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Representations with Iwahori-fixed vectors

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- Generic algebras
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Using the ideas of [Casselman 1980] descended from the Borel-Matsumoto theorem on admissible representations of p-adic reductive groups containing Iwahori-fixed vectors, it is possible to give an easily verifiable sufficient criterion for irreducibility of degenerate principal series. This result is not comparable to irreducibility results such as [Muić-Shahidi 1998], but is easily proven and easily applied.

Let G be a p-adic reductive group, P a minimal parabolic, N its unipotent radical, B the Iwahori subgroup matching P, and K a maximal compact subgroup containing B. As usual, a character $\chi : P/N \to \mathbf{C}^\times$ is *unramified* if it is trivial on $P \cap K$. Let $\delta = \delta_P$ be the modular function on P, and $\rho = \rho_P = \delta_P^{1/2}$ the square root of this modular function.

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1. Generic algebras

Let (W, S) be a Coxeter system, and fix a commutative ring R. We consider S-tuples of pairs (a_s, b_s) of elements of R, subject to the requirement that if $s_1 = ws_2w^{-1}$ for $w \in W$ and $s_1, s_2 \in S$, then $a_{s_1} = a_{s_2}$ and $b_{s_1} = b_{s_2}$. Refer to the constants a_s, b_s as *structure constants*. Let $\mathcal{A}$ be a free R-module with R-basis $\{T_w : w \in W\}$.

Theorem: Given a Coxeter system (W, S) and structure constants a_s, b_s $(s \in S)$ there is exactly one associative algebra structure on $\mathcal{A}$ so that

$$\begin{array}{rcll} T_sT_w &=& T_{sw} & \text{for } \ell(sw) > \ell(w) \\ T_s^2 &=& a_sT_s + b_sT_1 & \text{for all } s \in S \end{array}$$

and with the requirement that T_1 is the identity in $\mathcal{A}$. With this associative algebra structure, we also have

$$T_sT_w = a_sT_w + b_sT_{sw} \qquad \text{for } \ell(sw) < \ell(w)$$

Further, the right-handed version of these identities hold

$$\begin{array}{rcll} T_wT_s &=& T_{ws} & \text{for } \ell(ws) > \ell(w) \\ T_wT_s &=& a_sT_w + b_sT_{ws} & \text{for } \ell(ws) < \ell(w) \end{array}$$

Granting the theorem, for given structure constants define the **generic algebra**

$$\mathcal{A} = \mathcal{A}(W, S, \{(a_s, b_s) : s \in S\})$$

to be the associative R-algebra determined by the theorem.

Remark: If all $a_s = 0$ and $b_s = 1$ then the associated generic algebra is the *group algebra* of the group W over the ring R. When (W, S) is *affine* suitable structure constants yield the *Iwahori-Hecke algebra* of associated p-adic groups. Most often, this is

$$a_s = q - 1 \qquad b_s = q$$

where q is the residue field cardinality of a discrete valuation ring.

Proof: First, we see that the right-handed versions of the statements follow from the left-handed ones. Let $\ell(wt) > \ell(w)$ for $w \in W$ and $t \in S$. Since $\ell(wt) > \ell(w)$, there is $s \in S$ such that $\ell(sw) < \ell(w)$. Then

$$\ell(sw) + 1 = \ell(w) = \ell(wt) - 1 \leq \ell(swt) + 1 - 1 = \ell(swt) \leq \ell(sw) + 1$$

Thus,

$$\ell(w) = \ell(swt) > \ell(sw)$$

Then $T_w = T_s\, T_{sw}$ and

$$T_w T_t = (T_s T_{sw}) T_t = T_s T_{swt} = T_{wt}$$

where the first equality follows from $\ell(w) = \ell(ssw) > \ell(sw)$, the second follows by induction on length, and the third follows from $\ell(sswt) > \ell(swt)$. This gives the desired result. If $\ell(wt) < \ell(w)$, then by the result just proven $T_{wt} T_t = T_w$. Multiplying both sides by T_t on the right yields

$$T_w T_t = T_{wt} T_t^2 = T_{wt}(a_t T_t + b_t T_1) = a_t T_{wt} T_t + b_t T_{wt} = a_t T_w + b_t T_{wt}$$

where we computed T_t^2 by the defining relation. Thus, the right-handed versions follow from the left-handed.

Next, suppose that $\ell(sw) < \ell(w)$ and prove that

$$T_s T_w = a_s T_w + b_s T_{sw}$$

If $\ell(w) = 1$, then $w = s$, and the desired equality is the assumed equality

$$T_s^2 = a_s T_s + b_s T_1$$

Generally, from $\ell(s(sw)) = \ell(w) > \ell(sw)$ follows $T_s T_{sw} = T_w$. Then

$$T_s T_w = T_s^2 T_{sw} = (a_s T_s + b_s T_1) T_{sw} = a_s T_s T_{sw} + b_s T_{sw} = a_s T_w + b_s T_{sw}$$

as asserted. Thus, the multiplication rule for $\ell(sw) < \ell(w)$ follows from the rule for $\ell(sw) > \ell(w)$ and from the formula for T_s^2.

Uniqueness is easy. If $w = s_1 \ldots s_n$ is reduced, then

$$T_w = T_{s_1} \ldots T_{s_n}$$

Therefore, $\mathcal{A}$ is generated as an R-algebra by the T_s (for $s \in S$) and by T_1. The relations yield the rule for multiplication of any two T_{w_1} and T_{w_2}.

Now prove existence of this associative algebra for given data. Let $\mathcal{A}$ denote the free R-module on elements T_w for $w \in W$. In the ring $\mathrm{End}_R(\mathcal{A})$ we have **sl**left multiplications λ_s and *right multiplications* ρ_s for $s \in S$ given by

$$\begin{array}{rcccl} \lambda_s(T_w) & = & T_{sw} & \text{for} & \ell(sw) > \ell(w) \\ \lambda_s(T_w) & = & a_s T_w + b_s T_{sw} & \text{for} & \ell(sw) < \ell(w) \\ \\ \rho_s(T_w) & = & T_{ws} & \text{for} & \ell(ws) > \ell(w) \\ \rho_s(T_w) & = & a_s T_w + b_s T_{ws} & \text{for} & \ell(ws) < \ell(w) \end{array}$$

Grant for the moment that the λ_s commute with the ρ_t. Let λ be the subalgebra of $\mathcal{E}$ generated by the λ_s. Let $\varphi : \lambda \to \mathcal{A}$ by $\varphi(\lambda) = \lambda(T_1)$. Thus, for example, $\varphi(1) = T_1$ and, for all $s \in S$, $\varphi(\lambda_s) = T_s$. Certainly φ is a surjective R-module map, since for every reduced expression $w = s_1 \ldots s_n$

$$\varphi(\lambda_{s_1} \ldots \lambda_{s_n}) = (\lambda_{s_1} \ldots \lambda_{s_n})(1) = \lambda_{s_1} \ldots \lambda_{s_{n-1}} T_{s_n} = \lambda_{s_1} \ldots \lambda_{s_{n-2}} T_{s_{n-1} s_n} = \ldots = T_{s_1 \ldots s_n} = T_w$$

To prove injectivity of φ, suppose $\varphi(\lambda) = 0$. We will prove, by induction on $\ell(w)$, that $\lambda(T_w) = 0$ for all $w \in W$. By definition, $\varphi(\lambda) = 0$ means $\lambda(T_1) = 0$. Now suppose $\ell(w) > 0$. Then there is $t \in S$ so that $\ell(wt) < \ell(w)$. We are assuming that we already know that ρ_t commutes with λ, so

$$\lambda(T_w) = \lambda(T_{(wt)t}) = \lambda\rho_t T_{wt} = \rho_t \lambda T_{wt} = 0$$

by induction on length.

Thus, λ is a free R-module with basis $\{\lambda_w : w \in W\}$. This R-module isomorphism also implies that $\lambda_w = \lambda_{s_1} \ldots \lambda_{s_n}$ for any reduced expression $w = s_1 \ldots s_n$. The natural R-algebra structure on λ can be transported to $\mathcal{A}$, leaving only the checking of the relations.

To check the relations suppose that $\ell(sw) > \ell(w)$. For a reduced expression $w = s_1 \ldots s_n$ the expression $ss_1 \ldots s_n$ is a reduced expression for sw. Thus,

$$\lambda_s \lambda_w = \lambda_s \lambda_{s_1} \ldots \lambda_{s_n} = \lambda_{sw}$$

That is, we have the desired relation $\lambda_s \lambda_w = \lambda_{sw}$.

We check the other relation $\lambda_s^2 = a_s \lambda_s + b_s \lambda_1$ by evaluating at $T_w \in \mathcal{A}$. For $\ell(sw) > \ell(w)$,

$$\lambda_s^2(T_w) = \lambda_s(\lambda_s T_w) = \lambda_s(T_{sw}) = a_s T_{sw} + b_s T_w =$$

$$= a_s \lambda_s T_w + b_s \lambda_1 T_w = (a_s \lambda_s + b_s \lambda_1) T_w$$

If $\ell(sw) < \ell(w)$, then

$$\lambda_s^2(T_w) = \lambda_s(\lambda_s T_w) = \lambda_s(a_s T_w + b_s T_{sw}) =$$

$$= a_s \lambda_s T_w + b_s T_s T_{sw} = a_s \lambda_s T_w + b_s \lambda_1 T_w = (a_s \lambda_s + b_s \lambda_1) T_w$$

This proves that $\lambda_s^2 = a_s \lambda_s + b_s \lambda_1$, as desired.

The argument is complete except for the fact that the left and right multiplication operators λ_s and ρ_t commute with each other. A little exercise on Coxeter groups prepares for this.

Proposition: Let (W, S) be a Coxeter system, $w \in W$, and $s, t \in S$. If both $\ell(swt) = \ell(w)$ and $\ell(sw) = \ell(wt)$, then $swt = w$ (and $s = wtw^{-1}$.) In particular, $a_s = a_t$ and $b_s = b_t$, since s and t are conjugate.

Proof: Let $w = s_1 \ldots s_n$ be a reduced expression. On one hand, for $\ell(sw) > \ell(w)$

$$\ell(w) = \ell(s(wt)) < \ell(sw)$$

so the Exchange Condition applies. Namely, there is $v \in W$ such that $sw = vt$ and such that either $v = ss_1 \ldots \hat{s_i} \ldots s_n$ or $v = w$. But $v = ss_1 \ldots \hat{s_i} \ldots s_n$ is not possible, since this would imply that

$$\ell(wt) = \ell(s_1 \ldots \hat{s_i} \ldots s_n) < \ell(w)$$

contradicting the present hypothesis

$$\ell(wt) = \ell(sw) > \ell(w)$$

On the other hand, for $\ell(sw) < \ell(w) = \ell(s(sw))$, the hypotheses are met by sw in place of w, so the previous argument applies. Thus $s(sw) = (sw)t$, which gives $w = swt$. ///

Now the commutativity of the operators λ_s and ρ_t.

Lemma: For $s, t \in S$, the operators λ_s, ρ_t commute.

Proof: Prove that $\lambda_s \rho_t - \rho_t \lambda_s = 0$ by evaluating the left-hand side on T_w. There are few possibilities for the relative lengths of w, sw, wt, swt, and in each case the result follows by direct computation, although we need to use the *proposition* from above in two of them:

If $\ell(w) < \ell(wt) = \ell(sw) < \ell(swt)$, then by the definitions of the operators λ_s, ρ_t we have

$$\lambda_s \rho_t T_w = \lambda_s T_{wt} = T_{swt}$$

In the opposite case $\ell(w) > \ell(wt) = \ell(sw) > \ell(swt)$,

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) = a_t(a_s T_w + b_s T_{sw}) + b_t(a_s T_{wt} + b_s T_{swt})$$

which, by rearranging and reversing the argument with s and t interchanged and left and right interchanged, is

$$a_s(a_t T_w + b_t T_{wt}) + b_s(a_t T_{sw} + b_t T_{swt}) = \rho_t \lambda_s T_w$$

In the case that $\ell(wt) = \ell(sw) < \ell(swt) = \ell(w)$ invoke the *proposition* just above. We have $a_s = a_t$ and $b_s = b_t$ and $sw = wt$. Then compute directly

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) = a_t(a_s T_w + b_s T_{sw}) + b_t T_{swt}$$

$$= a_s(a_t T_w + b_t T_{wt}) + b_s T_{swt} = \rho_t(a_s T_w + b_s T_{sw}) = \rho_t \lambda_s T_w$$

as desired.

When $\ell(wt) < \ell(w) = \ell(swt) < \ell(sw)$

$$\lambda_s \rho_t T_w = \lambda_s(a_t T_w + b_t T_{wt}) = a_t T_{sw} + b_t T_{swt} = \rho_t(\lambda_s T_w)$$

A corresponding argument applies in the case opposite to the previous one wherein $\ell(sw) < \ell(w) = \ell(swt) < \ell(wt)$.

When $\ell(w) = \ell(swt) < \ell(sw) = \ell(wt)$ again invoke the *proposition* above to obtain $a_s = a_t$ and $b_s = b_t$ and also $sw = wt$. Then

$$\lambda_s \rho_t T_w = \lambda_s T_{wt} = a_s T_{wt} + b_s T_{swt} = a_t T_{sw} + b_t T_{swt} = \rho_t T_{sw} = \rho_t \lambda_s T_w$$

This finishes the proof of commutativity, and of the theorem on generic algebras. ///

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2. Strict Iwahori-Hecke algebras

This section demonstrates that Iwahori-Hecke algebras attached to groups acting suitably on buildings are generic algebras in the sense above. The argument depends only upon the *local finiteness* of the building.

Let G be a group acting strongly transitively on a thick building X, preserving a labelling. (The strong transitivity means that G is transitive upon pairs $C \subset A$ where C is a chamber in an apartment A in the given apartment system.) Let (W, S) be the Coxeter system associated to the apartments: each apartment is isomorphic to the Coxeter complex of (W, S). Conversely, a choice of apartment and chamber within it specifies (W, S). We assume always that S is finite. The subgroup B is the stabilizer of C.

The **local finiteness** hypothesis is that *we assume that for all $s \in S$ the cardinality*

$$q_s = \mathrm{card}(BsB/B) = \mathrm{card}(B\backslash BsB)$$

is finite. Recall that the subgroup of G stabilizing the facet F_s of C of type $\{s\}$ for $s \in S$ is

$$P = P_s = B\{1, s\}B = B \sqcup BsB$$

The subgroup B is the subgroup of P additionally stabilizing C, so BsB is the subset of $B\langle s\rangle B$ mapping C to *another* chamber s-adjacent to C (that is, with common facet F_s of type $\{s\}$.) Therefore, BsB/B is in bijection with the set of chambers s-adjacent to C (other than C itself) by $g \to gC$.

That is, the *local finiteness* hypothesis is that *every facet is the facet of only finitely-many chambers.* Equivalently, since S is finite, *each chamber is adjacent to only finitely-many other chambers.*

Fix a field k of characteristic zero. Let

$$\mathcal{H} = \mathcal{H}_k(G, B)$$

be the **Iwahori-Hecke algebra** in G over the field k, that is, the collection of left and right B-invariant k-valued functions on G supported on finitely-many cosets Bg in G.

The *convolution product* on $\mathcal{H}$ is

$$(f * \varphi)(g) = \sum_{h \in B\backslash G} f(gh^{-1})\varphi(h)$$

The hypothesis that φ is supported on finitely-many cosets Bx implies that the sum in the previous expression is *finite.* Since φ is left B-invariant and f is right B-invariant the summands are constant on cosets Bg, so summing over $B\backslash G$ makes sense. *Nevertheless, we must prove that the product is again in $\mathcal{H}$.* We do this in the course of the theorem.

Generally, let ch_E be the characteristic function of a subset E of G. By the Bruhat-Tits decomposition, *if indeed they are in $\mathcal{H}(G, B)$*, the functions ch_{BwB} form a k-basis for $\mathcal{H}(G, B)$. This Hecke algebra is visibly a free k-module.

Theorem: Each BgB is a finite union of cosets Bx, the algebra $\mathcal{H}$ is closed under convolution products, and

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = \mathrm{ch}_{BswB} \quad \text{for} \quad \ell(sw) > \ell(w)$$

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BsB} = a_s \mathrm{ch}_{BsB} + b_s \mathrm{ch}_B$$

with

$$a_s = q_s - 1 \;\text{ and }\; b_s = q_s$$

That is, *these Iwahori-Hecke operators form a generic algebra with the indicated structure constants.* Further, for a reduced expression $w = s_1 \ldots s_n$ (that is, with $n = \ell(w)$ and all $s_i \in S$), we have

$$q_w = q_{s_1} \ldots q_{s_n}$$

Proof: First prove that double cosets BwB are finite unions of cosets Bx at the same time that we study one of the requisite identities for the generic algebra structure. This also will prove that $\mathcal{H}$ is closed under convolution products. Do induction on the length of $w \in W$. Take $s \in S$ so that $\ell(sw) > \ell(w)$. At $g \in G$ where $\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB}$ does not vanish, there is $h \in G$ so that $\mathrm{ch}_{BsB}(gh^{-1})\mathrm{ch}_{BwB}(h) \neq 0$. For such h, we have $gh^{-1} \in BsB$ and $h \in BwB$. Thus, by the Bruhat cell multiplication rules,

$$g = (gh^{-1})h \in BsB \cdot BwB = BswB$$

Since this convolution product is left and right B-invariant

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = c \cdot \mathrm{ch}_{BswB}$$

for some *positive rational number* $c = c(s, w)$.

Compute the constant $c = c(s, w)$ by summing the previous equality over $B\backslash G$

$$c \cdot q_{sw} = c \cdot \mathrm{card}(B\backslash BswB) = c \cdot \sum_{g \in B\backslash G} \mathrm{ch}_{BswB}(g) =$$

$$= c \cdot \sum_{g \in B\backslash G} (\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB})(g) = \sum_{g \in B\backslash G} \sum_{h \in B\backslash G} \mathrm{ch}_{BsB}(gh^{-1})\mathrm{ch}_{BwB}(h) =$$

$$= \sum \sum \mathrm{ch}_{BsB}(g)\mathrm{ch}_{BwB}(h) = q_s\, q_w$$

(the latter by replacing g by gh, interchanging order of summation.)

Thus, $c = q_s q_w / q_{sw}$ and for $\ell(sw) > \ell(w)$

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = q_s q_w q_{sw}^{-1} \mathrm{ch}_{BswB}$$

This shows incidentally that the cardinality q_{sw} of $B\backslash BwB$ is finite for all $w \in W$, and therefore that the Hecke algebra is closed under convolution.

Now consider the other identity required of a generic algebra. Since

$$BsB \cdot BsB = B \sqcup BsB$$

we need evaluate $(T_s * T_s)(g)$ only at $g = 1$ and $g = s$. For $g = 1$ the sum defining the convolution is

$$(\mathrm{ch}_{BsB} * \mathrm{ch}_{BsB})(1) = \sum_{h \in B\backslash G} \mathrm{ch}_{BsB}(h^{-1})\mathrm{ch}_{BsB}(h) = q_s =$$

$$= (q_s - 1) \cdot 0 + q_s \cdot 1 = (q_s - 1)\mathrm{ch}_{BsB}(1) + q_s \mathrm{ch}_B(1)$$

For $g = s$

$$(\mathrm{ch}_{BsB} * \mathrm{ch}_{BsB})(s) = \sum_{h \in B\backslash G} \mathrm{ch}_{BsB}(sh^{-1})\mathrm{ch}_{BsB}(h) =$$

$$= \mathrm{card}(B\backslash(BsB \cap BsBs))$$

Let P be the parabolic subgroup $P = B \cup BsB$. This is the stabilizer of the facet F_s. The innocent fact that P is a group gives

$$BsB \cap BsBs = (P - B) \cap (P - B)s = (P - B) \cap (Ps - Bs) =$$

$$= (P - B) \cap (P - Bs) = P - (B \sqcup Bs)$$

Therefore, $BsB \cap BsBs$ consists of $[P : B] - 2$ left B-cosets. This number is $(q_s + 1) - 2 = q_s - 1$. Thus,

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BsB} = (q_s - 1)\mathrm{ch}_{BsB} + q_s \mathrm{ch}_B$$

Therefore, with $T_w = q_w^{-1}\mathrm{ch}_{BwB}$ we obtain a generic algebra with structure constants $a_s = (1 - q_s^{-1})$ and $b_s = q_s^{-1}$. However, this is a weaker conclusion than desired, as we wish to prove that for $\ell(sw) > \ell(w)$

$$q_s q_w = q_{sw}$$

If so, then our earlier computation would show that

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = \mathrm{ch}_{BswB}$$

Then taking simply $T_w = \mathrm{ch}_{BwB}$ would yield a generic algebra with structure constants $a_s = q_s - 1$ and $b_s = q_s$.

On one hand, (with $\ell(sw) > \ell(w)$) evaluate both sides of

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BwB} = q_s q_w q_{sw}^{-1} \mathrm{ch}_{BswB}$$

at sw. The left-hand side is

$$\sum_{h \in B\backslash G} \mathrm{ch}_{BsB}(swh^{-1})\mathrm{ch}_{BwB}(h) = \mathrm{card}(B\backslash(BsB(sw) \cap BwB)) =$$

$$= \mathrm{card}(B\backslash(BsBs \cap BwBw^{-1})) \geq \mathrm{card}(B\backslash(Bss \cap Bww^{-1})) = \mathrm{card}(B\backslash B) = 1$$

The right-hand side is $q_s q_w q_{sw}^{-1}$, so

$$q_s q_w \geq q_{sw}$$

On the other hand, invoking the theorem on generic algebras (with $\ell(sw) > \ell(w)$)

$$q_s^{-1}\mathrm{ch}_{BsB} * q_{sw}^{-1}\mathrm{ch}_{BswB} = (1 - q_s^{-1})q_{sw}^{-1}\mathrm{ch}_{BswB} + q_s^{-1}q_w^{-1}\mathrm{ch}_{BwB}$$

This gives

$$\mathrm{ch}_{BsB} * \mathrm{ch}_{BswB} = (q_s - 1)\mathrm{ch}_{BswB} + q_{sw}q_w^{-1}\mathrm{ch}_{BwB}$$

Evaluate both sides at w. The right side is $q_{sw}q_w^{-1}$ and the left is

$$\mathrm{card}(B\backslash(BsBw \cap BswB)) = \mathrm{card}(B\backslash(BsB \cap BswBw^{-1})) =$$

$$= \mathrm{card}(B\backslash(BsB \cap BsBBwB \cdot w^{-1})) \geq \mathrm{card}(B\backslash(BsB \cap BsBww^{-1})) = \mathrm{card}(B\backslash BsB) = q_s$$

by the cell multiplication rules. That is,

$$q_{sw} \geq q_s q_w$$

Combining these two computations yields $q_{sw} = q_s q_w$. Induction on length gives the assertion

$$q_{s_1 \ldots s_n} = q_{s_1} \ldots q_{s_n}$$

for a reduced expression $s_1 \ldots s_n \in W$. Thus, we obtain the generic algebra as claimed. ///

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3. Representations with Iwahori-fixed vectors

Here we prove the Borel-Matsumoto theorem. The structure of the Iwahori-Hecke algebras is the essential ingredient in this proof. Again, G is a p-adic reductive group and B an Iwahori subgroup. More specifically, let G_o be the label-preserving subgroup of G, and take B to be the subgroup in G_o stabilizing a chosen chamber in a chosen apartment in the associated (affine) building.

Theorem: *(Borel, Matsumoto, Casselman)* Let G be a reductive p-adic group with Iwahori subgroup B (unique up to conjugation) and corresponding minimal parabolic subgroup P. Let $M_o = B \cap M$ be a maximal compact subgroup of a chosen Levi component $M = M^P$ of P. Let π be a smooth representation of G with

$$\dim \pi^B < \infty$$

(Admissibility of π would assure the latter condition.) Under the quotient map $q : \pi \to \pi_N$ from π to its Jacquet module π_N with respect to P, the B-fixed vectors π^B in π map complex-linear isomorphically to the M_o-fixed vectors $(\pi_N)^{M_o}$ in π_N.

Proof: (See [Casselman 1980]) Let P be a minimal parabolic subgroup of G matching B. We grant the general fact that for *any* parabolic Q and matching parahoric B^Q, the B^Q-fixed vectors *surject* to the B^Q-fixed vectors in the Q-Jacquet module. Thus, the strength of the present assertion is the *injectivity.* Let $\pi(N)$ be the kernel of the quotient mapping to the P-Jacquet module. If $v \in \pi^B \cap \pi(N)$, then there is a large-enough compact open subgroup N_1 of N such that

$$\int_{N_1} n \cdot v \, dn = 0$$

Take $a \in M^P$ such that $aN_1a^{-1} \subset N_o$, where B has Iwahori factorization $B = N_1^{\mathrm{opp}} M_o N_o$. (The notation is potentially misleading: N_1 is *large* while N_1^{opp} is relatively *small.*) Further, we may take a to lie inside the label-preserving subgroup G_o of G. Then

$$0 = a \cdot 0 = \int_{N_1} anv \, dn = \int_{N_1} ana^{-1} \cdot av \, dn = \int_{aN_1a^{-1}} n \cdot av \, dn$$

Then

$$\int_{N_o} n \cdot av \, dn = \int_{N_o/aN_1a^{-1}} \int_{aN_1a^{-1}} n_o n \cdot av \, dn \, dn_o = \int_{N_o/aN_1a^{-1}} n_o \cdot 0 \, dn_o = 0$$

Normalize the measure of B to be 1. Then

$$0 = \int_{N_1^{\rm opp}} \int_{M_o} 0 \, dm \, dn^{\rm opp} = \int_{N_1^{\rm opp}} \int_{M_o} \int_{N_o} n^{\rm opp} \, m \, n \cdot av \, dn \, dm \, dn^{\rm opp} = \int_B b \cdot av \, db$$

Further, since $v \in \pi^B$,

$$\int_B b \cdot v \, db = v$$

Thus,

$$\int_{BaB} x \cdot v \, dx = \int_B \int_B ba' \cdot v \, db' \, db = \int_B ba \cdot v \, db = 0$$

from just above. That is,

$$\mathrm{ch}_{BaB} \, v = 0$$

For some reflections s_i there is a reduced expression $s_1 \ldots s_n$ such that

$$BaB = Bs_1 \ldots s_n B$$

By the structural results for the Iwahori-Hecke algebra,

$$\mathrm{ch}_{BaB} = \mathrm{ch}_{Bs_1B} * \ldots * \mathrm{ch}_{Bs_nB}$$

Each of the functions ch_{Bs_iB} acting on π stabilizes the *finite-dimensional* complex vector space π^B. For $s \in S$, the structural assertion

$$\mathrm{ch}_{BsB}^2 = q_s \, \mathrm{ch}_{BsB} + (q_s - 1) \, \mathrm{ch}_B$$

says that any eigenvalue λ of ch_{BsB} on π^B must satisfy

$$\lambda^2 = q_s \, \lambda + (q_s - 1)$$

so

$$\lambda = \frac{q_s \pm \sqrt{q_s^2 - 4(q_s - 1)}}{2} = \frac{q_s \pm (q_s - 2)}{2} = 1, \; q_s - 1$$

That is, no eigenvalue can be 0. Thus, each ch_{BsB} gives an invertible operator on π^B, and ch_{BaB} is necessarily invertible, contradicting our earlier conclusion unless $v = 0$. That is,

$$\pi^B \cap \pi(N) = 0$$

after all. ///

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4. Imbeddings to unramified principal series

The Borel-Matsumoto theorem on representations with Iwahori-fixed vectors gives several important corollaries about admissible spherical representations and the structure of unramified principal series.

Corollary: An irreducible admissible representation π with non-zero B-fixed vector imbeds into an unramified principal series representation.

Proof: Since the set π^B of B-fixed vectors in π is non-trivial, by the theorem the space of M_o-fixed vectors $\pi_N^{M_o}$ in the Jacquet module π_N is non-trivial. Since the Jacquet-module functor preserves admissibility,

$\pi_N^{M_o}$ is finite-dimensional. Since P is minimal, M/M_o is an abelian group. Thus, $\pi_N^{M_o}$ has an irreducible P-quotient χ which is a complex one-dimensional space upon which M/M_o acts. That is, M acts on χ by an unramified character (that is, a character trivial on M_o). By Frobenius Reciprocity, we obtain a non-zero G-map $\pi \to \mathrm{Ind}_P^G \chi$, which must be an isomorphism since π is irreducible. ///

Corollary: Every G-subrepresentation and every G-quotient of an unramified principal series representation of G is generated by its B-fixed vectors.

Proof: Let π be a subrepresentation of an unramified principal series

$$I_\chi = \mathrm{Ind}_P^G \rho \chi$$

where ρ is the square root of the modular function for P. From Frobenius Reciprocity

$$0 \neq \mathrm{Hom}_G(\pi, I_\chi) \approx \mathrm{Hom}_M(\pi_N, \rho\chi)$$

Since $\pi_N^{M_o} \approx \pi^B$ as complex vector spaces (from above), $\pi^B \neq 0$. That is, any subrepresentation π of an unramified principal series contains a non-zero B-fixed vector.

Taking contragredients, the inclusion

$$\pi \subset I_\chi$$

give rise to a surjection

$$I_{\check{\chi}} \approx \check{I}_\chi \to \check{\pi}$$

Let H be the kernel. We have shown that H^B is non-trivial.

///

Corollary: If an unramified principal series representation $I_\chi = i_P\chi$ is *not* generated by its unique (up to constant multiples) spherical vector φ (for fixed choice K of maximal compact), then there is a non-zero intertwining $T : I_\chi \to I_{\chi'}$ from I_χ to another unramified principal series $I_{\chi'}$ such that $T\varphi = 0$.

Proof: (Following Casselman) Let V be the proper submodule generated by the spherical vector. Then $I_\chi/V \neq 0$, so is still admissible (since an unramified principal series representation is admissible, and admissibility is preserved in quotients). And it is generated by its B-fixed vectors $(I_\chi/V)^B$, by the theorem, which then necessarily form a finite-dimensional subspace. Thus, I_χ/V is finitely-generated, so has an irreducible quotient σ, which (by the corollary) still has a non-zero B-fixed vector. Thus, σ imbeds into an unramified principal series $I_{\chi'}$. That is, we have a non-zero intertwining

$$I_\chi \to I_\chi/V \to \sigma \to I_{\chi'}$$

This proves the corollary. ///

5. Irreducibility criteria

First recall some relatively elementary standard facts about principal series representations. Let W be the (spherical) Weyl group of G, acting on P/N by conjugation, where N is the unipotent radical of the minimal parabolic P. Let W act on one-dimensional group homomorphisms

$$\chi : P/N \to \mathbf{C}^\times$$

by

$$(w\chi)(x) = \chi(w^{-1}xw)$$

As usual, say that χ is *regular* if $w\chi = \chi$ only for $w = 1$. Let $I_\chi = i_P\chi$ denote the (smooth) normalized unramified principal series

$$I_\chi = i_P\chi = \mathrm{Ind}_P^G \rho_P \chi$$

Let φ_χ be the unique K-spherical vector in I_χ such that $\varphi_\chi(1) = 1$.

Theorem: Let χ be an unramified character.
- If $T : I_\chi \to I_{\chi'}$ is a non-zero intertwining, then $\chi' = w\chi$ for some $w \in W$.
- If χ is regular, then for all $w \in W$

$$\dim \mathrm{Hom}_G(I_\chi, I_{w\chi}) = 1$$

///

In particular, for fixed regular χ and $w \in W$, the non-zero map

$$T_{\chi,w} : I_\chi \to I_{w\chi}$$

defined (for suitable inequalities imposed upon χ to ensure convergence, and then by analytic continuation) by

$$T_{\chi,w} f(g) = \int_{N \cap w^{-1}Nw \backslash N} f(w^{-1}ng)\, dn$$

is the only such intertwining, up to a constant.

Theorem: The holomorphically parametrized family $\chi \to T_{\chi,w}$ (for fixed $w \in W$) of intertwining operators $I_\chi \to I_{w\chi}$ has a holomorphic continuation to all regular χ. The holomorphically continued intertwining is not the zero intertwining. ///

Theorem: *([Casselman 1980])* Let χ be a regular character, let φ_χ be the spherical vector in $\mathrm{Ind}_P^G\chi$ normalized so that $\varphi_\chi(1) = 1$. There is an explicit not-identically-zero rational function $\lambda_w(\chi)$ of χ so that

$$T_{\chi,w}(\varphi_\chi) = \lambda_w(\chi) \cdot \varphi_{w\chi}$$

///

Corollary: *([Casselman 1980])* Let $\lambda(\chi) = \lambda_{w_o}(\chi)$ be the function attached to the longest Weyl element w_o in the spherical Weyl group. For *regular* characters χ, the spherical vector φ_χ generates I_χ if and only if $\lambda(\chi) \neq 0$.

Proof: Let V be the subrepresentation generated by the spherical vector. If V is not all of I_χ, then the quotient I_χ/V is non-zero. By the corollary of the Borel-Matsumoto theorem, this quotient has a non-zero intertwining to an unramified principal series $I_{\chi'}$. Necessarily $\chi' = w\chi$ for some $w \in W$. Either by looking at the rational functions $\lambda_w(\chi)$ directly, or by realizing that in a finite Coxeter group W for any w there is w' such that the longest element w_o is expressible as $w_o = w'w$ with

$$\ell(w_o) = \ell(w') + \ell(w)$$

Thus,

$$T_{w',\chi'} \circ T_{w,\chi} = T_{w_o,\chi}$$

and therefore

$$\lambda_{w_o,\chi} = \lambda_{w'}(\chi') \cdot \lambda_w(\chi) = \lambda_{w'}(\chi') \cdot 0 = 0$$

Thus, when the spherical vector fails to generate I_χ, $\lambda_{w_o}(\chi) = 0$. On the other hand, if $\lambda_{w_o}(\chi) = 0$, the spherical vector is in the kernel of the non-zero intertwining operator $T_{w_o}(\chi)$, so the spherical vector cannot generate I_χ. ///

Remark: These intertwining operators are well understood. A conceptual proof of meromorphic continuation can be given by some form of Bernstein's continuation principle. One version is in [Garrett 1997].

In fact, since the compositions of these intertwinings must fall back into the same class, one can make a stronger assertion about the relationships among the intertwinings and among the λ_w's for $w_1 w_2 = w_3$, etc.

Corollary: *(Casselman)* For regular χ, the unramified principal series I_χ is irreducible if and only if $\lambda(\chi) \neq 0$ and $\lambda(\chi^{-1}) \neq 0$.

Proof: Suppose V is a proper G-submodule of I_χ. On one hand, if V contains φ_χ, then φ_χ fails to generate all of I_χ, so $\lambda(\chi) = 0$. On the other hand, if V does not contain φ_χ, then the (smooth) contragredient $\check{I}_\chi \approx I_{\chi^{-1}}$ has a proper submodule

$$X = \{x \in \check{I}_\chi : x(V) = 0\}$$

which necessarily contains the spherical vector $\varphi_{\chi^{-1}}$. That is, $\check{I}_\chi$ is not generated by its spherical vector, so $\lambda(\chi^{-1}) = 0$. ///

Now let Q be any other parabolic subgroup containing P, with unipotent radical N^Q. Let $\sigma : Q/N^Q \to \mathbf{C}^\times$ be an unramified character. Let

$$i_Q\sigma = \mathrm{Ind}_Q^G\, \rho_Q\, \sigma$$

be the normalized degenerate principal series. By restriction, such σ gives an unramified character on P and we have an injection

$$i_Q\sigma = \mathrm{Ind}_Q^G\, \rho_Q\, \sigma \to \mathrm{Ind}_P^G\, \rho_Q\, \sigma = \mathrm{Ind}_P^G\, \rho_P \cdot \rho_P^{-1} \rho_Q\, \sigma = i_P(\rho_P^{-1}\rho_Q\, \sigma)$$

The following is an obvious extension to degenerate principal series of Casselman's results for unramified principal series, though the condition we obtain ceases to be provably *necessary* for irreducibility.

Corollary: For an unramified character σ of Q, if $\rho_P^{-1}\rho_Q\, \sigma$ is regular, and if $\lambda(\rho_P^{-1}\rho_Q\, \sigma) \neq 0$ and $\lambda(\rho_P^{-1}\rho_Q\, \sigma^{-1}) \neq 0$, then the (normalized) degenerate principal series $i_Q\sigma$ is irreducible.

Proof: First, we verify that if the spherical vector φ_σ in $i_Q\sigma$ generates $i_Q\sigma$, and if the same is true for the contragredient $\check{i}_Q\sigma \approx i_Q\check{\sigma}$, then $i_Q\sigma$ is irreducible. Indeed, suppose that $i_Q\sigma$ had a proper submodule V. On one hand, if V contains the spherical vector, then the spherical vector fails to generate $i_Q\sigma$. On the other hand, if V does not contain the spherical vector, then in the contragredient the submodule

$$X = \{x \in \check{i}_Q\sigma : x(V) = 0\}$$

is proper and contains the spherical vector, so the spherical vector fails to generate $\check{i}_Q\sigma$.

Now we relate this to the generation of unramified principal series by the spherical vector. From the obvious inclusion

$$\check{i}_Q\sigma \approx i_Q\check{\sigma} \to i_P(\rho_P^{-1}\rho_Q\, \check{\sigma})$$

by dualizing we obtain a surjection of the contragredients, which (by choice of the normalizations) gives a surjection

$$i_P(\rho_P\rho_Q^{-1}\, \sigma) \to i_Q\sigma$$

If the spherical vector in $i_Q\sigma$ fails to generate $i_Q\sigma$, then surely the same is true of $i_P(\rho_P\rho_Q^{-1}\, \sigma)$. The obvious parallel remark applies to the contragredient. If $\rho_P\rho_Q^{-1}\, \sigma$ is regular, then we may invoke Casselman's criterion for generation of unramified principal series by the spherical vector. ///

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