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Simple Proof of the Prime Number Theorem, etc.

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The point here is the relatively simple argument that non-vanishing of an L-function on the line $\mathrm{Re}(s) = 1$ implies an asymptotic result parallel to the application of $\zeta(s)$ to the Prime Number Theorem. This is based upon [Newman 1980]. In particular, this argument avoids estimates on the zeta function at infinity and also avoids Tauberian arguments.

For completeness, we recall the standard clever *ad hoc* argument for the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$, thus giving a complete proof of the Prime Number Theorem here.

However, the larger intent is to prove non-vanishing results for L-functions by capturing the L-functions in constant terms of Eisenstein series (after Langlands and Shahidi), and then apply the present argument to obtain the most immediate asymptotic corollary.

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1. Non-vanishing of L-functions on $\mathrm{Re}(s) = 1$

As the simplest example, the Riemann zeta function

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

does not vanish on the line $\mathrm{Re}(s) = 1$. This is not obvious! (The usual simple but *ad hoc* proof is given just below, for completeness.) As a consequence, using the Euler product expansion over primes, its logarithmic derivative

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{d}{ds} \log(1 - p^{-s}) = -\sum_p \frac{\log p}{p^s} - \sum_p \sum_{m \geq 2} \frac{\log p}{p^{ms}}$$

is holomorphic (except for the pole at $s = 1$) on an open set containing $\mathrm{Re}(s) \geq 1$. From this we prove (below) the *Prime Number Theorem*

$$\lim_{x \to \infty} \frac{\text{number of primes} \leq x}{x/\log x} = 1$$

or, at it is usually written,

$$\pi(x) \sim \frac{x}{\log x}$$

As is well known, a form of this was conjectured by Gauss, and the theorem was proven independently by Hadamard and by de la Valleé Poussin.

The methodology below is perhaps the clearest proof of the Prime Number Theorem, using simplifications found by D.J. Newman about 1980. However, the simplified form does not give any indication of the relation between zero-free regions and the error term in the Prime Number Theorem.

In general, non-vanishing of an L-function (with Euler product) on a vertical line implies an analogous asymptotic result.

The following proof is standard, but is objectionable for its *ad hoc* nature.

Proposition: The zeta function $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$.

Proof: The trick is to note that for arbitrary real θ

$$3 + 4\cos\theta + \cos 2\theta \geq 0$$

This follows from $\cos 2\theta = 2\cos^2\theta - 1$ and the fact that then

$$3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\,\cos^2\theta = 2\,(1 + \cos\,\theta)^2 \geq 0$$

Suppose that $\zeta(1 + it) = 0$, and consider

$$D(s) = \zeta(s)^3 \cdot \zeta(s + it)^4 \cdot \zeta(s + 2it)$$

At $s = 1$ the pole of $\zeta(s)$ at $s = 1$ would cancel some of the alleged vanishing of $\zeta(s + it)$ at $s = 1$, and $1 + 2it$ may be a 0 of $\zeta(s)$, but is certainly not a pole. Thus, in fact, if we can prove that $D(s)$ does not have a zero at $s = 1$, then we will have proven that $\zeta(s)$ has no zero on the line $\mathrm{Re}(s) = 1$.

For $\mathrm{Re}(s) > 1$, taking the logarithmic derivative of $D(s)$ gives

$$\frac{d}{ds} \log D(s) = -\sum_p \sum_{m\geq 1} \frac{(3 + 4p^{-mit} + p^{-2mit}) \log p}{p^{ms}}$$

The limit of this multiplied by $(s - 1)$, as $s \to 1$ from the right (on the real axis), is the order of vanishing of $D(s)$ at $s = 1$, including as usual poles as negative ordersof vanishing. The real part of $3 + 4p^{-mit} + p^{-2mit}$ is non-negative, as noted above. Thus, as $s \to 1$ along the real axis from the right, the real part of the latter expression is non-positive (due to the leading minus sign). In particular, this limit cannot be a positive integer. Thus, $D(s)$ does not have a genuine zero at $s = 1$. As noted, this implies that $\zeta(1 + it) \neq 0$. ///

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2. Convergence theorems

The two theorems in this section are two simple special cases of a general result. The first version has obvious relevance to Dirichlet series, but in fact the second version is what we will use to prove the Prime Number Theorem. A unified proof is given.

Theorem: *(Version 1)* Suppose that c_n is a bounded sequence of complex numbers. Define

$$D(s) = \sum_n \frac{c_n}{n^s}$$

Suppose that $D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 1$. Then the sum $\sum_n \frac{c_n}{n^s}$ converges for $\mathrm{Re}(s) \geq 1$.

Theorem: *(Version 2)* Suppose that $S(t)$ is a bounded locally integrable complex-valued function.

$$f(s) = \int_0^\infty S(t)\, e^{-st}\, dt$$

Suppose that $f(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 0$. Then the integral $\int_0^\infty S(t)\, e^{-st}\, dt$ converges for $\mathrm{Re}(s) \geq 0$ and equals $f(s)$.

Proof: The boundedness of the sequence of constants c_n assures that the sum $D(z) = \sum_n \frac{c_n}{n^z}$ is holomorphic for $\mathrm{Re}(z) > 1$. In this case define $f(z) = D(z+1)$. Thus, in either case we have a function $f(s)$ holomorphic on an open set containing $\mathrm{Re}(z) \geq 0$.

Let $R \geq 1$ be large. Depending on R, choose $0 < \delta > 1/2$ so that $f(z)$ is holomorphic on the region $\mathrm{Re}(z) \geq -\delta$ and $|z| \leq R$, and let $M \geq 0$ be a bound for it on that (compact) region.

Let γ be the (counter-clockwise) path bounded by the arc $|z| = R$ and $\mathrm{Re}(z) \geq -\delta$, and by the straight line $\mathrm{Re}(z) = -\delta$, $|z| \leq R$. Let A be the part of γ in the right half-plane and let B be the part of γ in the left half-plane.

By residues

$$2\pi i f(0) = \int_\gamma f(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

Indeed, the integral of $f(z)$ against the $N^z z/R^2$ term is simply 0 (by Cauchy's theorem), since $f(z) \cdot N^z z/R^2$ is holomorphic on a suitable region. On the other hand, the integral of $f(z)N^z$ against $1/z$ is $2\pi i$ times the value of $f(z)N^z$ at $z = 0$, which is $f(0)$.

The N^{th} partial sum or truncated integral (respectively)

$$S_N(z) = \sum_{n<N} \frac{c_n}{n^z}$$

$$S_N(z) = \int_0^N S(t)\, e^{-zt}\, dt$$

of $f(z)$ is an entire function of z, so we can express $S_N(0)$ as an integral over the whole circle of radius R centered at 0, rather than having to use the path along $\mathrm{Re}(z) = -\delta$ as for $f(z)$, namely

$$2\pi i S_N(0) = \int_{A \cup -A} S_N(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

where $-A$ denotes the left half of the circle of radius R. Breaking the integral into A and $-A$ pieces and replacing z by $-z$ in the $-A$ integral gives

$$\int_A S_N(z) N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz = 2\pi i S_N(0) - \int_A S_N(-z) N^{-z} \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

On the arc A, $f(z)$ is equal to its defining series, which we split into the N^{th} partial sum $S_N(z)$ and the corresponding N^{th} tail $T_N(z)$. Therefore, the N-tail $T_N(0) = f(0) - S_N(0)$ of the series/integral for $f(0)$ has an expression

$$2\pi i(f(0) - S_N(0)) = \int_A \left(T_N(z)N^z - S_N(-z)N^{-z}\right) \left(\frac{1}{z} + \frac{z}{R^2}\right) dz + \int_B f(z) N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

Essentially elementary estimates will now show that this goes to 0 as N becomes large.

We carry out these estimates in some detail. Use $a \ll b$ to mean $a = O(b)$, and let $x = \mathrm{Re}(z)$. We'll need some obvious and elementary inequalities:

$\frac{1}{z} + \frac{z}{R^2}$	$=$	$\frac{2x}{R^2}$	on $\lvert z\rvert = R$
$\frac{1}{z} + \frac{z}{R^2}$	$\ll$	$\frac{2}{\delta}$	along B, for fixed R, for δ sufficiently small
$T_N(z)$	$\ll$	$\int_N^\infty \frac{dn}{n^{x+1}} = \frac{1}{xN^x}$	
$S_N(-z)$	$\ll$	$\int_0^N n^{x-1}\, dn = N^x \left(\frac{1}{N} + \frac{1}{x}\right)$	

On the contour A

$$T_N(z) \cdot N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) \ll \frac{1}{xN^x} \cdot \frac{2x}{R^2} \ll \frac{1}{R^2}$$

and also on A

$$S_N(z) \cdot N^{-z} \left(\frac{1}{z} + \frac{z}{R^2} \right) \ll N^x \left(\frac{1}{N} + \frac{1}{x} \right) \cdot \frac{2x}{R^2} \ll \frac{1}{R^2} + \frac{1}{N\,R}$$

with constants independent of N, R, etc. Thus, estimating the integral over A by the sup of the absolute value of the integrand multiplied by the length of the path,

$$\int_A \left(T_N(z) N^z - S_N(-z) N^{-z} \right) \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \ll \frac{1}{R} + \frac{1}{N}$$

On the path B,

$$\int_B f(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \le \int_B M \cdot N^x \cdot \left(\frac{1}{|z|} + \frac{|z|^2}{R^2} \right) |dz| \le M \cdot \int_{-R}^{R} N^{-\delta} \cdot \frac{2}{\delta}\, dy + 2 \cdot M \cdot \int_{-\delta}^{0} N^x \cdot \frac{1}{R}\, dx$$

$$\le \frac{4M}{\delta N^\delta} + \frac{2M}{R \log N}$$

Thus, altogether,

$$f(0) - S_N(0) \ll \frac{1}{R} + \frac{1}{N} + \frac{RM}{\delta N^\delta} + \frac{M}{R \log N}$$

In this expression, for given positive ε take $R = 1/\varepsilon$, (with corresponding choice of δ, and then of bound M) obtaining

$$f(0) - S_N(0) \ll \varepsilon \cdot \left(1 + \frac{1}{\varepsilon N} + \frac{M}{\varepsilon \delta N^\delta} + \frac{M}{\log N} \right)$$

for all N. By now it is clear that for sufficiently large N the expression inside the parentheses is smaller than (for example) 2, proving that the sum/integral for $f(0)$ converges by proving that the partial sums/integral $S_N(0)$ converge to the value $f(0)$ of the holomorphic function f at 0. ///

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3. Corollary on asymptotics

This corollary of the convergence theorem is sufficient to prove the Prime Number Theorem. It is also used to prove a variant (below) in which the coefficients c_n are merely assumed to be bounded complex numbers and non-zero only for n prime.

Corollary: Let c_n be a sequence of non-negative real numbers. Define

$$D(s) = \sum_n \frac{c_n \cdot \log n}{n^s}$$

Suppose that

$$S(x) = \sum_{n \le x} c_n \cdot \log n$$

is $O(x)$, and that $(s-1)D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \ge 1$. That is, except for a possible simple pole at $s = 1$, $D(s)$ is holomorphic on $\mathrm{Re}(s) \ge 1$. Let ρ be the residue of $D(s)$ at $s = 1$. Then

$$\sum_{n \le x} c_n \cdot \log n \sim \rho\, x$$

Proof: Writing the sum as a Stieltjes integral and integrating by parts,

$$D(s) = \int_1^\infty t^{-s}\, dS(t) = s \cdot \int_1^\infty S(t)\, t^{-s-1}\, dt = s \cdot \int_0^\infty S(e^t)\, e^{-ts}\, dt$$

by replacing t by e^t. For $\mathrm{Re}\,(s) > 0$, from the definition we have

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; e^{-st}\, dt = \frac{f(s+1)}{s+1} - \frac{\rho}{s}$$

Note that $S(e^t)e^{-t}$ is bounded, and that the right-hand side is holomorphic on an open set containing $\mathrm{Re}\,(s) \geq 0$. Thus, the convergence theorem applies, and we conclude that

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; e^{-st}\, dt$$

is convergent for $\mathrm{Re}\,(s) \geq 0$. In particular, the integral for $s = 0$, namely

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; dt$$

is convergent. Changing variables back, replacing e^t by t, we conclude that

$$\int_1^\infty \frac{S(t) - \rho t}{t^2}\, dt$$

is convergent.

To complete the proof, note that $S(x)$ is positive real-valued and non-decreasing. Suppose now that there is $\varepsilon > 0$ so that there exist arbitrarily large x with $S(x) > (1+\varepsilon)\rho x$. Then

$$\int_x^{(1+\varepsilon)x} \frac{S(t) - \rho t}{t^2}\, dt \geq \int_x^{(1+\varepsilon)x} \frac{(1+\varepsilon)\rho x - \rho t}{t^2}\, dt = \rho \cdot \int_1^{1+\varepsilon} \frac{(1+\varepsilon) - t}{t^2}\, dt$$

by replacing t by tx, using the non-decreasing feature of $S(x)$. For $\rho \neq 0$, the latter expression is strictly positive and does not depend upon x, contradicting the convergence of the integral. Similarly, suppose that there is $\varepsilon > 0$ so that there exist arbitrarily large x with $S(x) < (1-\varepsilon)\rho x$. Then

$$\int_{(1-\varepsilon)x}^{x} \frac{S(t) - \rho t}{t^2}\, dt \leq \int_{(1-\varepsilon)x}^{x} \frac{(1-\varepsilon)\rho x - \rho t}{t^2}\, dt = \rho \cdot \int_{1-\varepsilon}^{1} \frac{(1-\varepsilon) - t}{t^2}\, dt$$

which is negative and independent of x, contradicting the convergence. ///

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4. Elementary lemma on asymptotics

The lemma here is elementary but used over and over, so deserves to be understood clearly apart from other issues.

Lemma: Let $f(x)$ be some function and suppose that

$$\sum_{p \leq x} f(p) \cdot \log p \sim rx$$

Then

$$\sum_{p \leq x} f(p) \sim \frac{rx}{\log x}$$

Proof: Let

$$\theta(x) = \sum_{p \le x} f(p) \cdot \log p$$

$$\varphi(x) = \sum_{p \le x} f(p)$$

Use a '*' to denote a sufficiently large but fixed lower limit of integration, whose precise nature is irrelevant to these asymptotic estimates. Integrating by parts

$$\varphi(x) \sim \int_*^x d\varphi(t) = \int_*^x \frac{1}{\log t} \cdot d\theta(t) = \left[\frac{1}{\log t}\theta(t)\right]_*^x + \int_*^x \theta(t)\, \frac{1}{t\log^2 t}\, dt$$

We can estimate the integral of $1/\log^2 t$ via

$$\int_*^x \frac{1}{\log^2 t}\, dt = \int_*^{\sqrt{x}} \frac{1}{\log^2 t}\, dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 t}\, dt$$

$$= \int_*^{\sqrt{x}} \frac{t}{t\log^2 t}\, dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 t}\, dt \le \sqrt{x} \cdot \int_*^{\sqrt{x}} \frac{1}{t\log^2 t}\, dt + \frac{1}{\log^2 \sqrt{x}} \cdot \int_{\sqrt{x}}^x 1\, dt$$

$$\sim \frac{2\sqrt{x}}{\log x} + \frac{4x}{\log^2 x} = o\left(\frac{x}{\log x}\right)$$

Thus,

$$\varphi(x) \sim \frac{rx}{\log x} - \int_*^x \theta(t)\, \frac{1}{t\log^2 t}\, dt \sim \frac{rx}{\log x}$$

This gives the asymptotics for $\varphi(x)$ as claimed. ///

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5. The Prime Number Theorem

This is the simplest example of application of the analytical results above. As always, $\pi(x)$ is the number of primes less than x. Using Chebycheff's traditional notation, let

$$\theta(x) = \sum_{p < x} \log p$$

where the notation is meant to imply that the sum is over primes less than x.

Theorem: *(Prime Number Theorem)*

$$\pi(x) \sim \frac{x}{\log x}$$

Proof: First, we'll use properties of $\zeta(s)$ and the convergence theorem's corollary to prove that

$$\theta(x) \sim x$$

Taking the logarithmic derivative of the zeta function gives

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{d}{ds} \log(1 - p^{-s}) = -\sum_p \frac{\log p}{p^s} - \sum_p \sum_{m \ge 2} \frac{\log p}{p^{ms}}$$

The second sum in the latter expression is readily estimated to give a holomorphic function in the region $\mathrm{Re}(s) > \frac{1}{2}$, so the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$ (and the simple pole with residue 1 at $s = 1$) implies that

$$f(s) = \sum_p \frac{\log p}{p^s}$$

has a simple pole with residue 1 at $s = 1$ and is otherwise holomorphic on $\mathrm{Re}(s) \geq 1$. This Dirichlet series has coefficients

$$c_n = \begin{cases} \log p & \text{(for } n = p \text{ prime)} \\ 0 & \text{(otherwise)} \end{cases}$$

The corollary of the convergence theorem immediately gives

$$\sum_{p \leq x} \log p \sim x$$

Then application of the lemma above gives the asymptotics on $\pi(x)$. ///

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6. Second corollary on asymptotics

Corollary: Let c_p be a bounded sequence of complex numbers indexed by primes p. Define

$$D(s) = \sum_p \frac{c_p \cdot \log p}{p^s}$$

Suppose that

$$S(x) = \sum_{p \leq x} c_p \cdot \log p$$

is $O(x)$, and that $(s-1)D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 1$. That is, except for a possible simple pole at $s = 1$, $D(s)$ is holomorphic on $\mathrm{Re}(s) \geq 1$. Let ρ be the residue of $D(s)$ at $s = 1$. Then

$$\sum_{p \leq x} c_p \log p \sim \rho\, x$$

Proof: First, consider the case that

$$S(x) = \sum_{p < x} c_p \cdot \log p$$

is real-valued (but not necessarily non-decreasing). Let C be a sufficiently large positive constant so that $C + c_p \geq 0$ for every prime index p. Then the first corollary applies to $S_1(x) = \sum_{p \leq x} (C + c_p) \cdot \log p$ and to the associated Dirichlet series

$$D_1(s) = \sum_p \frac{(C + c_p) \cdot \log p}{p^s} = C \cdot \sum_p \frac{\log p}{p^s} + D(s)$$

We already know that $(s-1) \sum_p \frac{\log p}{p^s}$ is holomorphic on $\mathrm{Re}(s) \geq 1$, has a simple pole with residue 1 at $s = 1$. And we have already proven the asymptotic assertion

$$\sum_{p \leq x} \log p \sim x$$

from the first corollary. Thus,

$$\sum_{p \leq x} (C + c_p) \cdot \log p \sim (C + \rho) \cdot x$$

from which we obtain

$$\sum_{p \le x} c_p \cdot \log p \sim \rho \cdot x$$

by subtracting the asymptotics for $\sum_p \frac{\log p}{p^s}$. This proves the corollary for real-valued bounded c_p. For complex-valued bounded c_p, simply break everything into real and imaginary parts. This proves the second corollary. ///

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7. A general asymptotic result

Let

$$L(s) = \prod_p \frac{1}{\det(1_n - p^{-s} \cdot \Phi_p)}$$

be a general Euler product expansion, where Φ_p is a semi-simple n-by-n complex matrix. Assume that *the eigenvalues of* Φ_p *are bounded* (as p varies over primes). The boundedness assures that this Dirichlet series converges for $\mathrm{Re}(s) > 1$. The non-vanishing of $L(s)$ on $\mathrm{Re}(s) = 1$ would in many cases be implied by the the behavior of a related *Eisenstein series* (in whose *constant term* the L-function appears). The proof is entirely parallel to the analogous proof of the Prime Number Theorem.

Theorem: Assume that $(s-1)L(s)$ is holomorphic and non-zero for $\mathrm{Re}(s) \ge 1$, and that $L(s)$ itself has a simple pole at $s = 1$ with residue ρ (possibly 0). Then

$$\sum_{p \le x} \log p \cdot \mathrm{tr}\, \Phi_p \sim \begin{cases} x & (\text{for } \rho \ne 0) \\ 0 & (\text{for } \rho = 0) \end{cases}$$

Proof: First suppose that $\rho \ne 0$. Let

$$\theta(x) = \sum_{p \le x} \log p \cdot \mathrm{tr}\, \Phi_p$$

First, use properties of $L(s)$ and the convergence theorem's corollary to prove that

$$\theta(x) \sim x$$

Taking the logarithmic derivative of $L(s)$ gives

$$\frac{d}{ds} \log L(s) = \frac{L'(s)}{L(s)} = \sum_p \frac{\mathrm{tr}\, \Phi_p \log p}{p^s} + \sum_p \sum_{m \ge 2} \frac{\mathrm{tr}\, \Phi_p \log p}{p^{ms}}$$

The second sum in the latter expression is a holomorphic function in the region $\mathrm{Re}(s) > \frac{1}{2}$, so the non-vanishing of $L(s)$ on $\mathrm{Re}(s) = 1$ (and the simple pole with residue $\rho \ne 0$ at $s = 1$) implies that the Dirichlet series

$$f(s) = \sum_p \frac{\mathrm{tr}\, \Phi_p \log p}{p^s}$$

has a simple pole with residue 1 at $s = 1$ and is otherwise holomorphic on $\mathrm{Re}(s) \ge 1$. This Dirichlet series has coefficients

$$c_n = \begin{cases} \mathrm{tr}\, \Phi_p \log p & (\text{for } n = p \text{ prime}) \\ 0 & (\text{otherwise}) \end{cases}$$

The corollary of the convergence theorem immediately gives

$$\theta(x) = \sum_{p \le x} \mathrm{tr}\, \Phi_p \, \log p \sim x$$

In the previous discussion, if $\rho = 0$, then the logarithmic derivative has no pole whatsoever at $s = 1$, and instead of $\sim x$ we have $\sim 0 \cdot x$, meaning that

$$\lim_{x \to \infty} \frac{\theta(x)}{x} = 0$$

From the lemma above, an asymptotic relation $\theta(x) \sim rx$ implies

$$\sum_{p \leq x} \mathrm{tr}\, \Phi_p \sim \frac{rx}{\log x}$$

for arbitrary r. ///

[Newman 1980] D.J. Newman, *Simple Analytic Proof of the Prime Number Theorem*, Amer. Math. Monthly 87 (180), no. 7, pp. 693-96.
