(*February 19, 2005*)

Automorphic Representations and L-functions

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

- Decomposition by central characters
- Square-integrable cuspforms
- Smoothness of cuspforms
- Eigen-cuspforms and automorphic representation
- Dirichlet series versus zeta and L-functions
- L-functions defined via local data
- Factoring unitary representations of adele groups
- Spherical representations and Satake parameters
- Local data, L-groups, higher L-functions

References and historical notes will be added later, maybe.

Many of the statements made here without proof are very difficult to prove! Just because no mention of proof is made it should not be presumed that it's 'just an exercise'!

---

1. Decomposition by central characters

Let Z be the center of a reductive linear group G defined over a number field k. Let G_k denote the k-valued points of G and let G_A denote the adele points of G. Likewise, let Z_k be the k-rational points of the center Z, and let Z_A denote the adele points of Z.

Since G_k, G_A, Z_k and Z_A are all *unimodular*, there exists a right G_A-invariant measure on $G_k\backslash G_A$ and on $Z_A G_k\backslash G_A$, unique up to constant multiples.

A **central character** is simply a continuous group homomorphism

$$\omega : Z_A \to \mathbf{C}^\times$$

Often we will want to assume that such ω is *trivial* on Z_k, so gives rise to

$$\omega : Z_A \to Z_k\backslash Z_A \to \mathbf{C}^\times$$

And often we will suppose that ω is **unitary**, meaning that for all $z \in Z_A$ we have $|\omega(z)| = 1$.

Let ωbe a central character. Let f be a complex-valued function on G_A so that

$$f(zg) = \omega(z)\ f(g)$$

for all $z \in Z_A$ and for all g. In this case, say that f **has central character** ω.

The space $L^2(G_k\backslash G_A)$ may be decomposed as a *direct integral* of Hilbert spaces, according to **central characters**, in the sense that $L^2(G_k\backslash G_A)$ is a direct integral of the Hilbert spaces

$$L^2(Z_A G_k\backslash G_A, \omega) = \{\ |f| \in L^2(Z_A G_k\backslash G_A) \text{ and } f \text{ has central character } \omega\}$$

---

2. Square-integrable cuspforms

Let G be a reductive linear group defined over a number field k. Let f be a complex-valued function on G_A.

The first hypothesis that we impose upon f is that f is **square-integrable with central character** ω, in the sense that

$$f \in L^2(Z_A G_k\backslash G_A, \omega)$$

for some character ω on the adele points Z_A of the center Z of G_A.

Next, for every k-parabolic subgroup P of G with unipotent radical N we suppose that for almost all $g \in G_A$ (in a measure-theoretic sense)

$$\int_{N_k\backslash N_A} f(ng)\ dn = 0$$

where the dn refers to a right N_A-invariant measure on the quotient $N_k\backslash N_A$. The space of $f \in L^2(Z_A G_k\backslash G_A, \omega)$ satisfying the latter condition for all k-parabolics P is denoted by

$$L^2_o(Z_A G_k\backslash G_A, \omega)$$

Functions in these spaces are called **square-integrable cuspforms with central character** ω.

---

3. Smoothness of cuspforms

A vector v in a complex-linear representation π of a *Lie group* H is **smooth** if the V-valued function

$$g \to \pi(g)v$$

on H is *infinitely differentiable.*

A vector v in a complex-linear representation π of a *totally disconnected group* H is **smooth** if the V-valued function

$$g \to \pi(g)v$$

is *uniformly locally constant*, meaning that there is an open subgroup N so that

$$f(g) = f(g\theta)$$

for all $g \in G$ and for all $\theta \in N$.

The adeles $\mathbf{A}$ are a product

$$\mathbf{A} = \mathbf{A}_{\text{inf}} \times \mathbf{A}_{\text{fin}}$$

where $\mathbf{A}_{\text{inf}}$ is the product of the *archimedean* (i.e., *infinite*) prime completions, and $\mathbf{A}_{\text{fin}}$ is the *finite* prime part of the adeles. Often $\mathbf{A}_{\text{fin}}$ is called **the finite adeles** and $\mathbf{A}_{\text{inf}}$ is called **the infinite adeles**.

An adele group G_A is a *product* of a Lie group and a totally disconnected group, namely,

$$G_A = G_{\text{inf}} \times G_{\text{fin}}$$

where G_{inf} is the product of the archimedean-prime completions, and where G_{fin} is the finite-adele points of G. We need to use coordinates

$$g_{\text{inf}} \in G_{\text{inf}} \qquad g_{\text{fin}} \in G_{\text{fin}}$$

on these two factors of G_A. A function on G_A is **smooth** if, as a function of the two variables g_{inf}, g_{fin} the function is smooth (in the two senses).

A vector φ in any complex-linear representation (π, V) of G_A is **smooth** or a **smooth vector** if

$$g_{\text{inf}} \times g_{\text{fin}} \to \pi(g_{\text{inf}} \times g_{\text{fin}})\varphi$$

as a function on G_A is smooth in the coordinates $g_{\text{inf}} \times g_{\text{fin}}$.

Due to the existence of *approximate identities* in the 'local groups' G_v for all completions k_v, *smooth vectors are dense* in unitary representations of G_A, whether irreducible or not. Thus, the smooth vectors are dense in $L^2_o(Z_A G_k \backslash G_A, \omega)$. Similarly, the smooth vectors are dense in the space of not-necessarily-cuspidal square-integrable function $L^2(Z_A G_k \backslash G_A, \omega)$, but this is of little consequence for us.

Thus, without loss of generality, we may suppose that a square-integrable cuspform f is **smooth**.

4. Eigen-cuspforms and automorphic representations

For a smooth square-integrable cuspform $\varphi \in L^2_o(Z_A G_k \backslash G_A, \omega)$ consider the collection of finite linear combinations of right translates

$$g \to \varphi(g\, g_o)$$

of φ by elements $g_o \in G_A$. The *completion* in $L^2_o(Z_A G_k \backslash G_A, \omega)$ of this space is the **subrepresentation generated by** φ.

One fundamental application of *reduction theory* to this situation is the fact that, *as a representation space for the adele group* G_A*, each space* $L^2_o(Z_A G_k \backslash G_A, \omega)$ *decomposes discretely and with finite multiplicities.* Thus, without loss of generality, we may suppose that a cuspform f *generates an irreducible unitary* *representation* π_f of G_A (under right translation). That is, the representation space of π_f is the completion in $L^2_o(Z_A G_k \backslash G_A, \omega)$ of the collection of all functions

$$g \to \sum_i c_i\, f(g \cdot g_i)$$

where the sum is finite, where $c_i \in \mathbf{C}$, and the g_i are fixed elements of G_A.

An irreducible unitary representation on a reductive adele group G_A which occurs as a subrepresentation of some $L^2_o(Z_A G_k \backslash G_A, \omega)$ is said to be an **automorphic cuspidal representation** or **cuspidal automorphic representation**. As an extension of classical terminology, we might say that such (square-integrable) cuspform f is an **eigen-cuspform**.

Further, as a corollary of the proof of discreteness and finite multiplicities in spaces of square-integrable cuspforms, we find that *a smooth square-integrable cuspform f generating an irreducible representation is of* ***rapid decay*** *at infinity in any Siegel set in G_A.*

5. Dirichlet series versus zeta and L-functions

Here we set up and clarify standard terminology, and then describe some *desiderata* for zeta and L-functions. As an overview: the class of all *Dirichlet series* includes and is *much* larger than the class of all *L-functions*, which usually is interpreted to include the class of all *zeta functions*. And the class of *Dirichlet L-functions* is a very tiny subclass of the class of all L-functions.

One kind of fairly general definition of L-function, in terms of so-called 'local data', is given in a following section.

We note the potential for confusion between the phrases *Dirichlet series* and *Dirichlet L-function. These two phrases are in no way synonymous.*

Any series of the form

$$\frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \frac{a_5}{5^s} + \ldots$$

is a **Dirichlet series**. The a_n are the **coefficients**. If for some exponent k we have

$$a_n = O(n^k)$$

then by elementary estimates the series is absolutely convergent (and uniformly so on compacta) for $\Re(s) > k+1$.

For that matter, the numbers n and n^s in the denominators can be replaced by

$$\lambda_1 < \lambda_2 < \lambda_3 \ldots \to +\infty$$

and corresponding s^{th} powers, giving what are sometimes called **generalized Dirichlet series**

$$\frac{a_1}{\lambda_1^s} + \frac{a_2}{\lambda_2^s} + \frac{a_3}{\lambda_3^s} + \frac{a_4}{\lambda_4^s} + \frac{a_5}{\lambda_5^s} + \ldots$$

But usually an object interpretable as a generalized Dirichlet series has a more useful aspect of another sort.

One basic point is that *every L-function and zeta function is a Dirichlet series, but not every Dirichlet series qualifies as an L-function or zeta-function.* And, for most purposes, the notion of 'L-function' includes all notions of 'zeta function' as special cases.

The potential source of confusion about the terminology is that there is a notion of Dirichlet *L-function.* Note that the phrase is 'Dirichlet *L-function*', not the more general 'Dirichlet *series'.* These Dirichlet L-functions are the most elementary of all L-functions, and include the **Riemann zeta-function**

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

as the most special and elementary case. Before giving the general definition of Dirichlet L-function, we need a little preparation. Fix a positive integer F, and let

$$\chi : \mathbf{Z}/F^\times \to \mathbf{C}^\times$$

be a group homomorphism from the multiplicative group $\mathbf{Z}/F^\times$ of the quotient ring $\mathbf{Z}/F$ to non-zero complex numbers. Extend χ to a function (still denoted by χ) on all of $\mathbf{Z}/F$ by defining it to be 0 off $\mathbf{Z}/F^\times$. By composing this extended χ with the quotient map

$$\mathbf{Z} \to \mathbf{Z}/F$$

we get a map (still denoted by χ)

$$\chi : \mathbf{Z} \to \mathbf{Z}/F \to \mathbf{C}^\times$$

The latter map is what is called a **Dirichlet character modulo** F. The most general *Dirichlet L-function* is of the form

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$$

for a Dirichlet character χ.

Assume an inequality $a_n = O(n^k)$ so that the function

$$D(s) = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \frac{a_5}{5^s} + \ldots$$

is holomorphic in the right half-plane $\Re(s) > k+1$. It is not hard to see that the coefficients a_n are completely determined by the holomorphic function $D(s)$. This is useful in what follows.

For convenience, let's suppose that $a_1 \neq 0$, so that we can divide through by it, and have

$$D(s) = 1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \ldots$$

With this normalization, we might demand an **Euler product factorization**

$$D(s) = 1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + + \ldots = \prod_p (1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \ldots)$$

where p runs over *primes*, at least in the region of absolute convergence $\Re(s) > k+1$. The factor

$$1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \ldots$$

is the p^{th} **Euler factor**.

It is not hard to see that such factorization is equivalent to a **weak multiplicativity** property of the coefficients a_n, namely

$$a_{mn} = a_m \cdot a_n \qquad \text{for } m,n \text{ relatively prime}$$

In practice, in any scenario in which the issue is not trivial, such weak multiplicativity is not verified directly, but is a corollary of some exogenous considerations. And here we are making implicit use of the unique factorization in the rational integers $\mathbf{Z}$. In more general situations where Euler factors are indexed by (finite) primes in a number field, we use the unique factorization into prime *ideals* rather than prime *numbers*.

For example, Riemann's zeta function has an Euler product

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots) = \prod_p \frac{1}{1-p^{-s}}$$

Similarly, because of unique factorization and the multiplicativity of Dirichlet characters χ, Dirichlet L-functions have Euler products

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \frac{\chi(p^3)}{p^{3s}} + \ldots) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

Assume that a Dirichlet series $D(s)$ has an Euler factorization over primes. Then we would hope further that for each prime p the p^{th} Euler factor is a *rational function* of p^{-s}, meaning that

$$1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \ldots = \frac{1 + b_1 p^{-s} + \ldots + b_m p^{-ms}}{1 + c_1 p^{-s} + \ldots + c_n p^{-ms}}$$

for some complex numbers $b_1, \ldots, b_m, c_1, \ldots, c_n$ of course depending upon p. (For that matter, it may be that m, n also depend upon p). For brevity, if this property holds, we would say that the Dirichlet series $D(s)$ **has rational Euler factors**.

For example, Riemann's zeta function and Dirichlet L-functions certainly have rational Euler factors, and the Euler factors have the desirable property of depending upon p in a very systematic way.

Without attempting to give a general description of what an 'L-function' or 'zeta function' might be, we *can* state as a general principle that *Euler factorization* and *rationality of the Euler factors* are *prerequisites.* Indeed, the *modern* definitions of most types of L-functions or zeta functions give them as Euler products from the start. Even the rationality of the Euler factors is sometimes part of the *definition*, it is just as likely that in another circumstance the issue of *proving* rationality may be fundamental.

---

6. L-functions defined via local data

Many L-functions and zeta functions fit into the following fairly elementary description, even if the ramifications are unclear.

In particular, this definition does *not* give any hints as to how to prove that the function so-defined has an analytic continuation, functional equation, and so on.

Fix a positive integer N, and fix a finite set S of *'bad'* primes. For a prime p not in the bad set S, let Ψ_p be an N-by-N invertible matrix. Then we have an **L-function attached to the local data** $\{\Psi_p\}$ defined by

$$L(s, \{\Psi_p\}) = \prod_{p\not\in S} \frac{1}{\det(1_N - p^{-s}\Psi_p)}$$

where *'det'* denotes determinant and 1_N is the N-by-N identity matrix.

Since Ψ_p only enters via its determinant, we could be a little coy about things and say just that we have an assignment of *conjugacy classes* $p \to \langle\Psi_p\rangle$, rather than specific matrices.

Note that in this definition the factorization over primes is certainly built in, and the rationality of the Euler factors (as functions of p^{-s}) is also built in. But without any further information we cannot even be confident that this series converges in a half-plane, much less that it has an analytic continuation, etc.

This definition gives no indication where one could expect the *'local data'* to come from, nor what might make a prime fall into the collection S of *'bad'* primes. In the case of Riemann's zeta, the set S is empty, and for every prime p, the local data at p is just $\Psi_p = 1$. In the case of Dirichlet L-functions for a Dirichlet character χ modulo F, we take S to be the set of primes dividing F, and the local data at all other primes is $\Psi_p = \chi(p)$.

An analogous definition of L-function can be given in which *rational* primes as above are replaced by the collection of all primes in a *number field.* However, such 'generalization' can be subsumed in the present set-up.

An issue of some consequence is that of defining Euler factors for *infinite* or *archimedean primes.* That is, ideally the assignment of *gamma factors* should proceed in perfect analogy with construction of all the other Euler factors.

7. Factoring unitary representations of adele groups

To define *automorphic* L-functions, we want to figure out how to attach *local data* Ψ_p (in the above sense) to an irreducible unitary representation π of a reductive adele group G_A. For the definition we will give, it will not matter whether π is automorphic or not! This section takes the first of two steps in associating *'local data'* to irreducible representations of these adele groups.

Since the local groups G_v (v running over primes of the global field k over which G is defined) are known to be *Type I*, any irreducible unitary representation π of G_A *factors over primes* into a completed restricted tensor product:

$$\pi \approx \widehat{\bigotimes_v}' \pi_v$$

where π_v is an irreducible unitary representation of G_v uniquely determined up to isomorphism. In particular, this factorization certainly applies to an irreducible unitary representation π_f generated by a square-integrable cuspform f.

Let K_v be a *'good' maximal compact subgroup* of G_v. Recall that an irreducible unitary representation of G_v is a **spherical representation** if it has a non-zero K_v-fixed vector, in which case there is a one-dimensional space of K_v-fixed vectors in the representation space.

With this terminology we can note that, further, for all but finitely-many primes v, the 'local' representations π_v are *spherical.* Let the set S of *bad primes* be at least large-enough to contain the finite set of primes v for which π_v is *non*-spherical.

Thus, we have turned the problem of acquisition of 'local data' into the problem of meaningfully associating 'local data' to spherical representations of the 'local' groups G_v. That is, we need to attach *'numerical invariants'* to spherical representations. This is done in the next section.

8. Spherical representations and Satake parameters

Now we begin to see how to attach *'local data'* Φ_v to spherical representations π_v of the 'local' groups G_v. To do so, we must introduce the *Satake parameters*, which arise in the *Satake transform.*

For simplicity of notation, we will suppress all the subscripts refering to the prime. This ought not create undue confusion, since all the considerations of this section are *local.* Also, in this context we do not need to think of groups as in any way being *functors*, so we can simplify our way of talking about them: rather than 'k-valued points of the k-group-scheme G', we will just say 'the group G'. And all subgroups will be presumed to be defined over whatever the base field is. In any case, in the basic example of $GL(n, \mathbf{Q}_p)$ these issues can be minimized.

So, let G be a reductive linear group over an ultrametric local field k. Let P be a minimal parabolic subgroup of G, with unipotent radical N and choice of Levi component M. When $G = GL(n, \mathbf{Q}_p)$, we take P to be upper-triangular matrices, N to be elements of P with 1's on the diagonal, and M just the diagonal matrices. Let K be a *'good'* maximal compact subgroup of G. Let $\mathcal{H}_{G,K}$ denote the **spherical Hecke algebra** defined as

$$\mathcal{H}_{G,K} = \text{ left and right } K\text{-invariant complex-valued functions on } G$$

Give G a right Haar measure so that the measure of K is 1. This is also a *left* Haar measure. Then $\mathcal{H}_{G,K}$ is a *convolution algebra* with the convolution product

$$(\eta * \varphi)(g) = \int_G \eta(gh^{-1})\, \varphi(h)\, dh$$

where dh refers to the Haar measure.

A crucial fact from elementary representation theory is that *the isomorphism class of a spherical representation is completely determined by the representation of the spherical Hecke algebra on the one-dimensional space of spherical vectors in the representation.* Since the space of spherical vectors is just one-dimensional, automorphisms are just complex scalars. Thus, to describe a spherical representation π is to describe a **C**-algebra homomorphism

$$\mathcal{H}_{G,K} \to \mathbf{C}$$

To give such a description, we first describe the structure of the spherical Hecke algebra itself.

Let δ_P be the *modular function* of P, meaning that for a right Haar measure μ_P *on* P we have

$$\mu_P(pE) = \mu_P(pEp^{-1}) = \delta_P(p) \cdot \mu_P(E)$$

for any $p \in P$ and measurable $E \subset P$. The **Satake transform** $S\eta$ of a function $\eta \in \mathcal{H}_{G,K}$ is defined by the integral formula

$$(S\eta)(m) = \delta_P(m)^{-1/2} \int_N \eta(nm)\, dn = \delta_P(m)^{1/2} \int_N \eta(mn)\, dn$$

where dn denotes a Haar measure on N normalized so that

$$\text{characteristic function of } K \to \text{ characteristic function of } K \cap M$$

Let $\mathcal{H}_{M,K\cap M}$ be the spherical Hecke algebra of the reductive group M (a fixed Levi component of the parabolic P). The *Weyl group* of M in G is

$$W = \text{ normalizer of } M \text{ in } G / \text{ centralizer of } M \text{ in } G$$

Satake's theorem is that *the Satake transform S gives an isomorphism from the spherical Hecke algebra $\mathcal{H}_{G,K}$ of G to the Weyl-group-invariant elements $\mathcal{H}^W_{M,K\cap M}$ of the spherical Hecke algebra $\mathcal{H}_{M,K\cap M}$ of M.* The isomorphism is called the **Satake isomorphism**.

Now generally the spherical Hecke algebra $\mathcal{H}_{M,K\cap M}$ of M is of the form

$$\mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

for indeterminates z_i, where n is the *split rank* of G. For example, for $GL(n, \mathbf{Q}_p)$ this rank is indeed n. The crucial point is that *the spherical Hecke algebra of the Levi component of a minimal parabolic is a commutative Noetherian ring.*

Next, the Weyl group W acts in such manner that the full spherical Hecke algebra of M is *integral* over the W-invariant elements. Therefore, always

$$\mathcal{H}^W_{M,K\cap M} = \text{ commutative Noetherian}$$

Thus, in particular, *the spherical Hecke algebra $\mathcal{H}_{G,K}$ is a commutative Noetherian ring.*

For example, in the case of $GL(n, \mathbf{Q}_p)$, the Weyl group W is the group of permutations of the indices $1, 2, \ldots, n$, so acts upon

$$\mathcal{H}_{M,K\cap M} \approx \mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

by permutation of these variables z_i (and inverses, correspondingly). Thus, the W-fixed subalgebra is

$$\mathcal{H}^W_{M,K\cap M} \approx \mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1} \ldots, z_n, z_n^{-1}]^W \approx \mathbf{C}[s_1, s_2, s_3, \ldots, s_n, s_n^{-1}]$$

That is, it is generated by the *elementary symmetric polynomials* $s_1, \ldots, s_n$ in the z_i, together with the single item s_n^{-1}..

Finally, for essentially elementary reasons the *integrality* assures that any algebra homomorphism

$$\lambda : \mathcal{H}_{G,K} \approx \mathcal{H}^W_{M,K\cap M} \to \mathbf{C}$$

extends to an algebra homomorphism

$$\tilde{\lambda} : \mathcal{H}_{M,K\cap M} \to \mathbf{C}$$

Thus, if we have an identification

$$\mathcal{H}_{M,K\cap M} \approx \mathbf{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

then the images

$$\tilde{\lambda}(z_1), \tilde{\lambda}(z_2) \ldots, \tilde{\lambda}(z_n)$$

are the **Satake parameters** associated to $\tilde{\lambda}$.

Thus, in summary, to a spherical representation we attach a one-dimensional representation of the spherical Hecke algebra of G, to which we associate a one-dimensional representation of the W-invariant spherical Hecke algebra of M, which we extend to a one-dimensional representation of the whole spherical Hecke algebra of M, which is completely determined by the images of the generators (denoted by $z_1, z_2, \ldots, z_n$ above).

---

9. Local data, L-groups, higher L-functions

So far, to any irreducible unitary representation π of a reductive adele group G_A, whether or not it arises from an automorphic form, we associate the list of 'local' representations π_v of the 'local' groups G_v. All but finitely-many of these local representations are spherical, so are specified by *Satake parameters.*

For each prime v of the global field we arrange the Satake parameters into a suitable diagonal matrix Φ_v: in the case of $GL(n, \mathbf{Q}_p)$ with Satake parameters $\tilde{\lambda}(z_1), \ldots, \tilde{\lambda}(z_n)$ (in the notation of the previous section) we simply take

$$\Phi_p = \begin{pmatrix} \tilde{\lambda}(z_1) & & & \\ & \tilde{\lambda}(z_2) & & \\ & & \ddots & \\ & & & \tilde{\lambda}(z_n) \end{pmatrix}$$

Then the **standard L-function** attached to the local data Φ_v coming from the Satake parameters is

$$L(s, \rho, \{\Phi_v\}) = \prod_{v \notin S} \frac{1}{\det(1 - q_v^{-s} \Phi_v)}$$

where q_v is the order of the residue class field for v, where the '1' denotes the identity matrix of appropriate size, and where S is the finite set of primes where the 'local representation' is not spherical. This also omits archimedean primes.

Further, once we have the 'local data' Φ_v coming from the Satake parameters, we can create *higher* L-functions by 'local date' as follows. Suppose that we have arranged so that all the Φ_v lie inside a group ${}^L G_v^o$ of matrices, for all possible spherical representations. Let

$$\rho :^L G_v^o \to GL(N, \mathbf{C})$$

be a finite-dimensional representation of ${}^L G_v^o$, *not* depending upon v. Then as 'local data' we might use

$$\Psi_v = \rho(\Phi_v)$$

and form a **higher L-function**

$$L(s, \rho, \{\Phi_v\}) = \prod_{v \notin S} \frac{1}{\det(1 - q_v^{-s} \rho(\Phi_v))}$$

where q_v is the order of the residue class field for v, where the '1' denotes the identity matrix of appropriate size, and where S is the finite set of primes where the 'local representation' is not spherical. This omits archimedean primes.

In the case of $GL(n, \mathbf{Q}_p)$, the L-group is just $GL(n, \mathbf{C})$, and the auxiliary representation ρ is just a finite-dimensional representation

$$\rho : GL(n, \mathbf{C}) \to GL(N, \mathbf{C})$$

In even more general situations, there is nevertheless a very systematic general prescription for arranging the Satake parameters in a diagonal matrix Φ_v. Further, this is arranged so that all possible local data Φ_v lie inside a group ${}^L G_v^o$ depending upon the 'local group' G_v. The group ${}^L G_v^o$ is the (connected component of the) **L-group** attached to G. The L-group idea can also be made to incorporate Galois groups and their representations.

It should be emphasized that the 'L-group formalism' is mostly just that, a formalism, and does not really circumvent fundamental issues, but mostly gives a unifying notation and language helpful in the general case. Anyway, in summary, the Satake parameters are used to specify a conjugacy class of semi-simple (i.e., diagonal) elements in the appropriate 'L-group'. For $GL(n, \mathbf{Q}_p)$ there is little to be gained, at the outset at least, from worrying about fancier definitions.

And, finally, if the local representations π_v are the tensor factors in an irreducible unitary representation π occuring inside a space $L^2_o(Z_A G_k \backslash G_A, \omega)$ of square-integrable cuspforms (for some ω), then the associated L-functions constructed as above are called **automorphic cuspidal L-functions**.