(*February 19, 2005*)

Simplest Example of Truncation and Maaß-Selberg Relations

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

According to Borel, Harish-Chandra gave the name *Maaß-Selberg relation* to the formula for the inner product of truncated Eisenstein series with cuspidal data, though apparently the systematic computation of these inner products is due to Langlands. A crucial technical issue is the precise notion of *truncation* of Eisenstein series.

We give the simplest possible example here, for clarity: spherical Eisenstein series for $SL(2,\mathbf{Z})$, and illustrate the consequences, namely restrictions on location and order possible poles of Eisenstein series, and square-integrability of residues of Eisenstein series.

We consider right $K = SO(2)$-invariant left $\Gamma = SL(2,\mathbf{Z})$-invariant functions on $G = SL(2,\mathbf{R})$. We may identify G/K with the upper half-plane $\mathbf{H}$ in the complex plane $\mathbf{C}$ if we wish, and use the usual coordinates $z = x + iy$ on $\mathbf{H}$ as coordinates on G/K. As usual, let

$$P = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\} \quad N = \{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G\} \quad M = \{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in G\}$$

be the standard parabolic subgroup, its unipotent radical, and its standard Levi component, respectively. For a left N-invariant left $P \cap \Gamma$-invariant function φ on G form the *Eisenstein series* $E(\varphi)$

$$E(\varphi)(g) = \sum_{\gamma \in P_{\mathbf{Z}} \backslash \Gamma} \varphi(\gamma g)$$

where $P_{\mathbf{Z}} = P \cap \Gamma$. The issue of convergence is non-trivial: this series does not necessarily converge for arbitrary φ. An important standard special case is where φ is right K-invariant and, in $z = x + iy$ coordinates on $\mathbf{H} \approx G/K$,

$$\varphi(x+iy) = y^s$$

With this φ, the Eisenstein series

$$E_s(g) = E(\varphi)(g)$$

is convergent for $\mathrm{Re}(s) > 1$. Further, by several different methods E_s may be shown to have a meromorphic continuation in $s \in \mathbf{C}$. (The range of possible precise senses of this meromorphic continuation is not the key question for the moment.)

It is well-known and not too hard to prove that the region

$$\mathcal{F} = \{z = x + iy : |x| \le \frac{1}{2}, |z| \ge 1\}$$

is a fundamental domain for the action of Γ on $\mathbf{H} \approx G/K$, meaning that $\Gamma \cdot \mathcal{F} = \mathbf{H}$, and that for $\gamma \neq \pm 1_2$ the measure of $\mathcal{F} \cap \gamma \cdot \mathcal{F}$ is 0. Define the usual Petersson inner product $\langle , \rangle$ on right K-invariant functions on $\mathbf{H} \approx G/K$ by

$$\langle f_1, f_2 \rangle = \int_{\Gamma \backslash \mathbf{H}} f_1(z)\, \overline{f_2(z)}\, \frac{dx\, dy}{y^2} = \int_{\mathcal{F}} f_1(z)\, \overline{f_2(z)}\, \frac{dx\, dy}{y^2}$$

A left Γ-invariant smooth function f on G has a Fourier expansion in the N-coordinate, where we may identify N with $\mathbf{R}$ and $N_{\mathbf{Z}} = N \cap \Gamma$ with $\mathbf{Z} \subset \mathbf{R}$:

$$f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \sum_{n \in \mathbf{Z}} c_n(g)\, e^{2\pi i n x}$$

The *constant term* $c_P f$ of f along P is its 0^{th} Fourier coefficient

$$c_P f(g) = c_0(g) = \int_{\mathbf{R}/\mathbf{Z}} f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\, dx$$

In $z = x + iy$ coordinates for right K-invariant functions, the constant term is a function of y alone, so we may write $c_P f(y)$ rather than $c_P f(g)$. Standard (but not trivial) estimates from *the theory of the constant term* show that all but the constant term (in the Fourier expansion) of an automorphic form are of *rapid decay* in $z = x + iy$ coordinates as $y \to \infty$, in the sense that for $n \neq 0$

$$\text{for all } \ell, \text{ as } y \to +\infty,\ y^\ell \cdot c_n(y) \text{ is bounded (with } n \neq 0)$$

The **truncation operators** Λ^T for large positive real T act on an automorphic form f by killing off f's constant term for large y. Thus, for a right K-invariant function, in $z = x + iy$ coordinates on $\mathbf{H} \approx G/K$,

$$(\text{naive } T\text{-truncation of } f)(x+iy) = \begin{cases} f(x+iy) & \text{for } y \leq T \\ f(x+iy) - c_P f(y) & \text{for } y > T \end{cases}$$

But this is not quite right. On a fundamental domain $\mathcal{F}$ this definition is acceptable, but it fails to correctly describe the truncated function on the whole domain $\mathbf{H}$ or whole group G, in the sense that the truncation is not properly described as an *automorphic form.* We want these truncation operators to yield automorphic forms, so for sufficiently large T (actually, $T \geq \sqrt{3}/2$ suffices) we can achieve the same effect by first defining the **tail** $c_P^T f$ of the constant term $c_P f$ of f by

$$c_P^T f(y) = \begin{cases} 0 & (\text{if } y < T) \\ C_P f(y) & (\text{if } y \geq T) \end{cases}$$

and then defining the **truncation operator** Λ^T

$$\Lambda^T f = f - E(c_P^T f)$$

The point is that

$$(\text{on the set } \mathcal{F}) \quad E(c_P^T f) = c_P^T f$$

since the support of any tail $c_P^T f$ is contained in the tail $\mathcal{F}^T = \{z = x + iy : y \geq T\}$ of the fundamental domain $\mathcal{F}$, and the proper translates of $\mathcal{F}^T$ by $\gamma \in \Gamma$ do not meet $\mathcal{F}$. In particular, for $z = x + iy$ in a fixed fundamental domain, for T large, the higher Fourier components are unaffected by truncation. One critical feature of the trunction procedure is the fact proven in the following proposition.

Proposition: The truncated Eisenstein series $\Lambda^T E_s$ is of rapid decay in all Siegel sets.

Proof: From the theory of the constant term, $f - c_P f$ is of rapid decay in a standard Siegel set. Thus, $f - c_P^T f$ is of rapid decay in a standard Siegel set. Thus, since on a fixed standard Siegel set $E(c_P^T E_s) = c_P^T E_s$ for T large enough, $\Lambda^T E_s$ is of rapid decay on a fixed standard Siegel set. Since $\Lambda^T E_s$ is an automorphic form, this implies rapid decay on all Siegel sets. ///

Let $\zeta(s)$ be the usual Euler-Riemann zeta function, and let $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ be the zeta function with its gamma factor attached. It is standard and relatively elementary that

$$c_P E_s = y^s + \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}$$

Abbreviate $c(s) = c_s = \xi(2s-1)/\xi(2s)$.

Remark: Whether or not one finds the following theorem interesting in its own right, the corollaries just below and analogous corollaries of the analogous theorem in a more general setting compel attention to the Maaß-Selberg relations.

Theorem: *(Maaß-Selberg relation)*

$$\langle \Lambda^T E_s, \Lambda^T E_r\rangle = \frac{T^{s+\bar r-1}}{s+\bar r-1} + c_s \frac{T^{(1-s)+\bar r-1}}{(1-s)+\bar r-1} + c_{\bar r}\frac{T^{s+(1-\bar r)-1}}{s+(1-\bar r)-1} + c_s c_{\bar r}\frac{T^{(1-s)+(1-\bar r)-1}}{(1-s)+(1-\bar r)-1}$$

Proof: In this the simplest possible case, the proof is is essentially a direct computation. First,

$$\langle \Lambda^T E_s, \Lambda^T E_r\rangle = \langle \Lambda^T E_s, E_r\rangle$$

because the tail of the constant term of E_r is orthogonal to the truncated version $\Lambda^T E_s$ of E_s. Then

$$\langle \Lambda^T E_s, \Lambda^T E_r\rangle = \langle \Lambda^T E_s, E_r\rangle = \langle E_s - E((y^r + c_r y^{1-r})^T), E_r\rangle = \langle E(\begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases}), E_r\rangle$$

The usual unwinding trick applied to the awkward Eisenstein series in the first argument of $\langle , \rangle$ transforms the last expression into

$$\int_{N_{\mathbf{Z}}\backslash \mathbf{H}} E(\begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases})\, c_P(E_{\bar r})\,\frac{dx\,dy}{y^2} = \int_0^\infty \begin{cases} -c_s y^{1-s} & (y \ge T) \\ y^s & (y < T)\end{cases} \cdot (y^{\bar r} + c_{\bar r} y^{1-\bar r}) \cdot \frac{1}{y}\,\frac{dy}{y}$$

$$= \int_0^T y^s \cdot (y^{\bar r} + c_{\bar r} y^{1-\bar r}) \cdot \frac{1}{y}\,\frac{dy}{y} - \int_T^\infty c_s y^{1-s}\,(y^{\bar r} + c_{\bar r} y^{1-\bar r}) \cdot \frac{1}{y}\,\frac{dy}{y}$$

Now we assume that $\mathrm{Re}\,(r)$ is bounded above and below (so that $\mathrm{Re}\,(1-r)$ is also bounded), and take $\mathrm{Re}\,(s)$ sufficiently large so that all the integrals converge. The above becomes

$$= \int_0^T y^{s+\bar r-1}\,\frac{dy}{y} + c_{\bar r}\int_0^T y^{s+(1-\bar r)-1}\,\frac{dy}{y} - c_s\int_T^\infty y^{(1-s)+\bar r-1}\,\frac{dy}{y} - c_s c_{\bar r}\int_T^\infty y^{(1-s)+(1-\bar r)-1}\,\frac{dy}{y}$$

which gives the expression of the theorem. By analytic continuation (in s and in $\bar r$) it is valid everywhere it makes sense. ///

Remark: The following corollaries certainly can be proven directly by use of explicit details such as the Fourier expansion of the Eisenstein series. However, the less elementary arguments here are part of an approach that generalizes.

Corollary: The only poles of E_s in the region $\mathrm{Re}\,(s) \ge \frac{1}{2}$ are on the segment $(\frac{1}{2}, 1]$. Any poles are *simple*. Any residues are square-integrable on $\Gamma\backslash\mathbf{H}$. Specifically,

$$\langle \mathrm{Res}_{\sigma_o} E_s, \mathrm{Res}_{\sigma_o} E_s\rangle = \mathrm{Res}_{\sigma_o} c_{\bar s}$$

Proof: We will only use the special case $r = s = \sigma + it$ of the theorem. In that case the Maaß-Selberg relation becomes

$$\langle E_s^T, E_s^T\rangle = \frac{T^{2\sigma-1}}{2\sigma-1} + c_s\frac{T^{-2it}}{-2it} + c_{\bar s}\frac{T^{2it}}{2it} + c_s c_{\bar s}\frac{T^{1-2\sigma}}{1-2\sigma}$$

Suppose that $s_o = \sigma_o + it_o$ were a pole of E_s of order ℓ with $t_o \ne 0$ and $\sigma_o > \frac{1}{2}$. From the theory of the constant term applied to this case, this is equivalent to the assertion that $c(s)$ has a pole at s_o of order ℓ. Also,

$$c(\bar s) = \overline{c(s)}$$

so $c(s)$ has a pole at $\overline{s_o}$ as well, of the same order as that at s_o, with leading Laurent term the complex conjugate of that at s_o. Thus, the function $s \to E_s^T$ also has a pole (as a meromorphic $L^2(\Gamma\backslash G/K)$-valued function) exactly at poles of c_s, of the same order, and so on. (Truncation alters neither the location nor the order of the poles.)

Take $s = \sigma_o + it$ in the above. In the real variable t, the left-hand side is asymptotic to a *positive* constant multiple of $(t - t_o)^{-2\ell}$ as $t \to t_o$, since the pole is of order ℓ and inner products are positive. The first term on the right-hand side is bounded as $t \to t_o$, and the second and third terms are asymptotic to non-zero constant multiples of $(t - t_o)^{-\ell}$. Thus, the first three terms on the right can be ignored as $t \to t_o$. The fourth term on the right-hand side is asymptotic to a positive constant multiple of $(t - t_o)^{-2\ell}$ from $c_s c_{\bar{s}}$, multiplied by $T^{1-2\sigma_o}/(1 - 2\sigma_o)$. Note that the denominator is *negative*, so that, altogether, the fourth term on the right-hand side is asymptotic to a *negative* constant multiple of $(t - t_o)^{-2\ell}$. The positivity of the left-hand side and negativity of the right-hand side (as $t \to t_o$) give a contradiction to the hypothesized pole. Thus, no poles can occur off the real axis in the region $\mathrm{Re}(s) > 1/2$.

Next, let $s_o = \sigma_o$ be a pole on $(\frac{1}{2}, 1]$. (We have convergence of the Eisenstein series for $\mathrm{Re}(s) > 1$, so no poles can occur in that region.) Take $r = s = \sigma_o + it$, obtaining

$$\langle E_s^T, E_s^T \rangle = \frac{T^{2\sigma_o - 1}}{2\sigma_o - 1} + c_s \frac{T^{-2it}}{-2it} + c_{\bar{s}} \frac{T^{2it}}{2it} + c_s c_{\bar{s}} \frac{T^{1-2\sigma_o}}{1 - 2\sigma_o}$$

Unlike the case where the pole is off the real axis, in which case $t \to t_o \neq 0$, here $t \to 0$. Thus, the second and third terms on the right-hand side blow up with order $\ell + 1$ (not merely ℓ, as in the previous case). Thus, the same argument as just above gives a contradiction *unless* $\ell = 1$, in which case the second and third terms' blow-up is of the same order as the left-hand side and the fourth term on the right-hand side. This proves that any pole on $(\frac{1}{2}, 1]$ is simple.

For a simple pole $\sigma_o \in (\frac{1}{2}, 1]$, let $s = \sigma_o + it$. Multiply the Maaß-Selberg relation through by t^2

$$t^2 \cdot \langle E_s^T, E_s^T \rangle = t^2 \cdot \frac{T^{2\sigma_o - 1}}{2\sigma_o - 1} + t^2 \cdot c_s \frac{T^{-2it}}{-2it} + t^2 \cdot c_{\bar{s}} \frac{T^{2it}}{2it} + t^2 \cdot c_s c_{\bar{s}} \frac{T^{1-2\sigma_o}}{1 - 2\sigma_o}$$

Take the limit of this as $t \to 0$. The first term on the right-hand side goes to 0, and everything else compute by residues, the middle two terms giving the same thing, yielding

$$\langle \mathrm{Res}_{\sigma_o} E_s^T, \mathrm{Res}_{\sigma_o} E_s^T \rangle = \mathrm{Res}_{\sigma_o} c_{\bar{s}} + \mathrm{Res}_{\sigma_o} c_s \, \overline{\mathrm{Res}_{\sigma_o} c_s} \, \frac{T^{1-2\sigma_o}}{1 - 2\sigma_o}$$

From this some elementary considerations give the square-integrability of the residue of the Eisenstein series. (General considerations about meromorphic vector-valued functions assure that taking residues commutes with taking the limit as $T \to \infty$.) Further, as $T \to +\infty$, since $1 - 2\sigma_o < 0$ we obtain

$$\langle \mathrm{Res}_{\sigma_o} E_s, \mathrm{Res}_{\sigma_o} E_s \rangle = \mathrm{Res}_{\sigma_o} c_{\bar{s}}$$

which gives the computation of the L^2-norm as desired. ///

Remark: In fact, in this simplest example, by properties of the zeta function we know that the only pole of $c(s)$ on $(\frac{1}{2}, 1]$ is at $s = 1$, and that the residue is a constant.

Corollary: The volume V of the fundamental domain for $SL(2, \mathbf{Z})$ acting on the upper half-plane $\mathbf{H}$ has inverse

$$V^{-1} = \mathrm{Res}_{s=1}\, c(s) = \mathrm{Res}_{s=1} \frac{\xi(2s-1)}{\xi(2s)} = \frac{1/2}{\pi^{-s/s}\Gamma(2/2)\,\zeta(2)} = \frac{3}{\pi}$$

Thus, the volume is

$$V = \frac{1}{\mathrm{Res}_{s=1}\, c(s)} = \pi/3$$

Proof: Use the auxiliary fact that the residue of E_s at $s = 1$ is a constant function, which is necessarily $\rho = \mathrm{Res}_{s=1}\, c(s)$. By the Maaß-Selberg relation the L^2 norm squared of this constant is $\rho = \mathrm{Res}_{s=1} c(s)$. Thus,

$$\rho = \rho^2 \cdot \text{ volume of } \mathcal{F}$$

so the volume is $1/\rho$, computed explicitly via standard facts about $\zeta(s)$. ///
