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From moments to pointwise estimates

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[0.0.1] Theorem: For a degree n L-function $L(s)$ with suitable analytic continuation and functional equation, a second-moment bound

$$\int_0^T |L(\tfrac{1}{2}+it)|^2\,dt \;\ll\; T^A \qquad\qquad (\text{for all } T \gg 1)$$

implies a pointwise bound

$$L(\sigma_o+it) \;\ll_{\sigma_o,\varepsilon}\; (1+|t|)^{\frac{A}{2}+\varepsilon} \qquad\qquad (\text{for all } \sigma_o \ge \tfrac{1}{2} \text{ and for all } \varepsilon > 0)$$

[0.0.2] Remark: The convexity bound has exponent $\frac{n}{4}$. Thus, a moment bound with $A < \frac{n}{2}$ would produce a subconvex bound.

Proof: Fix $\sigma_o > \frac{1}{2}$. For $0 < t_o \in \mathbb{R}$, let $s_o = \sigma_o + it_o$. Let R be the rectangle in $\mathbb{C}$ with vertices $\frac{1}{2} \pm iT$ and $2 \pm iT$, for $T > t_0$. By Cauchy's theorem,

$$L(s_o)^2 \;=\; \frac{1}{2\pi i}\int_R \frac{e^{(s-s_o)^2}}{s-s_o}\cdot L(s)^2\,ds$$

Since the L-function has polynomial vertical growth, we can push the top and bottom of R to infinity, giving

$$L(s_o)^2 \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{((\frac{1}{2}-\sigma_o)+i(t-t_o))^2}}{(\frac{1}{2}-\sigma_o)+i(t-t_o)}\cdot L(\tfrac{1}{2}+it)^2\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{((2-\sigma_o)+i(t-t_o))^2}}{(2-\sigma_o)+i(t-t_o)}\cdot L(2+it)^2\,dt$$

$$=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{((\frac{1}{2}-\sigma_o)+i(t-t_o))^2}}{(\frac{1}{2}-\sigma_o)+i(t-t_o)}\cdot L(\tfrac{1}{2}+it)^2\,dt \;+\; O(1) \qquad (\text{for } \tfrac{1}{2} < \sigma_o < 1)$$

The part of the integral where $|t-t_o| \ge t_o$ is visibly $\ll_{n,\sigma_o} e^{-t_o}$:

$$|e^{((\frac{1}{2}-\sigma_o)+i(t-t_o))^2}| \;=\; e^{(\frac{1}{2}-\sigma_o)^2-(t-t_o)^2} \;\ll_{\sigma_o}\; e^{-t_o^2/2}e^{-(t-t_o)^2/2} \;\ll\; e^{-t_o} \qquad (\text{for } |t-t_o| \ge t_o)$$

Squaring the convexity bound for $L(\frac{1}{2}+it)$ gives

$$L(\tfrac{1}{2}+it)^2 \;\ll\; |t|^{\frac{n}{2}+\varepsilon} \qquad\qquad (\text{for all } \varepsilon > 0)$$

Thus,

$$\int_{2t_o}^{\infty} \frac{e^{(\frac{1}{2}-\sigma_o)+i(t-t_o))^2}}{(\frac{1}{2}-\sigma_o)+i(t-t_o)}\cdot L(\tfrac{1}{2}+it)^2\,dt \;\ll_{\sigma_o}\; e^{-t_o^2/2}\int_{2t_o}^{\infty} e^{-(t-t_o)^2/2}\,t^{\frac{n}{2}+\varepsilon}\,dt \;\ll_{n,\varepsilon}\; e^{-t_o}$$

The other half of the tail, where $t < 0$, is estimated similarly, and more easily. For $0 < t < 2t_o$, use the moment estimate, and the trivial estimate

$$\frac{e^{((\frac{1}{2}-\sigma_o)+i(t-t_o))^2}}{(\frac{1}{2}-\sigma_o)+i(t-t_o)} \;\ll_{\sigma_o}\; e^{(\frac{1}{2}-\sigma_o)^2-(t-t_o)^2} \;\ll_{\sigma_o}\; 1$$

Then

$$\int_0^{2t_o} \frac{e^{(\frac{1}{2}-\sigma_o)+i(t-t_o))^2}}{(\frac{1}{2}-\sigma_o)+i(t-t_o)}\cdot L(\tfrac{1}{2}+it)^2\,dt \;\ll_{\sigma_o}\; \int_0^{2t_o} |L(\tfrac{1}{2}+it)|^2\,dt \;\ll\; t_o^A$$

Thus,

$$L(s_o)^2 \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(\frac{1}{2}+it-s_o)^2}}{\frac{1}{2}+it-s_o}\cdot L(\tfrac{1}{2}+it,f)^2\,dt \;+\; O(1) \;\ll_{n,\sigma_o}\; t_o^A$$

Then a standard convexity argument gives the asserted $|t_o|^{\frac{A}{2}+\varepsilon}$ on $\sigma_o = \frac{1}{2}$, for every $\varepsilon > 0$. ///
