(*December 5, 2005*)

Jacquet theory

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

- Jacquet modules
- Jacquet's lemmas

In effect, the present discussion is merely a slight abstraction of [Jacquet 1971]. This material has been treated in a number of places, but merits reiteration. In effect, this is an elaboration of the consequences of the fact that unipotent radicals of parabolic subgroups of p-adic reductive groups are ascending unions of compact open subgroups, and that the parabolic can act to contract any such subgroup to the identity.

We assume familiarity with simple general results concerning smooth representations of totally disconnected (locally compact, Hausdorff, separable) topological groups. As usual, the condition of smoothness of a representation π of a group G is that the fixing subgroup

$$G_x = \{g \in G : g \cdot x = x\}$$

be *open* in G for every x in the representation space π.

These results apply very broadly to reductive p-adic groups G and parabolic subgroups P, but the general apparatus of reductive groups is a separate issue from our points here. Indeed, it is already instructive to consider $G = GL(n, \mathbb{Q}_p)$ and maximal proper parabolics

$$P = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL(n_1, \mathbb{Q}_p), d \in GL(n_1, \mathbb{Q}_p)\}$$

or even simply the invertible 2-by-2 matrices $GL(2, \mathbb{Q}_p)$ and the parabolic P of upper triangular matrices.

All vector spaces will be over a fixed field k of characteristic zero, which may be taken to be the complex numbers without much loss.

---

1. *Jacquet modules*

Let π be a smooth representation of a p-adic reductive group G on a k-vectorspace also denoted π. We may suppress explicit reference to π, and write

$$g \times v \longrightarrow g \cdot v$$

for the action of $g \in G$ on $v \in \pi$. Let P be a parabolic subgroup with unipotent radical N and choice of Levi component M. The **Jacquet module** π_N (or $J_P\pi$) of π is the N-co-isotype of π for the trivial representation of N. That is, it is the largest quotient of π on which N acts trivially. Since P normalizes N, π_N is still a representation of P, and the quotient map

$$q : \pi \longrightarrow \pi_N$$

is a P-intertwining.

Proposition: The Jacquet module π_N is the quotient of π by the P-subrepresentation $\pi(N)$ generated by all expressions $v - n \cdot v$ for $v \in \pi$ and $n \in N$.

Proof: Under any P-map $r : \pi \longrightarrow V$ where N acts trivially on V, certainly

$$r(v - nv) = rv - r(nv) = rv - n(rv) = rv - rv = 0$$

so these elements are in the kernel of the quotient map to the Jacquet module. On the other hand, it is easy to check that the linear span of these elements is stable under P, hence under N, so we may form the quotient $\pi/\pi(N)$ as a P-space. By construction, N acts trivially. ///

Proposition: The Jacquet module π_N is a smooth P-representation.

Proof: Given $v \in \pi_N$, let $u \in \pi$ be such that $q(u) = v$. Invoking the smoothness, let G_u be the open subgroup of G fixing u. Then $P_o = G_u \cap P$ is a compact open subgroup of P, and (since the quotient map is a P-morphism) v is P_o-fixed. ///

Proposition: A vector $v \in \pi$ is in the kernel of the quotient $q : \pi \longrightarrow \pi_N$ if and only if there is a compact open subgroup N_o of N such that

$$\int_{N_o} n \cdot v \, dn = 0$$

Proof: If there is such an N_o, let

$$N_1 = N_o \cap G_v$$

This is open, and because G is totally disconnected it is closed, hence compact. Then

$$0 = \int_{N_o} n \cdot v \, dn = \int_{N_o/N_1} \int_{N_1} nn_1 \cdot v \, dn_1 \, dn = \text{meas}\,(N_1) \sum_{n \in N_o/N_1} n \cdot v$$

since N_o/N_1 is a finite set, say with t elements. Then

$$v = v - 0 = v - \frac{1}{t} \sum_{n \in N_o/N_1} n \cdot v = \frac{1}{t} \sum_{n \in N_o/N_1} v - n \cdot v$$

expressing v as a linear combination of the desired form.

On the other hand, given a finite collection of expressions $v - nv$ with $n \in N$ and $v \in \pi$, there is a compact open subgroup N_o of N containing all the finitely-many n. Then

$$\int_{N_o} n'(v - nv) \, dn' = \int_{N_o} n'v \, dn' - \int_{N_o} n'nv \, dn' = \int_{N_o} n'v \, dn' - \int_{N_o} n'v \, dn' = 0$$

by replacing n' by $n^{-1}n'$ in the second integral. ///

Note that for given G-morphism

$$f : A \longrightarrow B$$

the composite

$$q \circ f : A \longrightarrow B_N$$

is a map to a trivial N-space, so factors through A_N, giving a P-map $f_N : A_N \longrightarrow B_N$ such that

$$f_N \circ q = q \circ f$$

We may suppress the subscript N.

Proposition: The functor $\pi \longrightarrow \pi_N$ from G-representations to P-representations is *exact*, in the sense that short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

are sent to exact sequences

$$0 \longrightarrow A_N \xrightarrow{f} B_N \xrightarrow{g} C_N \longrightarrow 0$$

Proof: The right half-exactness is a more general property of co-isotypes. That is, the surjectivity of $g : B_N \longrightarrow C_N$ is easy, since $q \circ f : B \longrightarrow C_N$ is a surjection. Likewise, since the composite $g \circ f : A \longrightarrow C$ is 0, certainly

$$q \circ g \circ f : A \longrightarrow C_N$$

is 0, so the composite $A_N \longrightarrow B_N \longrightarrow C_N$ is 0.

The injectivity of $A_N \longrightarrow B_N$ and the fact that the image of A_N in B_N is the whole kernel of $B_N \longrightarrow C_N$ are less general, depending upon the special nature of the subgroup N, as manifest in the previous proposition. Let $a \in A$ such that $q(fa) = 0 \in B_N$. Then there is a compact open subgroup N_o of N such that

$$\int_{N_o} n \cdot fa \, dn = 0$$

Since f commutes with the action of N, this gives

$$f\left(\int_{N_o} n \cdot a \, dn\right) = 0$$

By the injectivity of f

$$\int_{N_o} n \cdot a \, dn = 0$$

so $qa = 0 \in A_N$. This proves exactness at the left joint.

Suppose $g(qb) = 0$. Then $q(gb) = 0$, so there is a compact open subgroup N_o in N such that

$$\int_{N_o} n \cdot gb \, dn = 0$$

and then $ng = gn$ gives

$$g\left(\int_{N_o} n \cdot b \, dn\right) = 0$$

Thus, the integral is in the kernel of g, so is in the image of f. Let $a \in A$ be such that

$$fa = \int_{N_o} n \cdot b \, dn$$

Without loss of generality, $\mathrm{meas}\,(N_o) = 1$. Then

$$\int_{N_o} n' \cdot fa \, dn' = \int_{N_o} \int_{N_o} n'n \cdot b \, dn \, dn' = \int_{N_o} \int_{N_o} n \cdot b \, dn \, dn'$$

by replacing n by $n'^{-1}n$. Then this gives

$$\int_{N_o} n \cdot (fa - b) \, dn = 0$$

So $q(fa - b) = 0$ and $f(qa) = qb$. This finishes the proof of exactness at the middle joint. ///

---

2. *Jacquet's lemmas*

Let $K = N_1^{\text{opp}} M_o N_o$ be a compact open subgroup admitting an Iwahori factorization with respect to the parabolic P, where M_o is a compact subgroup of a Levi component M of P, N_o is a compact open subgroup of the unipotent radical N of P, and N_1^{opp} is a compact open subgroup of an opposite unipotent radical N^{opp} to N. Further, this presumes that M_o normalizes N_o and N_1^{opp}. Let A be the maximal split torus in the center of the Levi component M. Let

$$A^- = \{a \in A : aN_o a^{-1} \subset N_o\}$$

Let $q : \pi \longrightarrow \pi_N$ be the quotient map to the Jacquet module of a smooth representation π of G. For a compact (not necessarily open) subgroup H of G, and a smooth representation π of G, let

$$\text{pr}_H : \pi \longrightarrow \pi^H$$

be the map to the H-fixed vectors given by

$$\text{pr}_H(v) = \frac{1}{\text{meas}\,(H)} \int_H h \cdot v \, dh$$

Lemma: *(Jacquet's First Lemma)* Given $v \in \pi^{M_o N_1^{\text{opp}}}$,

$$\text{pr}_K v = \text{pr}_{N_o} v$$

and therefore

$$\int_{N_o} n \cdot (v - \text{pr}_K v)\, dn = \int_{N_o} n \cdot (v - \text{pr}_{N_o} v)\, dn = 0$$

Thus, under the quotient map $q : \pi \longrightarrow \pi_N$ to the Jacquet module,

$$q(v - \text{pr}_K v) = q(v - \text{pr}_{N_o} v) = 0$$

Proof: The Iwahori factorization of K and $M_o N_1^{\text{opp}}$-invariance of v yields

$$\int_{N_o} n \cdot v \, dn = \int_{N_o} \int_{M_o} \int_{N_1^{\text{opp}}} nmn' \cdot v \, dn \, dm \, dn' = \int_K k \cdot v \, dk$$

That is, $\text{pr}_{N_o} v = \text{pr}_K v$. ///

Lemma: *(Jacquet's Second Lemma)* Let N_1, N_2 be compact open subgroups of N, with $m \in M$ such that $mN_1m^{-1} \subset N_2$. Then

$$\int_{N_1} n \cdot v \, dn = 0$$

implies

$$\int_{N_2} n \cdot mv \, dn = 0$$

Proof: This is by direct computation.

$$\int_{N_2} n \cdot mv \, dn = \int_{m^{-1}N_2 m} mn \cdot v \, dn = \int_{m^{-1}N_2 m/N_1} \int_{N_1} mnn_1 \cdot v \, dn_1 \, dn = \int_{m^{-1}N_2 m/N_1} mn \cdot 0 \, dn = 0$$

as asserted. ///

Proposition: Under the quotient map to the Jacquet module π_N, π^K maps surjectively to $(\pi_N)^{M_o}$. In particular, π_N is admissible if π is. Further, if π^K generates π then $(\pi_N)^{M_o}$ generates π_N.

Proof: Let V be a finite-dimensional complex subspace of $(\pi_N)^{M_o}$, and take a finite-dimensional complex subspace U of π mapping surjectively to V. There is a sufficiently small compact open subgroup N_2^{opp} of N^{opp} such that

$$U \subset \pi^{M_o N_2^{\mathrm{opp}}}$$

Take a in A such that

$$a N_2^{\mathrm{opp}} a^{-1} \subset N_1^{\mathrm{opp}}$$

Then

$$a \cdot U \subset \pi^{M_o N_1^{\mathrm{opp}}}$$

Then, on one hand,

$$q(a \cdot U) \subset q(\pi^{M_o N_1^{\mathrm{opp}}}) = q(\pi^K)$$

by Jacquet's first lemma. On the other hand, because q is a P-morphism

$$q(a \cdot U) = \pi_N(a) \cdot q(U) = \pi_N(a) \cdot V$$

Thus,

$$\pi_N(a) \cdot V \subset q(\pi^K)$$

Thus, for all finite-dimensional subspaces V of $\pi_N^{M_o}$

$$\dim V \leq \dim U^K \leq \dim \pi^K < \infty$$

by the assumed admissibility of π, giving the bound

$$\dim \pi_N^{M_o} \leq \dim \pi^K$$

As $a \in A$ centralizes M and hence M_o, $\pi_N(a)$ stabilizes $\pi_N^{M_o}$ and gives an automorphism of it. Thus, taking V to be the whole $\pi_N^{M_o}$ shows that π^K maps surjectively to $\pi_N^{M_o}$. ///

3. *Bibliography*

[Jacquet 1971] H. Jacquet, *Representations des groupes lineaires p-adiques*, Theory of Group Representations and Harmonic Analysis, CIME, II Ciclo, Mentcatini Terme 1970, 119-220, Edizioni Cremonese, Roma 1971.