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Primer of Unramified Principal Series

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

indispensable stuff with few prerequisites
- Recollection of some definitions
- Irreducibility of unramified principal series: GL(n)
- Irreducibility of unramified principal series: Sp(n)
- Relation to spherical representations
- Intertwinings of unramified principal series: GL(n)

An important but seldom-heard remark is that *there is essentially just one irreducible representation of a p-adic group*. Thus, rather than opening a Pandora's box of disastrous complication, introduction of representation-theoretic methods in study of automorphic forms and L-functions is exactly the right thing: to the extent that there *can* be simplicity here, it is in the context of representation-theory.

By the claim that there is 'just one' representation I mean that 'almost all the time' in the study of automorphic forms one considers only *spherical* representations, which miraculously are (essentially) *unramified principle series representations*, which by definition are the simplest *induced* representations, and which are described via structure constants depending upon a finite list of complex parameters. The point is that in terms of these parameters it makes sense to talk about 'generic' unramified principal series representations, *so there is just one.*

(Making the idea of 'generic' precise is part of the point of [Garrett 1994]. One should be aware that in representation theory proper the phrase 'generic representation' sometimes has a different meaning: it sometimes means that the representation 'has a Whittaker model', which is to say that it imbeds in a certain induced representation whose definition is descended from study of Fourier coefficients of automorphic forms. The above use of the word 'generic' is, instead, in the spirit of algebraic geometry.)

Further, not only do all the unramified principal series fit together into one family, but the fact that unramified principal series are *induced* makes available many standard techniques, especially Frobenius Reciprocity and further methods involving *orbit filtrations* and other 'physical' reasoning.

The point of these notes is to give clear statements of results which give precise details about irreducibility of unramified principal series, and thus tell when exactly it is legitimate to presume that spherical representations are unramified principal series, and vice-versa. All that we really do here is explicate the assertions of [Casselman 1980] and [Borel 1976] for $GL(n)$ and $Sp(n)$. The first of these is the simplest possible example, and the second is included in order to achieve a modicum of perspective. More precisely: one must understand **spherical representations** of p-adic groups, since they are the most important by far in any intelligent discussion of automorphic forms. In particular, in the list $\{\pi_{\mathbf{p}}\}$ of representations of **p**-adic groups attached to a 'cuspform', *all but finitely-many of the* $\pi_{\mathbf{p}}$ *must be spherical.*

But the defining features of spherical representations do *not* make clear at all how to address any issues about them. In [Satake 1963] it is shown how to attach a list of complex numbers to a spherical representation, thereby parametrizing the family of all spherical representations. (This was the genesis of **Satake parameters**), and was great progress in understanding spherical representations.

However, Satake's parametrization of spherical representations still does not give any techniques for studying them *as representations*, and implicitly expresses a viewpoint which does not really take representation-theory seriously.

More effective by far, once the general effectiveness of representation theory is appreciated, is the result of [Casselman 1980], which gives a precise criterion for irreducibility of unramified principal series (with *regular* charater: see below), and gives a complete result about the appearance of spherical representations as subrepresentations, quotient representations, or sub-quotients, of the unramified principal series. The latter part of the discussion begins from the result of [Borel 1976].

The proofs in [Casselman 1980], as well as [Borel 1976], are written so as to depend upon never-published notes [Casselman 1975], and also upon the general theory of buildings and BN-pairs, not to mention the general theory of reductive algebraic groups over p-adic fields. While the last of these may be dispensable, it is unavoidably the case that the prerequisites *for the proofs* are serious. Even if one limits one's attention to $GL(n)$, for the proofs one must have knowledge of enough building-theory to know about the basic structure of Iwahori-Hecke algebras, which are *not* quite the more familiar items from the elementary theory of modular forms.

So we will not give the proofs here, but only try to explain the *phenomena* to an extent to provide some motivation to embark upon the much more serious project of understanding the *causality.*

The result of [Casselman 1980] is fundamental, along with [Borel 1976], and desire to understand the proof ought to be sufficient reason to study buildings.

On the other hand, if one indulges in *excessively* compulsive representation-theoretic thinking, then the appealing simplicity of the ('regular') unramified principal series (and spherical representations) becomes objectionable, or one might start insisting that *all* irreducible representations be 'understood' before proceeding further, or insist upon addressing the delicacies involved when the character in the unramified principal series is *not* regular (see [Reeder ?]). Such impulses are mostly misguided except as issues in themselves, since the approaches necessary to address questions about *all* irreducibles, as in [Gross 1991], are quite unwieldy by comparison to the methods which quite effectively cope with far more delicate questions about unramified principal series. And, if one cannot answer a question about regular unramified principal series, it is unlikely that general irreducibles can be treated.

Finally, in the theory of automorphic forms, the p-factors in Euler products of automorphic L-functions are mirrors of an underlying phenomenology of unramified principal series. See [Garrett 1994].

1. Recollection of some definitions

Let k be an ultrametric local field of characteristic zero. (Thus, k is a finite extension of some $\mathbf{Q}_p$). Let q be the residue class field cardinality. The most accessible **classical group** is $GL_n(k)$, the group of invertible n-by-n matrices with entries in k.

Another 'popular' classical group is $Sp_n(k)$, the group of isometries of a non-degenerate alternating form on a $2n$-dimensional k-vectorspace. Here, as is often done, in coordinates we use the alternating form

$$J = J_n = \begin{pmatrix} 0 & & & -1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & -1 \\ 1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 0 \end{pmatrix}$$

where we have broken the matrix into n-by-n blocks. Then the definition is

$$Sp_n(k) = \{g \in GL_{2n}(k) : g^\top J g = J\}$$

Let G be either one of these groups, and V be a complex vectorspace with a group homomorphism

$$\pi : G \to \mathrm{Aut}_{\mathbf{C}}(V)$$

Such π is a **representation** of G on V. As usual, a representation π of G is said to be **smooth** if for every v in the representation space V the *isotropy group*

$$G_v = \{g \in G : \pi(g)v = v\}$$

of v is *open*. Such π is **irreducible** if there is no proper G-stable $\mathbf{C}$-subspace of V.

2. Irreducibility of unramified principal series: GL_n

Here we define the unramified principal series for GL_n and state the theorem giving necessary and sufficient condition for their *irreducibility*, under the hypothesis that the character is *regular*. In the followng section we do the same in the only slightly more complicated case of Sp_n, for a little further perspective.

These are the most important representations of $GL_n(k)$ and $Sp_n(k)$ and all groups in this class, and it is fortunate that they fit together via the parametrization by $s \in \mathbf{C}^n$, as seen below.

The **standard minimal parabolic** subgroup of $GL_n(k)$ is the subgroup P consisting of upper-triangular matrices. Let

$$s = (s_1, \ldots, s_n) \in \mathbf{C}^n$$

We consider one-dimensional representations $\chi = \chi_s$ of P of the form

$$\chi_s(\begin{pmatrix} p_{11} & & * \\ & \ddots & \\ 0 & & p_{nn} \end{pmatrix}) = |p_{11}|^{s_1} |p_{22}|^{s_2} \ldots |p_{nn}|^{s_n}$$

where $|x|$ is the normalization of the norm on k so that $|\varpi| = q^{-1}$, where ϖ is a local parameter. Any such χ is called an **unramified character** of P, since it is trivial ($= 1$) when all the p_{ii} are local units.

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The **modular function** δ_P of P (whose definition in terms of invariant measures need not concern us at the moment) is

$$\delta(\begin{pmatrix} p_{11} & & * \\ & \ddots & \\ 0 & & p_{nn} \end{pmatrix}) = |p_{11}|^{n-1} |p_{22}|^{n-3} |p_{33}|^{n-5} \ldots |p_{n-1,n-1}|^{3-n} |p_{nn}|^{1-n}$$

That is, in the notation above, this is the character χ_s with

$$s = (n-1, n-3, n-5, \ldots, 3-n, 1-n)$$

Let

$$I_\chi = I_s = I_{\chi_s}$$

be the **induced representation space** of complex-valued functions f on $GL_n(k)$ so that for $p \in P$ and $g \in GL_n(k)$ we have

$$f(pg) = \delta^{\frac{1}{2}}(p)\chi(p) \cdot f(g)$$

and so that there is some compact open subgroup K of $GL_n(k)$ so that

$$f(g\theta) = f(g)$$

for all $g \in GL_n(k)$ and for all $\theta \in K$. The latter condition is the condition of being *uniformly locally constant.* The insertion of the $\delta^{\frac{1}{2}}$ is the 'correct' normalization to achieve most symmetrical statement of results below.

The group $GL_n(k)$ acts upon such functions by the **right regular representation**

$$g \to R_g$$

defined by

$$(R_g f)(x) = f(xg)$$

The vectorspace I_χ, together with the action of $GL_n(k)$ upon it by the right regular representation, is an **unramified principal series** representation of $GL_n(k)$.

If $s_i = s_j \bmod \frac{2\pi i}{\log q}$ only for $i = j$, then the character $\chi = \chi_s$ is **regular**. This notion also has a more intrinsic form. Let W be the group of permutation matrices in $GL_n(k)$, i.e., the collection of matrices with just one 1 in each row and column, with all other entries 0. Then W normalizes the subgroup M of diagonal matrices. Thus, for $\chi = \chi_s$ as above, for $w \in W$, and for $m \in M$, we define

$$\chi^w(m) = \chi(w^{-1}mw)$$

Thus, such χ is *regular* if and only if $\chi^w \neq \chi$ unless $w = 1$. (The group W is essentially the **Weyl group** of the **Levi component** M of P).

Theorem: ([Casselman 1980]) For *regular* character χ: the unramified principal series representation I_χ is *irreducible* if and only if for all $i < j$ we have

$$s_i - s_j \neq \pm 1 \bmod \frac{2\pi i}{\log q}$$

Granting all these inequalities,

$$I_\chi \approx I_{\chi^w}$$

for all $w \in W$.

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3. Irreducibility of unramified principal series: Sp_n

Now we redo the description of the unramified principal series for Sp_n: The **standard minimal parabolic** subgroup P of $Sp_n(k)$ is not quite upper-triangular matrices in $Sp_n(k)$, but rather is the subgroup of matrices of the form

$$\begin{pmatrix} p_{11} & & * & & & \\ & \ddots & & & * & \\ 0 & & p_{nn} & & & \\ & & & p_{11}^{-1} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & p_{nn}^{-1} \end{pmatrix}$$

That is, in blocks, this subgroup consists of elements of the shape

$$\begin{pmatrix} A & * \\ 0 & (A^\top)^{-1} \end{pmatrix}$$

with A an n-by-n matrix of the form

$$A = \begin{pmatrix} p_{11} & & * \\ & \ddots & \\ 0 & & p_{nn} \end{pmatrix}$$

We could choose a different non-degenerate alternating form on k^{2n} to make the minimal parabolic actually upper-triangular, but in fact nothing is gained by such machination.

The **unramified characters** χ_s for $s \in \mathbf{C}^n$ are

$$\chi_s : \begin{pmatrix} p_{11} & & * & & & \\ & \ddots & & & * & \\ 0 & & p_{nn} & & & \\ & & & p_{11}^{-1} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & p_{nn}^{-1} \end{pmatrix} = |p_{11}|^{s_1} |p_{22}|^{s_2} \ldots |p_{nn}|^{s_n}$$

where again $|x|$ is the normalization of the norm on k so that $|\varpi| = q^{-1}$, where ϖ is a local parameter. Any such χ is called an **unramified character** of P, since it is trivial ($= 1$) when all the p_{ii} are local units.

The **modular function** δ_P of P is

$$\delta(\begin{pmatrix} p_{11} & & * & & & \\ & \ddots & & & * & \\ 0 & & p_{nn} & & & \\ & & & p_{11}^{-1} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & p_{nn}^{-1} \end{pmatrix} = |p_{11}|^{2n} |p_{22}|^{2n-2} |p_{33}|^{2n-4} \ldots |p_{n-1,n-1}|^4 |p_{nn}|^2$$

That is, in the notation above, this is the character χ_s with

$$s = (2n, 2n-2, 2n-4, \ldots, 4, 2)$$

Let

$$I_\chi = I_s = I_{\chi_s}$$

be the **induced representation space** of complex-valued functions f on $Sp_n(k)$ so that for $p \in P$ and $g \in Sp_n(k)$ we have

$$f(pg) = \delta^{\frac{1}{2}}(p)\chi(p) \cdot f(g)$$

and so that there is some compact open subgroup K of $Sp_n(k)$ so that

$$f(g\theta) = f(g)$$

for all $g \in GL_n(k)$ and for all $\theta \in K$. The latter condition is the condition of being *uniformly locally constant.* The insertion of the $\delta^{\frac{1}{2}}$ is the 'correct' normalization to achieve most symmetrical statement of results below.

The group $Sp_n(k)$ acts upon such functions by the **right regular representation**

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$$(R_g f)(x) = f(xg)$$

The vectorspace I_χ, together with the action of $Sp_n(k)$ upon it by the right regular representation, is an **unramified principal series** representation of $Sp_n(k)$.

If $s_i \neq \pm s_j \bmod \frac{2\pi i}{\log q}$ for $i \neq j$ and if $s_i \neq 0 \bmod \frac{2\pi i}{\log q}$ for all i, then the character $\chi = \chi_s$ is **regular**. The instrinsic version of this condition is as follows. Let W be the group of signed permutation matrices in $Sp_n(k)$, i.e., the collection of matrices in $Sp_n(k)$ with just one ± 1 in each row and column, with all other entries 0. Then W normalizes the subgroup M of P consisting of diagonal matrices. Thus, as for $GL_n(k)$, for $\chi = \chi_s$ as above, for $w \in W$, and for $m \in M$, we define

$$\chi^w(m) = \chi(w^{-1}mw)$$

A modest amount of reflection reveals that χ is *regular* if and only if $\chi^w = \chi$ implies $w = 1$. The group W modulo $W \cap M$ is the **Weyl group** of the **Levi component** M of P.

Theorem: For *regular* character χ: the unramified principal series representation I_χ is *irreducible* if and only if for all $i < j$ we have

$$s_i \pm s_j \neq \pm 1$$

and for all i

$$s_i \neq \pm 1$$

Granting all these inequalities,

$$I_\chi \approx I_{\chi^w}$$

for all $w \in W$.

Remarks: If the character is allowed to be not-necessarily regular, things instantly become very complicated. The case of $SL(2)$ is treated in [Casselman 1975].

Remarks: The proof of this result is highly non-trivial. In the context of 'the intrinsic general theory of reductive p-adic groups', even the *statement* of the result is rather complicated. The truly essential non-elementary ingredient in the proof is the fine structure theory of affine BN-pairs, which *can* be understood for the classical groups without worrying too much over the general theory of reductive algebraic groups.

4. Relation to spherical representations

Let **o** be the ring of integers of the local field k. The subgroup $GL_n(\mathbf{o})$ is readily seen to be a compact open subgroup of $GL_n(k)$. In fact, it is not so hard to prove that it is *maximal* among compact open subgroups. Analogously, the subgroup $Sp_n(\mathbf{o})$ of $Sp_n(k)$ consisting of matrices with entries in **o** is a maximal compact subgroup of $Sp_n(k)$, and is *open*.

Let G, K be $GL_n(k), GL_n(\mathbf{o})$ or $Sp_n(k), SpL_n(\mathbf{o})$. An irreducible smooth representation π of G is K-**spherical** if there is a non-zero vector v_o in the representation space which is a K-fixed vector, i.e., so that $\pi(\theta)v_o = v_o$ for all $\theta \in K$.

There are several relatively elementary things that can be proven about these spherical representations without knowing anything tangible about them. However, the crucial fact, which provided the impetus for this discussion, is the following. (With conventions as just above, this holds for both $GL_n(k)$ and $Sp_n(k)$).

Corollary: ([Satake 1963]) Every spherical representation π is a *subrepresentation* of an unramified principal series $I_\chi = I_s$, and is also a *quotient* of an unramified principal series. A list $s = (q^{-s_1}, \ldots, q^{-s_n})$ of complex numbers so that $\pi \subset I_s$ is the list of **Satake parameters**.

Remarks: Although this result was effectively contained in [Satake 1963], the ideas of [Borel 1976] give a more incisive and memorable proof.

Corollary: Let π be a spherical representation with Satake parameters $s = (q^{-s_1}, \ldots, q^{-s_n})$. If the unramified principal series $I_\chi = I_s$ is irreducible, then

$$\pi \approx I_\chi$$

In fact, for any $w \in W$,

$$\pi \approx I_\chi \approx I_{\chi^w}$$

That is, under this irreducibility assumption, the Satake parameters are ambiguous up to (the permutations in) the Weyl group W.

Remarks: By the criterion of the previous section, if χ is regular and if a certain finite list of inequalities is satisfied, then we have the isomorphism of this last corollary. Thus, for the Satake parameters off a finite collection of 'hypersurfaces', the spherical representation *is* an unramified principal series, and the Satake parameters are *ambiguous* up to *permutations*.

5. Intertwinings of unramified principal series: GL(n)

The result about irreducibility of unramified principal series can be refined usefully. Since the statement involves the **Weyl groups**, we treat the simplest case, GL_n, in this section, and redo everything in the slightly more complicated case of Sp_n in the next. Again, all that we present is an explication in more elementary terms of the result of [Casselman 1980].

For $GL_n(k)$ we asserted above that, for χ *regular*, I_χ and I_{χ^w} are isomorphic when certain inequalities modulo $\frac{2\pi i}{\log q}$ are met. With or without these inequalities, assuming only the regularity of the character χ, [Casselman 1980] proves via an *orbit filtration* argument that *always* the dimension of the space of G-homomorphisms (=intertwining operators) from I_χ to I_{χ^w} is 1:

$$\dim_{\mathbf{C}} \operatorname{Hom}_{GL_n(k)} (I_\chi, I_{\chi^w}) = 1$$

Further, there is an integral formula for such intertwining, convergent for suitable values of s, which then can be *analytically continued*. The precise form of the integral is not essential at the moment, only the fact that there is a *normalization* possible.

We observe that, by the **Iwasawa decomposition**

$$GL_n(k) = P \cdot GL_n(\mathbf{o})$$

in $GL_n(k)$, there is only a one-dimensional space of $GL_n(\mathbf{o})$-invariant vectors in any I_χ, a basis for this subspace being given by the obvious candidate

$$f_\chi(p\theta) = (\delta^{1/2}\chi)(p)$$

for all $\theta \in GL_n(\mathbf{o})$ and $p \in P$. Here we use the fact that χ is trivial on the overlap $P \cap GL_n(\mathbf{o})$. As usual, any $GL_n(\mathbf{o})$-invariant vector in a representation is called a **spherical vector**.

The point of the following result is that we can see *exactly* what happens to the essentially unique spherical vector under these G-homomorphisms:

Theorem: Let

$$T_{\chi,w} : I_\chi \to I_{\chi^w}$$

be the normalized non-zero intertwining operator. Then

$$T_{\chi,w} f_\chi = f_{\chi^w} \cdot \prod_{(i,j): i<j \text{and} w(i)>j} \frac{1-q^{s_j-s_i+1}}{1-q^{s_j-s_i}}$$

where we view elements $w \in W$ as giving permutations of $\{1, 2, \ldots, n\}$.

Remarks: If one wanted, the product over i, j with $i > j$ and so that $w(i) < j$ as the product over *positive roots taken to negative roots by w*.

Corollary: If

$$s_j - s_i + 1 \neq 0 \mod \frac{2\pi i}{\log q}$$

for all i, j, then the spherical vector *generates I_χ*.

Remarks: The corollary does *not* follow immediately from the statement of the theorem alone, but depends on an underlying idea about representations with Iwahori-fixed vectors, as in [Borel 1976].

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