{th} Fourier term is

$$\int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n)\, E(ng)\, dn \;=\; \int_{N_k\backslash N_{\mathbb{A}}} \overline{\psi}(n) \sum_{\gamma\in P_k\backslash G_k} \eta(\gamma ng)\, dn$$

$$= \sum_{w \in P_k \backslash G_k / N_k} \int_{(N_k \cap w^{-1} P_k w) \backslash N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(wng)\, dn$$

$$= \int_{N_k \backslash N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(ng)\, dn \; + \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_o ng)\, dn$$

$$= 0 \; + \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_o ng)\, dn \qquad (\text{where } w_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

because ψ is non-trivial and η is left $N_{\mathbb{A}}$–invariant. Denote the ψ^{th} Fourier term by

$$W(g) \;=\; W_{\eta,\psi}(g) \;= \int_{N_{\mathbb{A}}} \overline{\psi}(n)\, \eta(w_o ng)\, dn$$

With parameters in a suitable range,

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \text{Pé}^*(g)\, \overline{E}(g)\, dg \;= \int_{Z_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \int_{N_k \backslash N_{\mathbb{A}}} \widehat{\varphi}_{ng}(\psi)\, \overline{E}(ng)\, dn\, dg \tag{3.7}$$

$$= \int_{Z_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \widehat{\varphi}_g(\psi) \int_{N_k \backslash N_{\mathbb{A}}} \psi(n) \overline{E}(ng)\, dn\, dg \;= \int_{Z_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \widehat{\varphi}_g(\psi)\, \overline{W}(g)\, dg$$

When η and φ are monomial in the respective (restricted) tensor products of local representations, the latter integral is an Euler product.

At finite primes ν, we evaluate the ν^{th} Euler factor. In this case, the local components of both η and φ are right K_ν–invariant with maximal compact K_ν. In Iwasawa coordinates nmk, with $n \in N_\nu$, $m \in M_\nu$ and $k \in K_\nu$, a Haar measure is

$$d(nmk) = \delta^{-1}(m)\, dn \cdot dm \cdot dk$$

Thus, suppressing subscripts, the local integral is

$$\int_{Z \backslash M} \widehat{\varphi}_m(\psi)\, \overline{W}(m)\, \delta^{-1}(m)\, dm \tag{3.8}$$

There is no residual contribution to the spectral decomposition of $\text{Pé}^*(g)$.

Now we describe the continuous part of the spectral decomposition. At every place ν, let η_ν be the spherical vector in the (non-normalized) principal series $\text{Ind}_{P_\nu}^{G_\nu} \chi_\nu$, normalized by $\eta_\nu(1) = 1$. Take $\eta = \bigotimes_{\nu \le \infty} \eta_\nu$. The corresponding Eisenstein series is

$$E_\chi(g) \;\;= \sum_{\gamma \in P_k \backslash G_k} \eta(\gamma g)$$

For any left $Z_{\mathbb{A}} G_k$–invariant and right $K_{\mathbb{A}}$–invariant square-integrable f on $G_{\mathbb{A}}$, write

$$\langle f, E_\chi \rangle = \int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} f(g)\, \overline{E_\chi(g)}\, dg$$

With suitable normalization of measures,

$$\text{continuous spectrum part of } f \;=\; \int\limits_{\Re(\chi)=\frac{1}{2}} \langle f, E_\chi\rangle\, E_\chi\, d\chi$$

This formula requires isometric extensions to L^2 of integral formulas that converge literally only on a smaller dense subspace (pseudo-Eisenstein series).

Happily, applying this to Pé*, for $\Re(\chi_0)$ sufficiently large in comparison to $\Re(\chi)$ the integrals $\langle \text{Pé}^*, E_\chi\rangle$ were computed above in (3.8), yielding the same form of local integrals (at all finite places).

Assume φ is defined by (2.11). Then the integrals $\langle \text{Pé}^*, E_\chi\rangle$ can be evaluated in terms of (3.2) and (3.3) at all infinite places. For fixed character χ, the corresponding integral of Eisenstein series can be meromorphically continued by shifting the vertical line as in [DG1]. (The normalization of the Poincaré series in the latter reference is slightly different than here: in that paper the Poincaré series was normalized by needlessly multiplying by the gamma factor $\pi^{-w/2}\Gamma(w/2)$.) The sum over the characters χ is absolutely convergent giving the meromorphic continuation of the Poincaré series Pé(g). If $v = 0$, the Poincaré series has, as a function of the variable w, a pole of order $r_1 + r_2 + 1$ at $w = 1$, otherwise being holomorphic for $\Re(w) > 1 - c$, with $c > 0$ sufficiently small. As usual, r_1 and r_2 denote the number of real and complex embeddings of k, respectively. We record this discussion as

Theorem 3.9. *Assume φ is defined by (2.11). The Poincaré series Pé(g), originally defined for $\Re(v)$ and $\Re(w)$ large, has meromorphic continuation to a region in $\mathbb{C}^2$ containing $v = 0$, $w = 1$. As a function of w, for $v = 0$, it is holomorphic in the half-plane $\Re(w) > 1 - c$ for small positive constant c, except when $w = 1$ where it has a pole of order $r_1 + r_2 + 1$.*

§4. Asymptotic formula

Let k be a number field with r_1 real embeddings and $2r_2$ complex embeddings. Assume that φ is specialized to (2.11). By Theorem 2.12, for $\Re(v)$ and $\Re(w)$ sufficiently large, the integral $I(\chi_0) = I(v, w)$ defined by (2.6) is

$$I(v,w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi i} \int\limits_{\Re(s)=\sigma} L(\chi^{-1}|\cdot|^{v+1-s},\, f_1)\cdot L(\chi|\cdot|^{s},\, \bar{f}_2)\, \mathcal{K}_\infty(s,\, v,\, w,\, \chi)\, ds \tag{4.1}$$

where $\mathcal{K}_\infty(s, v, w, \chi)$ is given by (2.9) and (2.10), and where the sum is over $\chi \in \widehat{C}_0$ unramified outside S and with bounded ramification, depending only on f_1 and f_2.

By Theorem 3.9, it follows that $I(v, w)$ admits meromorphic continuation to a region in $\mathbb{C}^2$ containing the point $v = 0$, $w = 1$. In particular, if $f_1 = f_2 = \bar{f}$, then for $c > 0$ sufficiently small, the function $I(0, w)$ is holomorphic for $\Re(w) > 1 - c$, except for $w = 1$, where it has a pole of order $r_1 + r_2 + 1$.

We will shift the line of integration to $\Re(s) = \frac{1}{2}$ in (4.1) and set $v = 0$. To do so, we need *some* analytic continuation and good polynomial growth in $|\Im(s)|$ for the kernel function $\mathcal{K}_\infty(s, v, w, \chi)$. In fact, it is desirable for applications to have *precise* asymptotic formulae as the parameters s, v, w, χ vary. By the decomposition (2.10), the analysis of the kernel $\mathcal{K}_\infty(s, v, w, \chi)$ reduces to the corresponding analysis of the local component $\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu)$, for $\nu|\infty$. When ν is complex, one can use the asymptotic formula already established in [DG2], Theorem 6.2. For coherence, we include a simple computation matching, as it should, the local integral (2.9), for ν complex, with the integral (4.15) in [DG2].

Fix a complex place $\nu|\infty$. Recall that any character χ_ν of $Z_\nu\backslash M_\nu \approx \mathbb{C}^\times$ has the form

$$\chi_\nu(m_\nu) \;=\; |z_\nu|_{\mathbb{C}}^{-\frac{\ell_\nu}{2}+it_\nu}\, z_\nu^{\ell_\nu} \qquad \left(m_\nu = \begin{pmatrix} z_\nu & 0\\ 0 & 1\end{pmatrix},\; t_\nu\in\mathbb{R},\; \ell_\nu\in\mathbb{Z}\right)$$

Assuming f_1 and f_2 have trivial K_∞–type, the local integral (2.9) becomes

$$\begin{aligned}\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; &\int_0^\infty\int_0^\infty\int_{\mathbb{C}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left(|x|^2+1\right)^{-w} e^{2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(y_1xe^{i\theta_1}-y_2xe^{i\theta_2})}\\ &\cdot y_1^{2v+1-2s-2it_\nu} K_{i\mu_1}(4\pi y_1)\, y_2^{2s+2it_\nu-1} K_{i\mu_2}(4\pi y_2)\, e^{i\ell_\nu\theta_1} e^{-i\ell_\nu\theta_2}\; d\theta_1 d\theta_2\, dx\, dy_1 dy_2\end{aligned}$$

Replacing x by x/y_1, we obtain

$$\begin{aligned}\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; &\int_0^\infty\int_0^\infty\int_{\mathbb{C}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left(\frac{y_1}{\sqrt{|x|^2+y_1^2}}\right)^{2w} e^{2\pi i\cdot \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\left(xe^{i\theta_1}-\frac{y_2}{y_1}xe^{i\theta_2}\right)}\\ &\cdot y_1^{2v-1-2s-2it_\nu} K_{i\mu_1}(4\pi y_1)\, y_2^{2s+2it_\nu-1} K_{i\mu_2}(4\pi y_2)\, e^{i\ell_\nu\theta_1} e^{-i\ell_\nu\theta_2}\; d\theta_1 d\theta_2\, dx\, dy_1 dy_2\end{aligned}$$

If we further substitute

$$y_1 = r\cos\phi, \quad x_1 = r\sin\phi\cos\theta, \quad x_2 = r\sin\phi\sin\theta, \quad y_2 = u\cos\phi$$

with $0\le\phi\le\frac{\pi}{2}$ and $0\le\theta\le 2\pi$, then

$$\begin{aligned}\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; &\int_0^\infty\int_0^\infty\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (\cos\phi)^{2w+2v-1}\, e^{2\pi i\cdot\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(r\sin\phi\cdot e^{i(\theta+\theta_1)}-u\sin\phi\cdot e^{i(\theta+\theta_2)})}\\ &\cdot r^{2v+1-2s-2it_\nu} K_{i\mu_1}(4\pi r\cos\phi)\, u^{2s+2it_\nu-1} K_{i\mu_2}(4\pi u\cos\phi)\, e^{i\ell_\nu\theta_1} e^{-i\ell_\nu\theta_2}\sin\phi\; d\theta_1 d\theta_2\, d\theta\, d\phi\, dr\, du\end{aligned}$$

Using the Fourier expansion

$$e^{it\sin\theta} \;=\; \sum_{k=-\infty}^{\infty} J_k(t)\, e^{ik\theta}$$

we obtain

$$\begin{aligned}\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = (2\pi)^3 &\int_0^\infty\int_0^\infty\int_0^{\frac{\pi}{2}} K_{i\mu_1}(4\pi r\cos\phi) K_{i\mu_2}(4\pi u\cos\phi) J_{\ell_\nu}(4\pi r\sin\phi) J_{\ell_\nu}(4\pi u\sin\phi)\\ &\cdot u^{2s+2it_\nu} r^{2v+2-2s-2it_\nu}(\cos\phi)^{2w+2v-1}\sin\phi\;\frac{d\phi dr du}{ru}\end{aligned}$$

In the notation of [DG2], equation (4.15), this is essentially $\mathcal{K}_{\ell_\nu}(2s+2it_\nu, 2v, 2w)$. It follows that $\mathcal{K}_\nu(s,v,w,\chi_\nu)$ is analytic in a region $\mathcal{D}:\Re(s)=\sigma>\frac{1}{2}-\epsilon_0$, $\Re(v)>-\epsilon_0$ and $\Re(w)>\frac{3}{4}$, with a fixed (small) $\epsilon_0>0$, and moreover, we have the asymptotic formula

$$\begin{aligned}\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = \pi^{-2v+1} A(2v, 2w, \mu_1, \mu_2)\cdot\left(1+\ell_\nu^2+4(t+t_\nu)^2\right)^{-w}&\\ \cdot\left[1 \;+\; \mathcal{O}_{\sigma,\, v,\, w,\, \mu_1,\, \mu_2}\left(\left(\sqrt{1+\ell_\nu^2+4(t+t_\nu)^2}\right)^{-1}\right)\right]&\end{aligned} \tag{4.2}$$

where

$$A(v,w,\mu_1,\mu_2) \;=\; 2^{2w-2v-4}\,\frac{\Gamma\left(\frac{w+v+i\mu_1+i\mu_2}{2}\right)\Gamma\left(\frac{w+v-i\mu_1+i\mu_2}{2}\right)\Gamma\left(\frac{w+v+i\mu_1-i\mu_2}{2}\right)\Gamma\left(\frac{w+v-i\mu_1-i\mu_2}{2}\right)}{\Gamma\left(w+v\right)} \tag{4.3}$$

For ν real, the corresponding argument is even simpler (see [DG1] and [Zh2]). In this case, the asymptotic formula of $\mathcal{K}_\nu(s,v,w,\chi_\nu)$ becomes

$$\begin{aligned}\mathcal{K}_\nu(s,\,v,\,w,\,\chi_\nu) = B(v,w,\mu_1,\mu_2)\cdot\big(1+|t+t_\nu|\big)^{-w}&\\ \cdot\left[1\;+\;\mathcal{O}_{\sigma,\,v,\,w,\,\mu_1,\,\mu_2}\left(\big(1+|t+t_\nu|\big)^{-\frac12}\right)\right]&\end{aligned} \tag{4.4}$$

where $B(v,w,\mu_1,\mu_2)$ is a similar ratio of gamma functions.

It now follows that for $\Re(w)$ sufficiently large,

$$I(0,w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} L\big(\chi^{-1}|\cdot|^{\frac12-it},\,f_1\big)\cdot L\big(\chi|\cdot|^{\frac12+it},\,\bar f_2\big)\,\mathcal{K}_\infty(\tfrac12+it,\,0,\,w,\,\chi)\,dt \tag{4.5}$$

In the above we assumed for simplicity that both local parameters $\mu_1=\mu_{f_1,\nu}$ and $\mu_2=\mu_{f_2,\nu}$ at ν are real.

Since $I(0,w)$ has analytic continuation to $\Re(w)>1-c$, a mean value result can already be established by standard arguments. For instance, assume $f_1=f_2=\bar f$, and choose a function $h(w)$ which is holomorphic and with sufficient decay (in $|\Im(w)|$) in a fixed vertical strip. One can choose a suitable product of gamma functions, for example. Consider the integral

$$\frac{1}{i}\int\limits_{\Re(w)=L} I(0,w)\,h(w)\,T^w\,dw \tag{4.6}$$

with L a large positive constant. Taking $h(1)=1$, we have the asymptotic formula

$$\sum_{\chi\in\widehat{C}_{0,S}}\;\int\limits_{-\infty}^{\infty}|L(\tfrac12+it,\,f\otimes\chi)|^2\cdot M_{\chi,\,T}(t)\,dt\;\sim\;A\,T\,(\log T)^{r_1+r_2} \tag{4.7}$$

for some computable positive constant A, where

$$M_{\chi,\,T}(t)\;=\;\frac{1}{i}\int\limits_{\Re(w)=L}\mathcal{K}_\infty(\tfrac12+it,\,0,\,w,\,\chi)\,h(w)\,T^w\,dw \tag{4.8}$$

For a character $\chi\in\widehat{C}_0$, put

$$\kappa_\chi(t)\;=\;\prod_{\substack{\nu|\infty\\ \nu\,\text{real}}}|t+t_\nu|\quad\cdot\quad\prod_{\substack{\nu|\infty\\ \nu\,\text{complex}}}\big(\ell_\nu^2+4(t+t_\nu)^2\big)\qquad(t\in\mathbb{R})$$

where t_ν and ℓ_ν are the parameters of the local component χ_ν of χ. Since χ is trivial on the positive reals,

$$\sum_{\nu|\infty}\alpha_\nu\,t_\nu=0$$

with $\alpha_\nu=1$ or 2 according as ν is real or complex. Then, the main contribution to the asymptotic formula (4.7) comes from terms for which $\kappa_\chi(t)\ll T$.

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