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Simple non-unitarizability estimate for principal series

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The point here is to gives a trivial but useful estimate on the parameter range in which unramified principal series representations of p-adic reductive groups are unitarizable. For simplicity, and for intelligibility to beginners, we consider only $GL(n)$, although the method obviously generalizes.

Remark: Some time after this argument occurred to me, the footnote at the end of [Harris 1981] caused me to realize that using boundedness of spherical functions (and similar vectors in principal series) goes back at least to [MacDonald 1971]. In particular, difficult (but still sub-Ramanujan) estimates on Hecke operators used in sources such as [Andrianov 1979] can be replaced by invocation of more natural, standard boundedness properties of unitarizable (irreducible) principal series.

[Andrianov 1979] A.N. Andrianov, *The multiplicative arithmetic of Siegel modular forms*, Russian Math Surveys **34** (1979), pp. 75-148.

[Harris 1981] M. Harris, *The rationality of holomorphic Eisenstein series*, Inv. Math. **63** (1981), pp. 305-310.

[MacDonald 1971] I.G. MacDonald, *Spherical Functions on a Group of p-adic Type*, Ramanujan Institute, Univ. of Madras Publications, 1971.

Let $G = GL(n, F)$ where F is a p-adic local field. Let P be the minimal parabolic subgroup of G consisting of upper-triangular matrices. Let N be the unipotent radical of P, consisting of upper-triangular unipotent matrices in G. Let K be the standard maximal compact subgroup of G consisting of matrices with entries in the local integers $\mathbf{o}$ of F, and with determinant which is in $\mathbf{o}^\times$. As usual, a $\mathbf{C}$-valued function f on G is said to be K-finite if the collection of functions $g \to f(gk)$ for $k \in K$ is finite-dimensional.

For an n-tuple $s = (s_1, \ldots, s_n)$ define a character (one-dimensional representation) χ_s on P/N by

$$\chi_s(\mathbf{m}) = |m_1|^{s_1} \ldots |m_n|^{s_n}$$

Let π_s be the (non-normalized) unramified principal series representation of G, consisting of $\mathbf{C}$-valued functions on G which are right K-finite and have the left P-equivariance property

$$f(p \cdot g) = \chi_s(p) \cdot f(g)$$

for $p \in P$ and $g \in G$. As usual, let $\rho \in \mathbf{C}^n$ be the n-tuple

$$\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \frac{n-5}{2}, \ldots, \frac{5-n}{2}, \frac{3-n}{2}, \frac{1-n}{2})$$

Thus, the modular function δ_P of P is given in this notation as

$$\delta(p) = \chi_{2\rho}(p)$$

A representation π of G is *unitarizable* if there exists a positive-definite hermitian inner product $\langle , \rangle$ on π with the G-invariance property

$$\langle gu, gv \rangle = \langle u, v \rangle$$

for $g \in G$, $u, v \in \pi$.

Theorem: If the non-normalized unramified principal series π_s is unitarizable then $\mathrm{Re}(s_1 + \ldots + s_n) = 0$ and

$$0 \le \mathrm{Re}(s_1 + s_2 + \ldots + s_i) \le 2(\rho_1 + \ldots + \rho_i)$$

for every index $i = 1, \ldots, n-2, n-1$. That is, if these conditions are *not* met then π_s is surely *not* unitarizable.

Remark: Note that if we use normalized rather than non-normalized principal series, which amounts to replacing each s_i by $s_i + \rho_i$, then the conditions become the more symmetrical

$$-(\rho_1 + \ldots + \rho_i) \leq \mathrm{Re}(s_1 + s_2 + \ldots + s_i) \leq \rho_1 + \ldots + \rho_i$$

and still (because $\sum_i \rho_i = 0$)

$$\mathrm{Re}(s_1 + \ldots + s_n) = 0$$

Proof: The trick used in the argument is the boundedness of (matrix) coefficient functions of unitary representations. That is, for a unitary representation τ, with pairing $\langle , \rangle$ of τ with $\bar{\tau}$, with $u, v \in \tau$ and $g \in G$, by Cauchy-Schwartz,

$$|\langle gu, v\rangle| \leq |gu| \cdot |v| = |u| \cdot |v|$$

which is independent of $g \in G$.

The first condition of the theorem comes from consideration of the central character. Namely, scalar matrices with diagonal entries $\omega \in F^\times$ act on π_s by the scalar

$$|\omega|^{s_1 + \ldots + s_n}$$

For this to be bounded as ω ranges over $F^\times$ it is necessary (and sufficient) that the sum of the real parts of the s_i's be 0. The second condition is somewhat more complicated to derive.

The quotient $P\backslash G$ does not have a right G-invariant measure, because the modular function of G (namely, the trivial character) restricted to P is different from the modular function of P. In other words, there is no right-G-invariant $\mathbf{C}$-linear functional on the non-normalized principal series π_0. Nevertheless, there *is* a right G-invariant $\mathbf{C}$-linear functional Λ on the non-normalized principal series $\pi_{2\rho}$, given indirectly as follows. For $\varphi \in C_c^\infty(G)$ and $s \in \mathbf{C}^n$, define an *averaging map* α_s by

$$\alpha_s \varphi(g) = \int_P \chi_s(p)^{-1} \, \varphi(p \cdot g) \, dp$$

Then Λ is defined by

$$\Lambda(\alpha_{2\rho}\varphi) = \int_K \alpha_{2\rho}\varphi(k) \, dk = \int_G \varphi(g) \, dg$$

where we have invoked the Iwasawa decomposition $G = P \cdot K$. Thus, there is the usual natural pairing

$$\langle , \rangle : \pi_s \times \pi_{2\rho - s} \to \mathbf{C}$$

with the G-invariance property

$$\langle gu, gv \rangle = \langle u, v \rangle$$

given by

$$\langle u, v \rangle = \Lambda(uv) = \int_K u(k) \cdot v(k) \, dk$$

Ignoring the positive-definiteness, the assumed unitarizability implies the existence of a non-degenerate G-invariant hermitian form on π_s. That is, the hypothesis gives a non-degenerate G-invariant $\mathbf{C}$-bilinear pairing

$$\pi_s \times \overline{\pi_s} \to \mathbf{C}$$

which yields an isomorphism

$$\overline{\pi_s} \approx \pi_s\check{} \approx \pi_{2\rho - s}$$

Thus, to compute values of the pairing $\pi_s \times \overline{\pi_s}$ it suffices to compute values of the pairing $\pi_s \times \pi_{2\rho - s}$, although we definitely do not attempt to make the isomorphism $\overline{\pi_s} \approx \pi_{2\rho - s}$ explicit. Thus, boundedness of the coefficient functions for $\pi_s \times \overline{\pi_s}$ is equivalent to boundedness of the coefficient functions for $\pi_s \times \pi_{2\rho - s}$.

Consider a special case

$$u = \alpha_s \,\mathrm{ch}_B \qquad v = \alpha_{2\rho - s}\,\mathrm{ch}_B$$

where ch_B is the characteristic function of the Iwahori subgroup $B \subset K$ consisting of matrices in the maximal compact $K = GL(n, \mathbf{o})$ whose sub-diagonal entries are in the maximal ideal of the local integers $\mathbf{o}$. Let

$$a_t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

and further assume that $|t_i/t_{i+1}| \le 1$ for all indices i. The point of the latter hypothesis is that then

$$a_t^{-1} \cdot B \cdot a_t \subset N \cdot B$$

Then

$$\langle a_t \cdot u, v\rangle = \int_K \alpha_s \mathrm{ch}_B(k a_t)\, \alpha_{2\rho-s}\mathrm{ch}_B(k)\, dk = \int_B \alpha_s \mathrm{ch}_B(b a_t)\, db$$

$$= \int_B \alpha_s \mathrm{ch}_B(a_t\, a_t^{-1} b a_t)\, db = \chi_s(a_t) \int_B \alpha_s \mathrm{ch}_B(a_t^{-1} b a_t)\, db = \chi_s(a_t) \int_B 1\, db$$

since $a_t^{-1} b a_t \in N \cdot B$ for a_t with all $|t_i/t_{i+1}| \le 1$. For $\chi_s(a_t)$ to be bounded for all a_t with $|t_i/t_{i+1}| \le 1$ it must be that $\mathrm{Re}\,(s_1 + \ldots + s_i) \ge 0$, which is the lower bound claimed.

For the upper bound, consider again

$$u = \alpha_s \,\mathrm{ch}_B \qquad v = \alpha_{2\rho - s}\,\mathrm{ch}_B$$

and

$$\langle a_t \cdot u, v\rangle$$

but now impose the conditions $|t_i/t_{i+1}| > 1$ for all indices i. Then

$$\langle a_t \cdot u, v\rangle = \int_K \alpha_s \mathrm{ch}_B(k a_t)\, \alpha_{2\rho-s}\mathrm{ch}_B(k)\, dk = \int_B \alpha_s \mathrm{ch}_B(b a_t)\, db$$

$$= \int_B \alpha_s \mathrm{ch}_B(a_t\, a_t^{-1} b a_t)\, db = \chi_s(a_t) \int_B \alpha_s \mathrm{ch}_B(a_t^{-1} b a_t)\, db = \chi_s(a_t) \cdot \mathrm{meas}\,(X_t)$$

where X_t is the subset of B consisting of elements b so that

$$a_t^{-1} b a_t \in N \cdot B$$

noting that (because a_t is diagonal)

$$B \cap a_t\,(P \cdot B)\, a_t^{-1} = B \cap a_t\,(N \cdot B)\, a_t^{-1}$$

Then the conditions $|t_i/t_{i+1}| > 1$ assure that

$$X_t = \{ b \in B : |\frac{t_i}{t_j} \cdot b_{ij}| < 1 \text{ for all } i > j\}$$

where b_{ij} is the ij^{th} entry of b), since the diagonal entries of $a_t^{-1} b a_t$ remain local units. Thus,

$$\mathrm{meas}\, X_t = \prod_{i>j} |\frac{t_i}{t_j}|^{-1} = \chi_{-2\rho}(a_t)$$

Thus, the boundedness condition becomes the condition that $\chi_{s-2\rho}(a_t)$ be bounded for $|t_i/t_{i+1}| > 1$, which implies that

$$\mathrm{Re}\,(s_1 + \ldots + s_i) \le 2(\rho_1 + \ldots + \rho_i)$$

as claimed. ///