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Truncation and Maaß-Selberg Relations

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According to Borel, Harish-Chandra gave the name *Maaß-Selberg relation* to the formula for the inner product of truncated Eisenstein series with cuspidal data. Apparently the systematic computation of these inner products is due to Langlands. A crucial technical issue is the precise notion of *truncation* of Eisenstein series.

We give simple but non-trivial examples, with illustrative examples of consequences: possible poles of Eisenstein series, square-integrability of certain residues of Eisenstein series.

This bears upon the construction of non-trivial residual square-integrable automorphic forms coming from cuspforms on smaller groups, anticipating that such automorphic forms occur as residues of Eisenstein series. For example, we can see why there is no interesting (i.e., non-constant) non-cuspidal discrete spectrum for $GL(2)$ nor for $GL(3)$, but only for $GL(4)$ and larger groups. Namely, the Eisenstein series of interest on $GL(3)$ have no poles at all in the right half-plane. This follows immediately from the Maaß-Selberg relations.

Ideas of Zagier and Casselman regarding non-classical extensions of integrals (à la Hadamard and M. Riesz) give alternative proofs of Maaß-Selberg relations which are perhaps more conceptual, but the issue of appropriate truncation operators does not disappear.

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1. Maximal proper parabolics, cuspidal data, for $GL(n,\mathbf{Z})$

The simplest non-trivial examples of Maaß-Selberg relations and corollaries concern spherical Eisenstein series on $GL(n)$ associated to cuspidal data on the Levi component of maximal (proper) parabolics. We will assume standard facts about the constant terms of such Eisenstein series.

Let $G = GL(n,\mathbf{R})$, $\Gamma = GL(n,\mathbf{Z})$, and $K = O(n,\mathbf{R})$. For n_1, n_2 positive integers so that $n_1 + n_2 = n$, define the corresponding standard maximal proper parabolic

$$P = P_{n_1,n_2} = \{\begin{pmatrix} n_1 \times n_1 & * \\ 0 & n_2 \times n_2 \end{pmatrix}\}$$

with unipotent radical

$$N^P = \{\begin{pmatrix} 1_{n_1} & * \\ 0 & 1_{n_2} \end{pmatrix}\}$$

and standard Levi component

$$M^P = \{\begin{pmatrix} n_1 \times n_1 & 0 \\ 0 & n_2 \times n_2 \end{pmatrix}\} \approx GL(n_1) \times GL(n_2)$$

Fix a standard parabolic P and N its unipotent radical. For f an $N_{\mathbf{Z}} = N \cap \Gamma$-invariant function, the *constant term* $c_P(f)$ of f along P is defined as usual to be

$$c_P f(g) = \int_{N_{\mathbf{Z}}\backslash N} f(ng)\, dn$$

We consider only left Γ-invariant, right K-invariant cuspforms on G with trivial central character. In the present discussion, *cuspforms* will be spherical Hecke algebra eigenfunctions at all finite primes, will be square-integrable, and will generate irreducible unitary representations of $G = GL(n,\mathbf{R})$. (The latter condition is stricter than merely requiring that cuspforms are eigenvectors for the center of the universal enveloping algebra.)

Fix integers n_1, n_2. For $i = 1, 2$ let f_i be *cuspforms* on $GL(n_i, \mathbf{R})$. Let $P = P_{n_1,n_2}$, and put

$$\varphi(nmk) = \varphi_{s,f}(nmk) = |\det m_1|^{n_2 s}\, |\det m_2|^{-n_1 s}\, f_1(m_1)\, f_2(m_2)$$

where

$$m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

with $m_i \in GL(n_i)$, so that m is in the standard Levi component M of the parabolic subgroup P, $n \in N$ its unipotent radical, and k in K. Let $P_{\mathbf{Z}} = \Gamma \cap P$. Define the associated Eisenstein series $E^P(\varphi) = E^P_\varphi$ by

$$E^P_\varphi(g) = E^P(\varphi)(g) == \sum_{\gamma \in P_{\mathbf{Z}} \backslash \Gamma} \varphi(\gamma g)$$

For $\mathrm{Re}(s)$ sufficiently positive, this series converges absolutely and uniformly on compacta. It is a left $GL(n_1 + n_2, \mathbf{Z})$-invariant right $O(n_1 + n_2)$-invariant function with trivial central character. More generally, for any left $P_{\mathbf{Z}}$-invariant function η on G, define Eisenstein series

$$E^P(\eta)(g) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \eta(\gamma g)$$

To describe the meromorphic continuation and functional equation of these Eisenstein series, we recall the form of their constant terms (after Selberg, Langlands, et alia). The results asserted here are the result of non-trivial computations. For maximal proper $P = P_{n_1,n_2}$ with $n_1 = n_2$ (that is, *self-associate*), the Eisenstein series E^P_φ as above, attached to cuspidal data f on the Levi component of P has constant term with values on $M = M^P$ given by (via a non-trivial computation, using the cuspidality of the data)

$$c_P E^P_\varphi = \varphi + c_s \cdot \varphi^w$$

where c_s is a ratio of L-functions attached to f whose precise nature does not immediately concern us, and

$$\varphi^w(nmk) = |\det m_1|^{n_2(1-s)}\, |\det m_2|^{-n_1(1-s)}\, f_2(m_1)\, f_2(m_2) = \varphi(wmw^{-1}) \cdot \delta_P^{-1}(m)$$

where $\delta_P(m) = |\det m_1|^{n_2} \cdot |\det m_2|^{-n_1}$ is the modular function of P, $m \in M$ is as above, $n \in N$, and $k \in K$. For all other standard parabolics the constant term is 0. For $n_1 \neq n_2$ (the *non-self-associate* case), let $Q = P_{n_2,n_1}$ be the associate of P. Then (via a non-trivial computation, using the cuspidality of the data)

$$\begin{aligned} c_P E^P_\varphi(m) &= \varphi(m) \\ c_Q E^P_\varphi(m) &= c_s \cdot \varphi^w(m) \end{aligned}$$

and where again c_s is a ratio of L-functions depending on the data φ. All other constant terms are 0.

Theorem: (*Selberg, Langlands, Bernstein, et alia*) Eisenstein series E^P_φ have meromorphic continuations in s, with functional equation

$$c_s^{-1} E^P_\varphi = E^{P^w}_{\varphi^w}$$

where P^w is the associate parabolic to P (whether or not P is self-associate).

Now define the truncation operators. For a standard maximal proper parabolic $P = P_{n_1,n_2}$ as above, for $g = nmk$ with $m \in M^P$ as above, $n \in N^P$, and $k \in O(n)$, define

$$h^P(g) = \frac{|\det m_1|^{n_2}}{|\det m_2|^{n_1}} = \delta^P(nm) = \delta^P(m)$$

where δ^P is the modular function on P. (The exponents make this function invariant under the center Z of G.) For fixed large real T, the T-**tail** of the P-constant term of a left $N_{\mathbf{Z}}^P$-invariant function F

$$c_P^T F(g) = \begin{cases} c_P F(g) & (\text{for } h^P(g) \geq T) \\ 0 & (\text{for } h^P(g) < T) \end{cases}$$

If $n_1 \neq n_2$, P is not self-associate, and has associate $Q = P_{n_2,n_1}$. Similarly, define the T-tail of the Q-constant term of left $N_{\mathbf{Z}}^Q$-invariant F by

$$c_Q^T F(g) = \begin{cases} c_Q F(g) & (\text{for } h^Q(g) \geq T) \\ 0 & (\text{for } h^Q(g) < T) \end{cases}$$

We want the truncations of the Eisenstein series under consideration to be square integrable (which could be accomplished a number of ways), and also so that their inner products are calculable in explicit and straightforward terms. Further, there should be no obstacle to meromorphic continuation of the *tail* in the truncation. These requirements are somewhat at odds with each other. Define

$$\Lambda^T E_\varphi^P = \begin{cases} E_\varphi^P - E^P(c_P^T E_\varphi^P) & (\text{for } n_1 = n_2, \text{ i.e., for } P \text{ self-associate}) \\ E_\varphi^P - E^P(c_P^T E_\varphi^P) - E^Q(c_Q^T E_\varphi^P) & (\text{for } n_1 \neq n_2, \text{ i.e., for } P \text{ not self-associate}) \end{cases}$$

Proposition: The truncated Eisenstein series $\Lambda^T E_\varphi^P$ is of rapid decay in Siegel sets.

Proof: As usual, for a root γ, let $g \to a^\gamma$ be the function which sends g to the value of γ on a, where $g = nak$ is an Iwasawa decomposition, with n in the unipotent radical of a minimal parabolic, a in the Levi component, and k in the maximal compact.

The argument is simpler in the self-associate case, which we carry out first. For any simple (positive) root α, let c_α be the constant term along the unipotent radical of the maximal proper parabolic attached to α. Then one basic result from the theory of the constant term is that on a standard Siegel set for any automorphic form (K-finite, $\mathbf{Z}$-finite, of moderate growth) $f - c_\alpha f$ is of rapid decay as $a^\alpha \to +\infty$. Also, from the theory of constant terms of Eisenstein series with cuspidal data, for self-associate maximal proper P in $GL(n)$ (and in classical groups) all such constant terms are 0 except that along P itself. That is, on standard Siegel sets $E_\varphi^P - c_P E_\varphi^P$ is of rapid decay. Thus, $E_\varphi^P - c_P^T E_\varphi^P$ is of rapid decay on standard Siegel sets, and then the automorphic form

$$\Lambda^T E_\varphi^P = E_\varphi^P - E^P(c_P^T E_\varphi^P)$$

is of rapid decay on *all* Siegel sets.

In the non-self-associate case, let $Q \neq P$ be the other associate of P, and let α, β be the simple positive roots corresponding to P and Q, respectively. Because f is a cuspform and P is not self-associate only a single Bruhat cell (corresponding to the trivial Weyl element) contributes to the constant term $c_P E_\varphi^P$ and

$$c_P E_\varphi^P(m) = \varphi(m)$$

which is rapidly decreasing on standard Siegel sets as $a^\lambda \to +\infty$ for any simple (positive) root $\lambda \neq \alpha$, because f is a cuspform. Similarly, only a single Bruhat cell (corresponding to the longest Weyl element) contributes to the constant term $c_Q E_\varphi^P$, which is rapidly decreasing on standard Siegel sets as $a^\lambda \to +\infty$ for any simple (positive) root $\lambda \neq \beta$, because f is a cuspform. Thus, the truncation

$$\Lambda^T E_\varphi^P = E_\varphi^P - E^P(c_P^T E_\varphi^P) - E^Q(c_Q^T E_\varphi^P)$$

has decay properties as follows. If $a^\gamma \to +\infty$ for γ other than α, β, then all three terms on the right-hand side are of rapid decay. If $\alpha \to +\infty$, then each of the two expressions $E_\varphi^P - E^P(c_P^T E_\varphi^P)$ and $E^Q(c_Q^T E_\varphi^P)$ is of rapid decay. And if $\beta \to +\infty$, then each of the two expressions $E_\varphi^P - E^Q(c_Q^T E_\varphi^P)$ and $E^P(c_P^T E_\varphi^P)$ is of rapid decay. Thus, as the value of any simple positive root goes to $+\infty$ in a standard Siegel set, the truncation goes rapidly to zero. ///

Proposition: Given a compact subset C of G, there are only finitely-many $\gamma \in P_{\mathbf{Z}}\backslash\Gamma$ such that for $g \in C$

$$c_P^T E_\varphi^P(\gamma g) \neq 0$$

Thus, the series expression for $E^P(c_P^T E_\varphi^P)$ is uniformly locally finite, and therefore has a meromorphic continuation in s.

Proof: This follows from the fact that there are only finitely-many $\gamma \in P_{\mathbf{Z}}\backslash\Gamma$ such that

$$\gamma \cdot C \cap (\text{support of } c_P^T E_\varphi^P) \neq \phi$$

The latter follows from the fact that for any maximal proper parabolic Q

$$h^Q(wnw^{-1}\cdot m) \leq h^Q(m)$$

with $m \in M^Q$, $n \in N^Q$, with w the longest Weyl element. ///

For two left Γ-invariant functions Φ and Ψ with the property that $\Phi \cdot \overline{\Psi}$ is Z-invariant, define

$$\langle \Phi, \Psi \rangle = \int_{Z\cdot\Gamma\backslash G} \Phi(g)\,\overline{\Psi(g)}\,dg$$

Let $h = h_1 \otimes h_2$ be a another cuspform on the Levi component M of P, let $r \in \mathbf{C}$, and define

$$\psi(nmk) = \psi_{r,h}(nmk) = |\det m_1|^{n_2 r}\,|\det m_2|^{-n_1 r}\,h_1(m_1)\,h_2(m_2)$$

where

$$m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

with $m_i \in GL(n_i)$, so that m is in M, $n \in N^P$, and k in K. Let $\langle f, h\rangle$ be the inner product on $GL(n_1) \times GL(n_2)$ modulo its center and

$$f^w(m) = f(wmw^{-1}) = f_1(m_2)\,f_2(m_1)$$

(without the renormalization by the modular function).

Theorem: *(Maaß-Selberg relations)* The inner product $\langle \Lambda^T E_\varphi^P, \Lambda^T E_\psi^P\rangle$ of truncations $\Lambda^T E_\varphi^P$ and $\Lambda^T E_\psi^P$ of two Eisenstein series E_φ^P and E_ψ^P attached to cuspidal data φ, ψ on a maximal proper parabolic P is given as follows. For P *self-associate* (i.e., for $n_1 = n_2$)

$$\langle \Lambda^T E_\varphi^P, \Lambda^T E_\psi^P\rangle = \langle f,h\rangle \frac{T^{s+\bar r-1}}{s+\bar r-1} + \langle f,h^w\rangle \overline{c_r^\psi} \frac{T^{s+(1-\bar r)-1}}{s+(1-\bar r)-1} + \langle f^w,h\rangle c_s^\varphi \frac{T^{(1-s)+\bar r-1}}{(1-s)+\bar r-1} + \langle f^w,h^w\rangle c_s^\varphi \overline{c_r^\psi} \frac{T^{(1-s)+(1-\bar r)-1}}{(1-s)+(1-\bar r)-1}$$

For P *not self-associate* (i.e., for $n_1 \neq n_2$)

$$\langle \Lambda^T E_\varphi^P, \Lambda^T E_\psi^P\rangle = \langle f,h\rangle \frac{T^{s+\bar r-1}}{s+\bar r-1} + \langle f^w,h^w\rangle c_s^\varphi \overline{c_r^\psi} \frac{T^{(1-s)+(1-\bar r)-1}}{(1-s)+(1-\bar r)-1}$$

Remark: That is, the expression for the not-self-associate case is identical to that for the self-associate case but with the middle two terms missing. In the non-self-associate case the inner products $\langle f^w, h\rangle$ and $\langle f, h^w\rangle$ would not make sense, in any case, because in that case $wMw^{-1} \neq M$, so the two functions live on different groups.

Corollary: For maximal proper parabolics P in $GL(n)$, on the half-plane $\mathrm{Re}(s) \geq 1/2$ an Eisenstein series E^P_φ has no poles whatsoever if P is not self-associate. If P is self-associate, the only possible poles are on the real line, and only occur if $\langle f, f^w\rangle \neq 0$. In that case, any pole is simple, and the residue is square-integrable. In particular, taking $f = f_o \otimes f_o$

$$\langle \mathrm{Res}_{s_o} E^P_\varphi, \mathrm{Res}_{s_o} E^P_\varphi\rangle = \langle f_o, f_o\rangle^2 \cdot \mathrm{Res}_{s_o} c^\varphi_s$$

Proof: (of theorem).The self-associate case admits a simple argument, which we give, despite the fact that this simplicity is misleading about what happens more generally. The point is that because of the self-associate-ness the truncated Eisenstein series $\Lambda^T E^P_\varphi$ is itself an Eisenstein series

$$\Lambda^T E^P_\varphi = E^P(\varphi) - E^P(c^T_P E^P_\varphi) = E^P(\varphi - c^T_P E^P_\varphi)$$

As may be intuitively plausible, the Eisenstein series made from the tail of the constant term integrates to zero against the truncated Eisenstein series, that is,

$$\langle \Lambda^T E^P_\varphi, E^P(c^T_P E^P_\varphi)\rangle = 0$$

This is essentially a computation, though not completely trivial, and uses the cuspidality of the data φ. Use this fact, and then unwind the truncated Eisenstein series to obtain

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E_\psi\rangle = \langle \Lambda^T E^P_\varphi, E_\psi\rangle = \int_{Z\cdot P_{\mathbf{Z}}\backslash G} \begin{cases} -c_s\varphi^w(g) & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T) \end{cases} \cdot \overline{E_\psi(g)}\, dg$$

where Z is the center of G. This is

$$\int_{Z\cdot(N\cdot P_{\mathbf{Z}})\backslash G} \begin{cases} -c_s\varphi^w(g) & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T) \end{cases} \cdot \overline{c_P E_\psi(g)}\; dg$$

$$= \int_{Z\cdot(N\cdot P_{\mathbf{Z}})\backslash G} \begin{cases} -c_s\varphi^w(g) & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T) \end{cases} \cdot (\overline{\varphi_\psi(g)} + \overline{c_r\varphi_{1-r,h^w}(g)})\, dg$$

Since the integrand is now left $N = N^P$-invariant and right K-invariant, this integral may be computed as an integral over the Levi component M^P of P, using the Iwasawa decomposition $G = N^P \cdot M^P \cdot K$. Of course, in these coordinates the Haar integral on G is

$$\int_G f(g)\, dg = \int_N \int_M \int_K f(nmk)\, \delta_P^{-1}(m)\, dn\, dm\, dk$$

Then

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\psi\rangle = \int_{Z\cdot M^P_{\mathbf{Z}}\backslash M^P} \begin{cases} -c_s\varphi^w(m) & (\text{for } h^P \geq T) \\ \varphi(m) & (\text{for } h^P < T) \end{cases} \cdot (\overline{\varphi_\psi(g)} + \overline{c_r\varphi_{1-r,h^w}(g)})\; \delta_P(m)^{-1}\, dm$$

This gives rise to the four terms of the theorem for the self-associate case.

In the non-self-associate case, the truncated Eisenstein series $\Lambda^T E^P_\varphi$ is not itself an Eisenstein series, and more serious attention is required to evaluate the inner product of truncated Eisenstein series. One computes (using the cuspidality of the data φ and ψ) that

$$\begin{aligned} \langle E^P_\varphi - E^P(c^T_P E^P_\varphi), E^P(c^T_P E^P_\varphi)\rangle &= 0 \\ \langle E^P_\varphi - E^Q(c^T_Q E^P_\varphi), E^Q(c^T_Q E^P_\varphi)\rangle &= 0 \\ \langle E^P(c^T_P E^P_\varphi), E^Q(c^T_Q E^P_\varphi)\rangle &= 0 \end{aligned}$$

Then the inner product of the truncated Eisenstein series is

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\psi \rangle = \langle E^P_\varphi - E^P(c^T_P E^P_\varphi), E^P_\psi \rangle + \langle E^Q(c^T_Q E^P_\varphi), E^Q(c^T_Q E^P_\psi)\rangle$$

Now the pairings unwind. First,

$$\langle E^P_\varphi - E^P(c^T_P E^P_\varphi), E^P_\psi \rangle = \langle E^P(c_P E^P_\varphi - c^T_P E^P_\varphi), E^P_\psi \rangle = \int_{Z\cdot P_{\mathbf{Z}}\backslash G} \begin{cases} 0 & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T)\end{cases} \cdot \overline{E_\psi(g)}\, dg$$

where Z is the center of G. This is

$$\int_{Z\cdot(N\cdot P_{\mathbf{Z}})\backslash G} \begin{cases} 0 & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T)\end{cases} \cdot \overline{c_P E_\psi(g)}\; dg = \int_{Z\cdot(N\cdot P_{\mathbf{Z}})\backslash G} \begin{cases} 0 & (\text{for } h^P \geq T) \\ \varphi(g) & (\text{for } h^P < T)\end{cases} \cdot \overline{\psi(g)}\, dg$$

Since the integrand is now left $N = N^P$-invariant and right K-invariant, this integral may be computed as an integral over the Levi component M^P of P, using the Iwasawa decomposition $G = N^P \cdot M^P \cdot K$. Then

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\psi \rangle = \int_{Z\cdot M^P_{\mathbf{Z}}\backslash M^P} \begin{cases} 0 & (\text{for } h^P \geq T) \\ \varphi(m) & (\text{for } h^P < T)\end{cases} \cdot \overline{\psi(g)}\, \delta_P(m)^{-1}\, dm$$

This gives one term for the not self-associate case.

Let c^φ_s and c^ψ_r be the ratios of L-functions occurring as

$$c_Q E^P_\varphi = c^\varphi_s \cdot \varphi^w$$

$$c_Q E^P_\psi = c^\psi_r \cdot \psi^w$$

Then the other pairing is unwound in similar fashion

$$\langle E^Q(c^T_Q E^P_\varphi), E^Q(c^T_Q E^P_\psi)\rangle = \int_{Z\cdot Q_{\mathbf{Z}}\backslash G} \begin{cases} c^\varphi_s \varphi^w(g) & (\text{for } h^P \geq T) \\ 0 & (\text{for } h^P < T)\end{cases} \cdot \overline{E_\psi(g)}\, dg$$

where Z is the center of G. This is

$$\int_{Z\cdot(N\cdot Q_{\mathbf{Z}})\backslash G} \begin{cases} c^\varphi_s \varphi^w(g) & (\text{for } h^Q \geq T) \\ 0 & (\text{for } h^Q < T)\end{cases} \cdot \overline{c_Q E_\psi(g)}\; dg = \int_{Z\cdot(N\cdot Q_{\mathbf{Z}})\backslash G} \begin{cases} c^\varphi_s\, \varphi^w(g) & (\text{for } h^Q \geq T) \\ 0 & (\text{for } h^Q < T)\end{cases} \cdot \overline{c^\psi_r\, \psi^w(g)}\, dg$$

Since the integrand is now left N^Q-invariant and right K-invariant, this integral may be computed as an integral over the Levi component M^Q of Q, using the Iwasawa decomposition $G = N^Q \cdot M^Q \cdot K$. Then

$$\langle E^Q(c^T_Q E^P_\varphi), E^Q(c^T_Q E^P_\psi)\rangle = \int_{Z\cdot M^Q_{\mathbf{Z}}\backslash M^Q} \begin{cases} 0 & (\text{for } h^Q \geq T) \\ c^\varphi_s\, \varphi(m) & (\text{for } h^Q < T)\end{cases} \cdot \overline{c^\psi_r\, \psi^w(g)}\, \delta_Q(m)^{-1}\, dm$$

This gives the second term of the theorem for the not self-associate case. ///

Proof: (of corollary). From the theory of the constant term, the only possible poles of the Eisenstein series are at poles of the constant terms, which in this case means a pole of c_s. Invoke the Maaß-Selberg relation with $r = s$ and $h = f$. In the non-self-associate case this is

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\varphi \rangle = \langle f, f\rangle \frac{T^{2\sigma-1}}{2\sigma - 1} + \langle f^w, h^w\rangle |c_s|^2 \frac{T^{1-2\sigma}}{1-2\sigma}$$

where $\sigma = \mathrm{Re}(s)$. The non-self-associate case is slightly unlike the simple case of $GL(2)$, in that the inner product of truncated Eisenstein series is missing the two middle terms which made it possible (in effect) for there to be a pole. Specifically, in the non-self-associate case, let $s_o = \sigma_o + it_o$ be an alleged pole s_o of c_s

of order ℓ in that half-plane. Letting $s = \sigma_o + it$ approach s_o vertically the left-hand side of the relation is asymptotic to a *positive* multiple of $t^{-2\ell}$, while on right-hand side only the second of the two terms blows up at all. In particular, that expression

$$|c_s|^2 \cdot \langle f^w, f^w \rangle \cdot \frac{T^{1-2\sigma}}{1-2\sigma}$$

is asymptotic to a *negative* multiple of $t^{-2\ell}$, since $\sigma = \mathrm{Re}(s) > \frac{1}{2}$. Thus, there is no pole in that half-plane.

Similarly, in the self-associate case, for there to be any pole at all the two middle terms on the right-hand side of the relation must not vanish, or the same contradiction occurs, so $\langle f, f^w \rangle$ must be non-zero, and the alleged pole must be on the real axis, and must be simple. (If any of these conditions fail, the middle terms cannot keep up with the negative value of the fourth term). For $f = f_o \otimes f_o$, $f^w = f$ and

$$\langle f, f \rangle = \langle f_o, f_o \rangle \cdot \langle f_o, f_o \rangle$$

Letting $s = \sigma + it$, and $c_s = c_s^\varphi$, in the self-associate case the Maaß-Selberg relation becomes

$$\langle \Lambda^T E_\varphi^P, \Lambda^T E_\varphi^P \rangle = \langle f_o, f_o \rangle^2 \frac{T^{2\sigma-1}}{2\sigma - 1} + \langle f_o, f_o \rangle^2 \overline{c_s} \frac{T^{2it}}{2it} + \langle f_o, f_o \rangle^2 c_s \frac{T^{-2it}}{-2it} + \langle f_o, f_o \rangle^2 |c_s|^2 \frac{T^{1-2\sigma}}{1-2\sigma}$$

Multiplying through by $t^2 = (it)(-it)$ and taking the limit as $t \to 0$ gives

$$\langle \mathrm{Res}_\sigma \Lambda^T E_\varphi^P, \mathrm{Res}_\sigma \Lambda^T E_\varphi^P \rangle = \langle f_o, f_o \rangle^2 \, \overline{\mathrm{Res}_\sigma c_s} \cdot \frac{1}{2} + \langle f_o, f_o \rangle^2 \, \mathrm{Res}_\sigma c_s \cdot \frac{1}{2} + \langle f_o, f_o \rangle^2 \, |\mathrm{Res}_\sigma c_s|^2 \frac{T^{1-2\sigma}}{1-2\sigma}$$

Letting $T \to +\infty$ causes the last term to go to zero, and yields the indicated finite limit in the self-associate case, since $c_{\bar{s}} = \overline{c_s}$ and the supposed pole is on the real axis. ///