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Proving Admissibility of Irreducible Unitaries

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Ideas from Kaplansky, Kolchin, Godement

Harish-Chandra, Bernstein, Kazhdan

Here we prove the *admissibility* of irreducible unitary representations of real reductive linear Lie groups *and* of linear p-adic reducive groups. This is a rewriting and collation of fragments of the above-named authors' work.

The Lie case of the admissibility assertion was known in the 1950's, but the p-adic case was not known until the mid 1970's.

The admissibility of irreducible unitaries of both p-adic reductive and linear reductive Lie groups are is one of the fundamental facts which make representation theory effective in the study of automorphic forms and L-functions.

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1. *Higher commutators*

The ideas here have their origin in work of Kaplansky, who also credits Kolchin. Godement later adapted these ideas and extended their application.

Let R be an associative ring, not necessarily with an identity. Define the **higher commutator** or n-**commutator** of n elements of R by

$$[x_1, \ldots, x_n] = \sum_\pi \operatorname{sgn}(\pi)\, x_{\pi(1)} \ldots x_{\pi(n)}$$

where π is summed over the symmetric group S_n permuting n things.

If all n-commutators $[x_1, \ldots, x_n]$ are 0 for elements $[x_1, \ldots, x_n]$ of a ring R, then say that R is n**-abelian**. Note that R is commutative if and only if it is 2-abelian.

Lemma: If R is r-abelian and $s > r$ then R is s-abelian.

Proof: Let G be the symmetric group on s letters. Let H be the symmetric group on $\{1, \ldots, r\}$, and let $J \approx \mathbf{S}_{s-r}$ be the symmetric group on $\{r+1, \ldots, s\}$. We imbed $H \times J$ into G in the obvious way. Let X be a set of representatives for $(H \times J)\backslash G$. Then

$$[y_1, \ldots, y_s] = \sum_{x \in X} \mathrm{sgn}(x) \sum_{\tau \in J} \mathrm{sgn}(\tau) \left(\sum_{\sigma \in H} \mathrm{sgn}(\sigma) y_{\sigma x(1)} \ldots y_{\sigma x(r)} \right) y_{\tau x(r+1)} \ldots y_{\tau x(s)}$$

Every inner sum over σ is zero. Note that the sgn-function behaves as indicated: for example $\mathrm{sgn}(\sigma)$ with σ viewed as an element of $\mathbf{S}_s$ is the same as $\mathrm{sgn}(\sigma)$ with σ viewed as an element of $\mathbf{S}_r$. ♣

Lemma: Let A be an associative algebra over a field k not of characteristic 2, of k-dimension $n < \infty$. Then A is $n+1$-abelian.

Proof: The assertion is multi-linear in the arguments, so it suffices to take the x_i to be k-basis elements for A. If it should happen that $x_i = x_j$ for $i \neq j$, then there is a transposition leaving the *value* of the higher commutator unchanged, but changing the sign on the sum. Thus, the expression is 0 since we are not in characteristic 2. ♣

Lemma (Kaplansky-Kolchin): Let k be a field of characteristic not 2. Given a positive integer n, let $r = r(n)$ be the least positive integer so that the ring $M_n(k)$ of $n \times n$ matrices with entries in the field k is $r(n)$-abelian. Then

$$r(n) \geq r(n-1) + 2 > r(n-1)$$

Proof: First, by the previous lemma, we see that there do indeed exist *finite* integers $r(n)$ so that $M_n(k)$ is $r(n)$-abelian.

There exist $x_0, \ldots, x_{r(n-1)-1} \in M_{n-1}(k)$ so that

$$[x_0, \ldots, x_{r(n-1)-1}] \neq 0$$

There exists a pair of indices (p,q) so that the $(p,q)^{th}$ entry of this higher commutator is non-zero. Then let

$$y_i = \begin{pmatrix} x_i & 0 \\ 0 & 0 \end{pmatrix}$$

Then we claim that

$$[y_0, \ldots, y_{r(n-1)-1}, e_{j_0,n}, e_{n,n}] \neq 0$$

where $e_{i,j}$ is the matrix with all 0's excepting a 1 at the $(i,j)^{th}$ place.

To prove the latter claim, we note that, with y, z from among the y_i, we have

$$\ldots y e_{n,n} z \ldots = 0$$

unless there are indices i, j so that $y_{i,n} \neq 0$ and $z_{n,j} \neq 0$. From this sort of consideration, we have

$$[y_0, \ldots, y_{r(n-1)-1}, e_{q,n}, e_{n,n}] = [y_0, \ldots, y_{r(n-1)-1}] e_{q,n} e_{n,n} =$$

$$= [y_0, \ldots, y_{r(n-1)-1}] e_{q,n}$$

which has $(p,q)^{th}$ entry the same as that of

$$= [x_0, \ldots, x_{r(n-1)-1}]$$

which is non-zero. This gives the assertion. ♣

Corollary: Fix a field k of characteristic not 2. Let $r(n)$ be as just above in the Kaplansky-Kolchin lemma. Let A be a ring with unit. If A is $r(n)$-abelian then every finite-dimensional irreducible representation of A (on a k-vectorspace) has dimension $\leq n$.

Proof: If A is $r(n)$-abelian, then certainly any homomorphic image of it is $r(n)$-abelian also. Therefore, for every finite-dimensional representation (π, V) of A, we conclude that $\pi(A)$ is $r(n)$-abelian. Since V is irreducible and finite-dimensional, by the Density Theorem $\pi(A)$ is the whole endomorphism algebra of V. By the Kaplansky-Kolchin lemma,

$$\dim_k(V) \leq n$$

This proves the corollary. ♣

Corollary: Fix a field k of characteristic not 2. Let $r(n)$ be as just above in the Kaplansky-Kolchin lemma. Let A be a ring with unit. If for every $0 \not\alpha \in A$ there is an *semi-simple* representation π of A with $\dim \pi \leq n$ and with $\pi\alpha \neq 0$ then A is $r(n)$-abelian.

Proof: The hypothesis assures that, in fact, for every $0 \not\alpha \in A$ there is an *irreducible* representation π of A with $\dim \pi \leq n$ and with $\pi\alpha \neq 0$.

Suppose that some r-commutator

$$\alpha = [x_1, \ldots, x_r]$$

in A were non-zero. Take an irreducible representation (π, V) of A with $\pi\alpha \neq 0$ and with $\dim \pi \leq n$. By the Density Theorem, $\pi(A) = \mathrm{End}(V)$. Then

$$0 \neq \pi[x_1, \ldots, x_r] = [\pi x_1, \ldots, \pi x_r]$$

By the Kaplansky-Kolchin lemma, $r < r(n)$. ♣

The following lemma arises in HarishChandra's (1969) reduction of the question of admissibility of irreducibles (p-adic case) to a related question concerning supercuspidal representations. (See below). This assertion is not utterly trivial to verify; indeed there are several less careful assertions 'in circulation' which are demonstrably false.

Key Lemma (Godement): In a ring R, consider r-commutators $[x_1, \ldots, x_r]$ where each x_i is an s-commutator $[y_{i,1}, \ldots, y_{i,s}]$. *Suppose that s is odd.* If every such $[x_1, \ldots, x_r]$ is 0, then R is rs-abelian.

Proof: We write rs elements of the ring as

$$x_{1,1}, x_{1,2}, \ldots, x_{1,s}, x_{2,1}, x_{2,2}, \ldots, x_{2,s}, \ldots, x_{r,1}, x_{r,2}, \ldots, x_{r,s}$$

Let G be the permutations on the set of all indices (i, j) with $1 \leq i \leq r$ and $1 \leq j \leq s$. Let H_i be the group of permutations of

$$\{(i, j) : 1 \leq j \leq s\}$$

We view H_i as the subgroup of G which permutes these indices and leaves all others fixed. Let $J \approx \mathbf{S}_r$ be the subgroup of G consisting of elements of the form

$$\tau_\pi : (i, j) \to (\pi(i), j)$$

where $\pi \in \mathbf{S}_r$ does not depend upon j. Note that J normalizes the subgroup $\Pi_i H_i$ of G. Let X be a set of representatives for $J\Pi_i H_i$.

Then

$$\mathrm{sgn}(\tau_\pi) = \mathrm{sgn}(\pi)^s = \mathrm{sgn}(\pi)$$

because τ_π is a copy of $\pi \in \mathbf{S}_r$ 'on the diagonal' in an s-fold cartesian power of $\mathbf{S}_r$, and because s is odd. Then

$$\mathrm{sgn}(\Pi_i \sigma_i \, \tau_\pi \, \xi) = \Pi_i \mathrm{sgn}(\sigma_i) \, \mathrm{sgn}(\pi) \mathrm{sgn}(\xi)$$

Then

$$[x_{(1,1)}, \ldots, x_{(r,s)}] = \sum_{\xi \in X} \mathrm{sgn}(\xi) \sum_{\tau = \tau_\pi \in J} \mathrm{sgn}(\pi) \left(\sum_{h \in H_1} \mathrm{sgn}(h) x_{h\tau\xi(1,1)} \ldots x_{h\tau\xi(1,s)} \right) \ldots$$

$$\ldots \left(\sum_{h \in H_r} \mathrm{sgn}(h) x_{h\tau\xi(r,1)} \ldots x_{h\tau\xi(r,s)} \right) =$$

$$= \sum_{\xi \in X} \mathrm{sgn}(\xi) \sum_{\tau = \tau_\pi} \mathrm{sgn}(\pi)[x_{\xi(1,\pi 1)}, \ldots, x_{\xi(1,\pi s)}] \ldots [x_{\xi(r,\pi 1)}, \ldots, x_{\xi(r,\pi s)}] =$$

$$= \sum_{\xi \in X} \mathrm{sgn}(\xi)[y_{\pi 1}, \ldots, y_{\pi r}]$$

where

$$y_i = [x_{\xi(i,1)}, \ldots, x_{\xi(i,s)}]$$

This expresses an rs-commutator *for s odd* as a linear combination of r-commutators of s-commutators. Thus, it must vanish. ♣

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2. *Godement's principle: bounding dimensions of irreducibles*

A Banach space representation

$$\pi : A \to \mathrm{End}(B)$$

of an associative algebra A is **completely irreducible** if every bounded operator on B is a *strong limit* of operators in $\pi(A)$.

Recall that the **strong topology** on the collection of bounded operators on a Banach space B is defined by the collection of semi-norms

$$\nu_x(T) = |Tx|$$

for $x \in B$ and a bounded operator T, where $|x|$ is the norm on B. Thus, $T_i \to T$ in the strong topology if and only if, for all $x \in B$, $T_i x \to Tx$ in the topology of B.

Note: The von Neumann Density Theorem (see next section) implies that for unitary Hilbert space representations, topologically irreducible representations are completely irreducible. In fact, I don't know of any other way that (infinite-dimensional) completely irreducible representations occur.

For present purposes, a **separating family $\mathcal{S}$ of representations** of an associative algebra A is a collection of representations so that, given a finite set of elements $x_0, \ldots, x_n$ of A, there is $\pi \in \mathcal{S}$ so that the images $\pi(x_i)$ are all distinct.

Theorem (Godement): Suppose that an associative algebra A has a separating family $\mathcal{S}$ of representations, each of which is of dimension $\leq N$ (with $N < \infty$). Then every *completely irreducible* Banach space representation of A (not necessarily assumed finite-dimensional) has dimension $\leq N$.

Proof: From the discussion of higher commutators, for each integer n there is a polynomial P_n in $r(n)$ (not necessarily commuting) variables so that

$$P_n(x_0, \ldots, x_{r(n)}) = 0$$

for all $x_0, \ldots, x_{r(n)} \in M_n(\mathbb{C})$, but so that there are $y_0, \ldots, y_{r(n)} \in M_{n+1}(\mathbb{C})$ so that

$$P_n(y_0, \ldots, y_{r(n)}) \neq 0$$

We have the explicit form for these polynomials:

$$P_n(x_0, \ldots, x_{r(n)}) = \sum_\sigma \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \ldots x_{\sigma(r)}$$

with σ summed over the symmetric group $\mathcal{S}_{r(n)}$ on $r(n)$ things. The crucial aspect of them which we need in this proof is the fact that, in every monomial

$$x_{i_0} \ldots x_{i_{r(n)}}$$

occuring,

$$i_j \neq i_k$$

for $j \neq k$. (And every index does occur in every monomial). That is, each monomial is *linear* in a given x_i.

Put $r = r(N)$. Let π be a completely irreducible representation of A on a Banach space B. We claim that it suffices to prove that

$$P_N(T_0, \ldots, T_r) = 0$$

for all bounded operators $T_0, \ldots, T_r$ on B. Indeed, let E be a (necessarily closed) finite-dimensional subspace of B. The Han-Banach theorem can be used to show that, given an endomorphism S of E, there is an endomorphism $\tilde{S}$ of B whose restriction to E is S. Therefore, for any $S_0, \ldots, S_r$ endomorphisms of E, we have

$$P_N(S_0, \ldots, S_r) = P_N(\tilde{S}_0, \ldots, \tilde{S}_r) = 0$$

Therefore, by the nature of P_N,

$$\dim E \leq N$$

Thus, to prove the theorem it does indeed suffice to prove that

$$P_N(T_0, \ldots, T_r) = 0$$

for all bounded operators $T_0, \ldots, T_r$ on B.

Recall that in the previous section we showed: If an associative algebra A has the property that there is an integer n so that, for every $0 \neq x \in A$ there is an irreducible representation π of A of dimension $\leq n$ so that $\pi(x) \neq 0$, then A is $r(n)$-abelian.

We emphasize that

$$P_N(T_0, \ldots, T_r)$$

is, after all, a sum of terms of the form

$$\pm T_{i_0} \ldots T_{i_N}$$

For brevity, write x instead of πx for $x \in A$. By the definition of complete irreducibility, T_0 is a strong limit of elements t_s with $t_s \in A$, and every expression of the form

$$x_{i_0} \ldots t_s \ldots x_{i_N}$$

has limit

$$x_{i_0} \ldots T_0 \ldots x_{i_N}$$

in the strong topology. Here we are using the fact that each monomial in P_N is linear in T_0. Therefore, for all $x_2, \ldots, x_r \in A$, in the strong topology

$$P_N(t_s, x_2, \ldots, x_r) \to P_N(T_0, x_2, \ldots, x_r)$$

But the left-hand side in this equality is always 0, so

$$P_N(T_0, x_2, \ldots, x_r) = 0$$

Suppose (inductively) that we have shown that

$$P_N(T_0, \ldots, T_{j-1}, x_j, \ldots, x_r) = 0$$

for all $x_j, \ldots, x_r \in A$. Consider a monomial

$$\pm M T_j N$$

of

$$P_N(T_0, \ldots, T_j, x_{j+1}, \ldots, x_r)$$

where M, N are monomials in the *other* T_i's and in the x_i's. Again, we are using the fact that each monomial in P_N is linear in T_j. Let t_s be elements of A which have strong limit T_j. Then in the strong topology

$$M t_s N \to M T_j N$$

Therefore, in the strong topology,

$$P_N(T_0, \ldots, T_{j-1}, t_s, x_{j+1}, \ldots, x_r) \to P_N(T_0, \ldots, T_{j-1}, T_j, x_{j+1}, \ldots, x_r)$$

The induction hypothesis implies that for all t_s we have

$$P_N(T_0, \ldots, T_{j-1}, t_s, x_{j+1}, \ldots, x_r) = 0$$

Therefore, we conclude that

$$P_N(T_0, \ldots, T_{j-1}, T_j, x_{j+1}, \ldots, x_r) = 0$$

Having completed this induction, we conclude that, for all $T_0, \ldots, T_r$ bounded operators on B,

$$P_N(T_0, \ldots, T_r) = 0$$

By the earlier remarks in this proof, this implies that the dimension of B is $\leq N$. ♣

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3. *The von Neumann (strong) density theorem*

In effect, the von Neumann density theory asserts that *topologically irreducible* $*$*-representations on Hilbert spaces are 'completely irreducible'* (in Godement's sense above).

Let H be a Hilbert space on which an associative algebra A acts by bounded operators. We may identify elements of A with their actions upon H. We suppose that A is closed under *adjoints*:

$$A^* = A$$

We suppose that H is *topologically irreducible*, i.e., that the only A-stable closed subspaces of H are 0 and H itself. Then, the Density Theorem asserts that, *for every bounded operator T on H, for every finite collection $v_0, \ldots, v_n$ of elements of H, and for every $\epsilon > 0$, there is $\alpha \in A$ so that*

$$|Tv_i - \alpha v_i| < \epsilon$$

for all i.

In other words, if H is an irreducible A-space and A is $*$-closed, then in the *strong topology* A is dense in the collection of all bounded (i.e., continuous) operators on H.

Proof: One important caution: many basic properties which the *uniform* norm on bounded operators possesses, and which are taken for granted, fail for the strong topology ...

We grant ourselves Schur's Lemma for bounded operators on a Hilbert space (a consequence of even the crudest version of a spectral theorem for bounded operators on Hilbert spaces). That is, we grant that if S is a bounded operator commuting with all elements of A, then S is necessarily a scalar.

Let $\mathrm{End}_A(H^n)$ be the ring of endomorphisms of

$$H^n = H \oplus \ldots \oplus H \qquad n \text{ copies}$$

commuting with all elements of A. We give this direct sum the inner product which is the sum of that on the summands. It is easy to check that H^n is a Hilbert space.

From Schur's Lemma it is not hard to show that every $S \in \mathrm{End}_A(H^n)$ is of the form

$$S(v_1 \oplus \ldots v_n) = \sum_j S_{1j} v_j \oplus \ldots \oplus \sum_j S_{nj} v_j$$

for some complex numbers S_{ij}. That is, this endomorphism ring is isomorphis to the ring $B = M_n(\mathbb{C})$ of $n \times n$ complex matrices.

On the other hand, one verifies directly that

$$S = T \times \ldots \times T \in \mathrm{End}_B(H^n)$$

Now put

$$v = v_1 \oplus \ldots \oplus v_n \in H^n$$

For any algebraic subspace E of any Hilbert space F, we have an orthogonal direct sum decomposition

$$F = \bar{E} \oplus E^\perp$$

Therefore,

$$H^n = \overline{(Av)} \oplus (Av)^\perp$$

Let e be the orthogonal projector to $\overline{(Av)}$. Then e commutes with all elements of A, basically because A is closed under $*$. Thus, $e \in B$.

Then

$$Sv = Sev = eSv \in \overline{(Av)}$$

since S is a B-endomorphism and $e \in B$. That is, for every $\epsilon > 0$ there is $\alpha \in A$ so that with respect to the norm on H^n

$$|Sv - \alpha v|^2 < \epsilon$$

Expressing this norm in terms of that on H, this is

$$|Tv_1 - \alpha v_1|^2 + \ldots + |Tv_n - \alpha v_n|^2 < \epsilon$$

This is the von Neumann density theorem. ♣

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4. *Irreducibles of connected solvable groups*

We recall Lie's Theorem on irreducible finite-dimensional representations of connected solvable groups. The **derived group** or **commutator subgroup**

$$G^{(1)} = G' = [G, G]$$

of a group G is the subgroup generated by all expressions

$$ghg^{-1}h^{-1}$$

(In some cases one takes *closure* as well).

A group G is **solvable** if

$$G \supset G' \supset G'' \supset \ldots G^{(n)} \supset \ldots$$

eventually terminates, i.e., if $G^{(n)} = 1$ for n large enough.

Theorem (Lie): Let B be a connected solvable real Lie group. Then all irreducible finite-dimensional representations of B are one-dimensional.

Proof (Godement): First we need

Lemma: If B is connected then so is B'.

Proof of Lemma: Let

$$X_1 = \{xyx^{-1}y^{-1} : x, y \in B\}$$

$$X_n = \{x_1 \ldots x_n : x_1, \ldots, x_n \in X_1\}$$

Then X_1 is connected, being the continuous image of the connected set $B \times B$, and similarly all the other X_n are connected. From this it is a little exercise to check that $B' = \bigcup X_i$ is connected:

If U, V were disjoint open sets in B with $U \cup V = B$, then for all indices i either $X_i \subset U$ or $X_i \subset V$. If $X_1 \subset U$ then $X_1 \subset X_i$ implies that $V \cap X_i = \emptyset$. Thus, we would conclude that $X_1 \subset U$ would imply $B \subset U$. Thus, B' is connected, as the Lemma asserts. (♣)

Proof of theorem: If B is abelian, this is the finite-dimensional Schur's lemma. We do induction on the 'height' of B, i.e., the least integer n so that $B^{(n)} = 1$. Let π be an irreducible finite-dimensional representation of B, and let $N = B'$.

Then $\mathrm{Res}^B_N \pi$ contains a one-dimensional representation λ of N, by finite-dimensionality, so there is a non-zero λ-eigenvector v_λ of N. Let V_λ be the space of all λ-eigenvectors in the representation space V of π. Let Λ be the collection of one-dimensional representations λ of N so that $V_\lambda \neq 0$. By the finite-dimensionality, Λ is finite. And, of course, if $v_i \in V_{\lambda_i}$ for distinct λ_i, then $\sum_i v_i = 0$ implies that all v_i are 0.

Since N is normal in B, B acts on the set Λ by the 'dual' of conjugation:

$$\lambda^b(n) = \lambda(b^{-1}nb)$$

. We give the collection of all one dimensional *complex-valued* characters on N the 'strong' topology: $\lambda_i \to \lambda$ if $\lambda(n) \to \lambda(n)$ for all $n \in N$. Since Λ is finite, it is discrete. On the other hand, since B is connected, its continuous image $\{\lambda^b : b \in B\}$ is also connected. Thus, Λ must be a discrete set with a single connected component, so consists of a single element.

Given $b \in B$, again using the finite-dimensionality, take a non-zero eigenvector v for b; suppose $bv = cv$ with $c \in \mathbb{C}$. Then, for all $x \in B$, the commutator $bxb^{-1}x^{-1}$ is in N, by definition of what N is. Thus,

$$\pi(b)(\pi(x)v) = \pi(bxb^{-1}x^{-1})\pi(x)(\pi(b)v) =$$

$$= \lambda(bxb^{-1}x^{-1})\pi(x)(cv) = \lambda(bxb^{-1}x^{-1})\, c\, \pi(x)v$$

That is, $\pi(x)v$ is also an eigenvector of b.

Further, the function $c\lambda(bxb^{-1}x^{-1})$ is a continuous function of x, so by connectedness of B does not depend upon x. Thus, its value is just c, since this is the value obtained for $x = 1$. Thus, the c-eigenspace for b in V is a B-stable non-zero subspace of V, so must be all of V.

In other words, there is a map $\phi : B \to \mathbb{C}$ so that for all $v \in V$ and for all $b \in B$

$$\pi(b)v = \phi(b)\, v$$

It follows that ϕ is a continuous group homomorphism. Thus, if V were not one-dimensional we would obtain a contradiction. So we conclude that π was indeed one-dimensional. ♣

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5. *Separating families of irreducibles*

We introduce the notion of a **separating family** of irreducible finite-dimensional representations. Let G be a closed subgroup of some $GL(n, \mathbb{R})$. That is, G is assumed to be a **linear group**.

Theorem: For all $0 \neq f \in C_c^\infty(G)$, there is an *irreducible finite-dimensional* representation (π, V) of G so that $\pi(f) \neq 0 \in \mathrm{End}_{\mathbb{C}}(V)$.

Proof: We may suppose without loss of generality that f is real-valued. By the Weierstrass approximation theorem, for every $\epsilon > 0$ there is a real polynomial Q on the space of real $n \times n$ matrices so that the supremum of $|f(x) - Q(x)|$ *over the support of* f is $< \epsilon$. Then

$$\int_G fQ \geq \int f \cdot (f - \epsilon) = \int |f|^2 - \epsilon \int |f|$$

which is positive (hence, non-zero) for ϵ small enough.

Let (ρ, V) be the representation space of G 'generated by' Q under right translations. That is, V consists of all complex linear combinations of right translates by G of Q:

$$(\sum_i c_i\, R_{h_i} Q)(g) = \left(g \to \sum_i c_i Q(gh_i)\right)$$

Since the action of G is linear, it preserves the total degree of such polynomials. Thus, ρ is necessarily finite-dimensional.

Now $\int_G fQ \neq 0$ is the same thing as

$$(\rho(f)Q)(1) = (R_f(Q))(1) \neq 0$$

where R is the right regular representation. Thus, $\rho(f)Q \neq 0$, so certainly $\rho f \neq 0$.

By Zorn's lemma, this ρ has an *irreducible* non-zero quotient. By induction, we can make a chain of G-subspaces

$$V = V_1 \supset V_2 \supset V_3 \supset \ldots \supset V_n = 0$$

where each quotient $\pi_i = V_i / V_{i+1}$ is an *irreducible* finite-dimensional G-representation.

If $\pi_{n-1} f \neq 0$ then we have the theorem. If, on the other hand, $\pi_{n-1} f = 0$, then $\pi_{n-1} f$ gives rise to a well-defined operator on W_{n-2}. It is easy to check that this operator is none other than $\pi_{n-2} f$. If this is non-zero, then we are done. If it is zero, then we continue inductively. The finite-dimensionality of ρ assures that, if $\pi f \neq 0$, then for some i we must have $\pi_i f \neq 0$. This gives the theorem. ♣

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6. *A Subrepresentation Theorem*

Here is a subrepresenation theorem for finite-dimensional representations of linear reductive real Lie groups. Let B be a connected solvable subgroup of the linear reductive real group G.

Proposition: Every finite-dimensional irreducible representation ρ of G is a subrepresentation of $\mathrm{Ind}_B^G \chi$ for some one-dimensional representation χ of B.

Proof: By finite-dimensionality, the restriction of ρ to B contains some irreducible of B, which is one-dimensional, by previous results. Fix a non-zero vector v in that subspace, and take λ in the linear dual V^* so that $\lambda(v) \neq 0$. Then the map

$$v \to c_{v,\lambda}$$

is a G-homomorphism to $\mathrm{Ind}_B^G \chi$, where $c_{v,\lambda}$ is the coefficient function

$$c_{v,\lambda}(g) = \langle \rho(g)v, \lambda \rangle$$

where $\langle , \rangle$ is the canonical pairing

$$V \times V^* \to \mathbb{C}$$

Since $\lambda(v) \neq 0$ this map is non-zero, and since ρ is irreducible this must be an injective map. ♣

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7. HarishChandra-Godement Admissibility Theorem

Now we have the HarishChandra-Godement theorem on admissibility for reductive real linear Lie groups. Let G be a reductive real linear Lie group with maximal compact subgroup K.

Theorem: Let G be a reductive linear real Lie group and let π be an irreducible unitary representation of G. For every irreducible δ of K, the multiplicity of δ in π is $\leq \dim_{\mathbb{C}} \delta$.

Proof: Let Ω be a separating family of *finite-dimensional* representations of G, meaning that for any $0 \neq \phi \in C_c^\infty(G)$ there is a $\rho \in \Omega$ so that $\pi(\phi) \neq 0$.

Proposition: Fix an irreducible δ of K. If there is a constant N so that the multiplicity of δ in $\rho|_K$ is $\leq N$ for all $\rho \in \Omega$, then for *every* irreducible *unitary* π of G the multiplicity of δ in $\pi|_K$ is $\leq N$.

Proof of proposition: Let $C_c^\infty(G, \delta)$ be the collection of test functions which are of left and right K-type δ, i.e., which under the right (or left) regular action of K generate a space which is δ-isotypic. We have

$$C_c^\infty(G, \delta) = \chi_\delta * C_c^\infty(G) * \chi_\delta$$

where the convolution is over K and χ_δ is the character of the (finite-dimensional!) representation δ. Recall that $\pi(\chi_\delta)$ acts as a non-zero scalar on any δ-isotypic space.

Let π^δ be the δ-isotype in π. Then for $f \in C_c^\infty(G, \delta)$ we have $\pi(f) = 0$ if and only if $\pi^\delta(f) = 0$. This follows from the convolution expression just above.

For $\pi \in \Omega$ we are assuming that $\dim \pi^\delta \leq N$. Therefore, for all $f_0, \ldots, f_{r(N)} \in C_c^\infty(G, \delta)$ we have a vanishing higher commutator

$$[\pi f_0, \ldots, \pi f_{r(N)}] = 0$$

where $r(N)$ as in the Kaplansky-Kolchin trick discussion. Then, by Godement's principle,

$$\dim \pi^\delta \leq N$$

This is the desired admissibility assertion. ♣

Proof of theorem: We apply the result of the proposition, invoking the Subrepresentation Theorem for finite-dimensional representations of G:

We have seen that the finite-dimensional representations of G are a separating family, and that they all lie inside $\mathrm{Ind}_B^G \chi$ for any fixed maximal connected solvable subgroup B, for some one-dimensional χ. The Iwasawa decomposition asserts that $G = BK$ and $B \cap K = 1$. Then, *as K-spaces* we have

$$\mathrm{Ind}_B^G \chi \approx \mathrm{Ind}_{B \cap K}^K \chi_{B \cap K} \approx$$

$$\approx \mathrm{Ind}_{\{1\}}^K \mathbb{C} = C_c^\infty(K) \subset L^2(K) \approx \hat{\bigoplus}_\delta\ \delta \otimes \check{\delta}$$

where we view either $C_c^\infty(K)$ or $L^2(K)$ as $K \times K$-spaces with respect to the biregular representation. Thus, δ occurs with multiplicity which is the dimension of $\check{\delta}$, which is the same as that of δ. This proves the theorem. ♣

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8. *The theorem on admissibility for p-adic reductive groups*

Let G be a p-adic reductive group in the sequel.

Theorem: Let K be a compact open subgroup of G, and let (δ, W) be an irreducible representation of K. Then δ occurs with finite multiplicity in any irreducible unitary representation (π, V) of G.

The proof of this result was completed over the course of about twenty years, starting with some ideas of Godement in the early '50's (based on the Kaplansky-Kolchin higher commutator business), systematically developed by HarishChandra in the '60's, and concluded by Bernstein's estimate in the early '70's. The latter result invokes in a serious way HarishChandra's general development of the representation theory in order to reduce the question to a relatively elementary issue. Thus, the main body of the result is due to HarishChandra, with some crucial elements contributed by Godement (before) and Bernstein (later).

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9. *Background for the p-adic case*

Let Z be a closed subgroup of G inside the center of G, sufficiently large so that the center of G modulo Z is compact. (E.g., we might take Z to be the whole center, but the present condition is more flexible and often more convenient). By Schur's Lemma (the unitary case), any irreducible unitary representation π of G, when restricted to the center of G, consists of scalar operators; in particular, $\pi(Z)$ consists of scalar operators. As usual, we say that π has 'central character' $\omega : Z \to \mathbb{C}^\times$ if $\pi(z)v = \omega(z)v$ for all $z \in Z$ and for all $v \in V$.

As usual, define $C_c^\infty(G)$ to be the collection of locally constant $\mathbb{C}$-valued functions f on G which are compactly-supported. And define $C_c^\infty(Z\backslash G, \omega)$ to be the collection of locally constant $\mathbb{C}$-valued functions f on G which are compactly-supported modulo Z, and so that

$$f(zg) = \omega(z)f(g)$$

for all $z \in Z$ and $g \in G$.

For a finite-dimensional (complex) representation (δ, V) of a 'good' maximal compact subgroup K of G, let $\mathcal{H}(G,^\delta\ )$ be the space of $End_{\mathbb{C}}(V)$-valued compactly-supported locally constant functions on G so that, for all $g \in G$ and for all $x, y \in K$

$$f(xgy) = \delta(x)f(g)\delta(y)$$

(The notation here is slightly abusive, but this is unavoidable.) This is a generalization of the Hecke algebras $\mathcal{H}(G, K)$ which are defined to be the left and right K-invariant $\mathbb{C}$-valued test functions on G.

Further, for a character ω of Z and representation (δ, V) of K, we define $\mathcal{H}_\omega(G, K, \delta)$ to be the collection of $\mathrm{End}(V)$-valued functions f on G so that for all $g \in G$, $z \in Z$, and $x, y \in K$

$$f(zxgy) = \omega(z)\delta(x)f(g)\delta(y)$$

Say that f in $C_c^\infty(G)$, $\mathcal{H}_\omega(G, K, \delta)$, or $\mathcal{H}(G, \delta)$ is a **supercuspform** if, for all $g \in G$ and for all unipotent radicals N of proper parabolics P we have

$$\int_N f(gn)\, dn = 0$$

In any space of functions on G in which the notion of supercuspform makes sense, we denote the subspace of such by appending left superscript 'o' to the symbol for the space. Thus, the subspace of $\mathcal{H}_\omega(G, K, \delta)$ consisting of all such is ${}^o\mathcal{H}_\omega(G, K, \delta)$.

An irreducible unitary representation (π, V) of G with central character ω is **supercuspidal** if there exist $v \in V, \lambda \in V^*$ so that *the coefficient function $c_{v,\lambda}$ is compactly-supported modulo the center.* (Again, by Schur's Lemma every irreducible *does* have a central character.)

There is another 'definition' of supercuspidal representation whose equivalence to the previous definition must be appreciated. Let N be the unipotent radical of a parabolic P in G, and let N' range over larger and larger compact open subgroups of N, so that the union of all the N' is all of N. Let (π, V) be an irreducible unitary representation of G. If

$$\lim_{N'} \int_{N'} \pi(n)v\, dn = 0$$

for a vector $v \in V$, then (only temporarily) say that v is *P-cuspidal.* If every *smooth* vector in an irreducible unitary π is P-cuspidal for every parabolic P, then say (only temporarily) that π is *'para-supercuspidal'.*

It is a non-trivial result, due to HarishChandra (and recreated by Jacquet in a different circumstance) that for irreducible unitary representations *supercuspidal and para-supercuspidal are equivalent.*

As a paraphrase of the latter theorem, we see that always the *coefficient functions of supercuspidal representations are supercuspforms.*

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10. *HarishChandra's reduction to the supercuspidal case*

The argument of this section reduces the general question of admissibility of irreducible unitaries to a question of admissibility of irreducible *supercuspidal* representations. However, we must prove something stronger than just admissibility for supercuspidal representations: we must show that, as π varies over irreducible supercuspidal unitary representations, that

$$\sup_\pi \dim_\mathbb{C} \mathrm{Hom}_K(\rho, \pi|_K) < \infty$$

for every irreducible ρ of every compact open subgroup K. That is, we need to show *bounded multiplicity* of ρ in π as π varies over supercuspidal representations.

Note that supercuspidal representations, being square integrable (modulo the center), are admissible. This result holds for any (unimodular) totally disconnected group. Also, note that it suffices to consider only the case that K is a 'good' maximal compact subgroup, since for any other compact open subgroup Θ and a (necessarily finite-dimensional) irreducible ρ of Θ, $\mathrm{Ind}_\Theta^K \rho$ is finite-dimensional, etc.

Proof of reduction: We do induction on the *semisimple rank* $srk(G)$ of G. If this rank is 0 then G/Z is compact and an irreducible representation of G is necessarily finite-dimensional. The bounded multiplicity of ρ in any irreducible of G follows from this.

When $srk(G) > 0$ then G has a proper parabolic subgroup P. A Levi component M of P has $srk(M) < srk(G)$. Let N be the unipotent radical of P.

Let V_P be the subspace of W consisting of elements w so that for all $\theta \in N \cap K$

$$\rho(\theta)w = w$$

Let E_P be the orthogonal projection of W to W_P. For $m \in M \cap K$ define

$$\rho_M(m) = \rho(m) \circ e_P$$

Then ρ_M is a representation of $M \cap K$ upon V_P.

For $f \in \mathcal{H}(G,\rho)$ and $m \in M$ define

$$(\mu_P f)(m) = f^P(m) = \delta_P^{1/2}(m) \int_N f(mn) E_P \, dn$$

where δ_P is the modular function on P. Then $f^P \in \mathcal{H}(M,\delta_M)$. The 'obvious' integration-theory computation proves directly that the map

$$\mu_P : f \to f^P$$

is a *convolution-algebra homomorphism*

$$\mu_P : \mathcal{H}(G,\delta) \to \mathcal{H}(M,\delta_M)$$

(Indeed, it would appear that the reason HarshChandra used $\mathcal{H}(G,\rho)$ rather than considering Hecke algebras $\mathcal{H}(G,K')$ of *scalar-valued* functions for *smaller* compact open subgroups is that this convolution-algebra homomorphism gets lost).

The compact support and local constancy give a standard sort of result:

$$\mathcal{H}(M,\delta_M) \subset \mathcal{H}(M,\delta_0) \otimes \mathrm{End}(V_P)$$

where δ_0 is the trivial representation of a small-enough compact open subgroup K_0 of G inside K. By induction, $\mathcal{H}(M,\delta_0)$ is r-abelian for all sufficiently large r, and certainly $\mathrm{End}(W)$ is r-abelian for all sufficiently large r, so the tensor product is r-abelian for all sufficiently large r. Then $\mathcal{H}(M,\delta_M)$ is r-abelian for all sufficiently large r.

There are finitely-many G-conjugacy classes of parabolics, and it follows (via the Iwasawa decomposition) that there are finitely-many K-conjugacy classes of parabolics. We note that $r(P)$ is constant on K-conjugacy classes.

Fix $s \geq r(P)$ for all P. Take $f_1, \ldots, f_s \in \mathcal{H}(G,\delta)$ and put

$$\phi = [f_1, \ldots, f_s]$$

Then since $\mathcal{H}(M,\delta_M)$ is s-abelian *for every proper parabolic* we certainly have $\mu_P(\phi) = 0$ *for all* P if s is sufficiently large. That is, such s-commutators are $\mathrm{End}(W)$-valued *supercuspforms* for all s sufficiently large.

The HarishChandra's Corollary to Bernstein's Theorem (see the following section) asserts that the convolution algebra ${}^o\mathcal{H}(G,\delta)$ of $\mathrm{End}(W)$-valued supercuspforms is q-abelian for all sufficiently large q.

Thus, for all sufficiently large s and q, q-commutators of s-commutators of elements of $\mathcal{H}(G,\delta)$ are all 0. From this, by Godement's Key Lemma on these higher commutators, $\mathcal{H}(G,\delta)$ itself is r-abelian for all sufficiently large r. This completes HarishChandra's proof. ♣

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11. *HarishChandra's corollary to Bernstein's estimate*

The following theorem was proven by Bernstein about 1972. A crucial role is played by a clever auxiliary estimate due to Kazhdan. Also necessary is the older fact that *discrete series representations of unimodular totally disconnected groups are admissible.* HarishChandra's speculative development of representation theory under the assumption that someone would prove a theorem (such as Bernstein did) not only showed remarkable foresight, but also was indispensable to the Bernstein result.

Indeed, here we state as 'Bernstein's Theorem' the result which Bernstein literally proved, without invoking the prior work of HarishChandra. By 'HarishChandra's corollary' we mean the deductions made by HarishChandra from an estimate of the sort proven by Bernstein.

Let $\hat{Z}$ be the unitary dual of Z, i.e., the collection of unitary characters on Z. As usual, let ${}^o\mathcal{H}_\omega(G,K)$ be the collection of left and right K-invariant elements of ${}^oC_c^\infty(Z\backslash G,\omega)$.

Theorem (Bernstein): For any compact open subgroup K of G, the supremum of the dimensions of the *finite-dimensional* representations of $\mathcal{H}_\omega(G,K)$ is finite.

Proof: Below, in succeeding sections.

The first of the following two corollaries is HarishChandra's non-trivial deduction (in advance) from the result just stated. The second corollary, which is an elementary corollary of the first corollary, is what is actually needed in HarishChandra's induction argument of the previous section.

Corollary (HarishChandra): For any compact open subgroup K of G, the Hecke algebra ${}^o\mathcal{H}_\omega(G,K)$ is p-abelian for sufficiently large p.

Proof: From general 'abstract representation theory', one knows the following (often called the Gelfand-Raikov Theorem): Let G be a unimodular, locally compact, second countable topological group. Then, *given* $0 \neq f \in C_c^o(G)$, *there is an irreducible unitary representation* π *of* G *so that* $\pi(f) \neq 0$.

From this, we know that, given an element $f \in^o \mathcal{H}_\omega(G,K)$ there is an irreducible unitary representation π of G so that $\pi(f) \neq 0$. But unless π is supercuspidal we would have $\pi(f) = 0$, since by HairshChandra's work (mentioned above) supercuspidal representations are 'para-supercuspidal'. Thus, for $f \in^o \mathcal{H}_\omega(G,K)$ we conclude that there is a *supercuspidal* irreducible unitary representation π of G so that $\pi(f) \neq 0$.

By the definition, supercuspidal representations have coefficient functions which are compactly supported modulo the center (and are continuous), so are square-integrable modulo the center. That is, they are in the *discrete series.* Again from 'general representation theory', any discrete-series representation of a locally compact unimodular totally disconnected group *is admissible.* Thus, the associated representations of $\mathcal{H}_\omega(G,K)$ on the K-fixed vectors π^K in discrete series π are *finite-dimensional.* It also follows from the admissibility that π^K is an (algebraically) *irreducible* representation of $\mathcal{H}_\omega(G,K)$, for π a discrete series representation.

Then, by the Bernstein estimate, there is an absolute bound (depending only upon K) for the dimension of the K-fixed vectors in discrete series of G.

Thus, we have a family of finite-dimensional representations, of absolutely bounded dimension, which 'separate' elements of ${}^o\mathcal{H}_\omega(G,K)$. By the Godement principle, we conclude that ${}^o\mathcal{H}_\omega(G,K)$ is indeed p-abelian for sufficiently large p. ♣

Corollary (of corollary): Let (ρ, V) be a finite-dimensional representation of a compact open subgroup K of G. Then the convolution algebra ${}^oC_c^\infty(G,\rho)$ of $\mathrm{End}(W)$-valued supercuspforms is p-abelian for all sufficiently large p.

Proof of corollary of corollary: We assume the assertion of HarishChandra's corollary. For a sufficiently small compact open subgroup K_0 of $K \subset G$, $\rho|_{K_0}$ is a multiple of the trivial representation. Thus,

$$ {}^o\mathcal{H}(G,\delta) \subset^o \mathcal{H}(G,K_0) \otimes_{\mathbb{C}} \mathrm{End}(W) $$

Now for an irreducible (π, V) of G with central character ω, for $f \in \mathcal{H}(G)$ we have

$$ \pi(f) = \pi_\omega(f_\omega) $$

where

$$ f_\omega(g) = \int_Z \omega(z)\, f(zg)\, dz \in C_c^\infty(Z\backslash G, \omega^{-1}) $$

and for $v \in V$

$$ \pi_\omega(f_\omega)v = \int_G /Z\ f(\bar{g})\, \pi(\bar{g})v\, d\bar{g} $$

Take $f_1, \ldots, f_p \in^o \mathcal{H}(G, K_0)$. Then

$$\pi[f_1, \ldots, f_p] = [\pi_\omega f_{1,\omega}, \ldots, \pi_\omega f_{p,\omega}] = 0$$

for all p large-enough so that

$$p > \sup_{\omega \in \hat{Z}} \dim {}^oC_c^\infty(K_0 \backslash G / K_0, \omega)$$

Therefore, we conclude that ${}^o\mathcal{H}(G, \delta)$ is p-abelian for such p. This proves the corollary. ♣

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12. *Kazhdan's estimate on commutative subalgebras*

Kazhdan's estimate on commutative subalgebras of matrix algebras is essential in Bernstein's argument and of independent interest. The reader might imagine that a sharper estimate can be obtained, but it seems difficult to do so.

[12.0.1] **Lemma:** *(Kazhdan)* Let k be a field. A commutative k-subalgebra (with unit 1) $A = k[\alpha_1, \ldots, \alpha_\ell]$ of $\mathrm{End}_k(k^n)$ with ℓ (non-scalar) k-algebra generators $\alpha_1, \ldots, \alpha_\ell$ has

$$\dim_k A \;\le\; (n^2)^{1-2^{-\ell}}$$

Proof: The argument is by induction on ℓ. For $\ell = 1$, the assertion is that the dimension is at most n, which follows from elementary divisor theory. For $\ell > 1$, we do an induction on n. Let $\varphi(\ell, n)$ be the maximum dimension of commutative algebras A of $\mathrm{End}_k(k^n)$ with ℓ generators for given n, and let

$$f(\ell, n) \;=\; (n^2)^{1-2^{-\ell}}$$

If V decomposes as a direct sum of A-modules

$$V \;=\; V_1 \oplus V_2$$

with $\dim_k V_i = n_i$, then, by induction in n,

$$\dim_k A \;\le\; \varphi(\ell, n_1) + \varphi(\ell, n_2) \;\le\; f(\ell, n_1) + f(\ell, n_2) \;\le\; f(\ell, n)$$

with the last inequality following because the exponent $2(1 - 2^{-\ell})$ satisfies $2(1 - 2^{-\ell}) \ge 1$.

Therefore, to accomplish the induction step in n, it suffices to consider V which does *not* decompose properly as a direct sum of A-submodules. We assume $\dim_k A > f(\ell, n)$ and reach a contradiction.

When the A-module V does not properly as a direct sum of proper A-submodules, there is a k-algebra homomorphism $\lambda : A \to k$ and an integer N such that for all $\alpha \in A$ and for all $v \in V$

$$(\alpha - \lambda(\alpha))^N(v) \;=\; 0$$

Replace each of the (non-scalar) generators $\alpha_1, \ldots, \alpha_\ell$ of A by $\alpha_i - \lambda(\alpha_i)$, to assume that these generators are *nilpotent*.

We claim that, when V does not decompose as a direct sum of proper A-submodules,

$$\dim_k A \;\le\; \phi(\ell, [n - \frac{\phi(\ell, n)}{n}]) + \phi(\ell - 1, n)$$

where $[x]$ is the greatest integer less than or equal a real number x.

To prove this claim, let I be the ideal in A generated by the nilpotent generators $\alpha_1, \ldots, \alpha_\ell$, and fix a subspace L in V complementary to $I \cdot V$ with $m = \dim_k L$. Since $I^i L$ generates $I^i \cdot V$ modulo $I^{i+1} \cdot V$, necessarily $A \cdot L = V$.

Any $T \in A$ is necessarily determined by its values on L, because $AL = V$ and for $\alpha_i \in A$ and $v_i \in L$ we have

$$T(\sum_i \alpha_i v_i) \;=\; \sum_i \alpha_i \, T v_i$$

Thus,

$$\dim_k A \leq nm$$

and

$$n - m \;\leq\; \big[n - \frac{\phi(\ell, n)}{n}\big]$$

Let $A' = A \cdot \alpha_1$ and let A'' be the subalgebra generated by $\alpha_2, \ldots, \alpha_\ell$. Then $A = A' + A''$ and $\alpha \cdot V \subset I \cdot V$ gives

$$\begin{aligned} \dim_k A \;&\leq\; \dim_k A' + \dim_k A'' \;\leq\; \dim_k A\big|_{I \cdot V} + \dim_k A'' \;\leq\; \varphi(\ell, n-m) + \varphi(\ell - 1, n) \\ &=\; \phi(\ell, \big[n - \frac{\phi(\ell,n)}{n}\big]) + \phi(\ell - 1, n) \end{aligned}$$

proving the claim.

Now we can reach our contradiction to the assumption that $\dim_k A > f(\ell, n)$. Using the assertion of the claim, applying the induction hypothesis,

$$\begin{aligned} f(\ell, n) \;<\; \dim_k A \;\leq\; f(\ell, \big[n - \frac{\phi(\ell,n)}{n}\big]) + f(\ell-1, n) \;&\leq\; f(\ell, n - \frac{\phi(\ell,n)}{n}) + f(\ell - 1, n) \\ &\leq\; f(\ell, n - \frac{f(\ell,n)}{n}) + f(\ell - 1, n) \end{aligned}$$

because f is monotone increasing in its second argument, and because of the assumption

$$f(\ell, n) \;<\; \dim_k A \;\leq\; \varphi(\ell, n)$$

Abbreviating $\varepsilon = 2^{1-\ell}$, the inequality we've apparently obtained

$$f(\ell, n) \;<\; f(\ell, n - \frac{f(\ell,n)}{n}) + f(\ell - 1, n)$$

is written more explicitly as

$$n^{2-\varepsilon} \;<\; (n - n^{1-\varepsilon})^{2-\varepsilon} + n^{2-2\varepsilon}$$

Removing a factor of $n^{2-\varepsilon}$ from both sides would give

$$1 \;<\; (1 - n^{-\varepsilon})^{2-\varepsilon} + n^{-\varepsilon}$$

or

$$1 - n^{-\varepsilon} \;<\; (1 - n^{-\varepsilon})^{2-\varepsilon}$$

This last inequality is impossible, because $2 - \varepsilon \geq 1$. Thus, the hypothesis that $\dim_k A > f(\ell, n)$ leads to a contradiction. This completes the induction step in n for fixed ℓ, and thereby completes the induction in ℓ. ///

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13. *Proof of Bernstein's estimate*

Proof of Theorem: More generally, suppose that we have a locally compact topological group G with a decomposition

$$G = K_0 A^+ \Omega Z K_0$$

where A^+ is a finitely-generated commutative semi-group with unit, Ω is a finite set, K_0 is a compact open subgroup, and Z is a closed subgroup of G contained in the center of G. (Here a *semi-group* is simply a set with an associative binary operation.)

Let $K \subset K_0$ be a small-enough compact open subgroup so that K has an 'Iwahori factorization'

$$K = \Gamma^- \cdot \Gamma^+ = \Gamma^+ \cdot \Gamma^-$$

where for all $a \in A^+$

$$a\Gamma^- a^{-1} \subset \Gamma^-$$
$$a^{-1}\Gamma^+ a \subset \Gamma^+$$

The Bruhat-Tits structure theory referred to as *buildings and BN-pairs* assures that these hypotheses apply to p-adic reductive groups. Further, for classical groups, these hypotheses can be verified directly.

The following proposition's conclusion is of the sort we want. Therefore, our labor will be to see that the hypotheses of this proposition apply to the Hecke algebras $\mathcal{H}(G, K)$.

Proposition: Let $\mathcal{L}$ be an associative algebra with unit, let $\mathcal{A} \supset \mathcal{Z}$ be subalgebras with $\mathcal{Z}$ inside the center of $\mathcal{L}$. Take $A_1, \ldots, A_\ell \in \mathcal{A}$ and $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \mathcal{L}$. Suppose further that $\mathcal{A}$ is commutative, generated by the A_i's and by $\mathcal{Z}$, and suppose that

$$\mathcal{L} = \sum_{i,j} X_i \mathcal{A} Y_j$$

Then every irreducible finite-dimensional representation of $\mathcal{L}$ has dimension $\leq (pq)^{(2^{\ell-1})}$. That is, there is an absolute bound on the dimension of finite-dimensional irreducibles.

Proof of proposition: Let $\rho : \mathcal{L} \to \mathrm{End}(\mathbb{C}^n)$ be irreducible. Then the finite-dimensional Schur's Lemma implies that $\rho(\mathcal{Z}) = \mathbb{C}$, since $\rho(\mathcal{Z})$ is central. And, by Burnside's theorem from the elementary theory of semisimple modules, $\rho(\mathcal{L}) = \mathrm{End}(\mathbb{C}^n)$. In particular,

$$\dim \rho(\mathcal{L}) = n^2$$

On the other hand, by Kazhdan's Lemma,

$$\dim \mathcal{A} \leq n^{2-2^{1-\ell}}$$

Then certainly

$$n^2 = \dim \rho(\mathcal{L}) \leq pqn^{2-2^{1-\ell}}$$

from which we have

$$1 \leq pqn^{-2^{1-\ell}}$$

or, taking $2^{\ell-1}$ powers,

$$n \leq (pq)^{2^{\ell-1}}$$

This proves the proposition. ♣

Lemma: Let $g, h \in G$. If either g or h normalizes K, or if $g, h \in A^+$, then

$$KgK \cdot KhK = KghK$$

Proof: When one of g, h normalizes K the assertion is immediate. For $g, h \in A^+$, we use the 'Iwahori decomposition' to compute:

$$KgK \cdot KhK = Kg\Gamma^-\Gamma^+hK = K(g\Gamma^-g^{-1})gh(h^{-1}\Gamma^+h)K \subset KghK$$

as desired. ♣

Let $\mathcal{H}(G,K) = C_c^\infty(K\backslash G/K)$ be the usual Hecke algebra 'at level K', i.e., the collection of left and right K-invariant compactly-supported $\mathbb{C}$-valued functions on G. The multiplication in this ring is convolution on G. Let $\mathcal{Z}$ be the subalgebra of $\mathcal{H}(G,K)$ generated by the characteristic function of sets of the form KzK for $z \in Z$. Let $\mathcal{A}$ be the subalgebra of $\mathcal{H}(G,K)$ generated by the characteristic functions of the sets $K\alpha zK$ for $\alpha \in A^+$, $z \in Z$.

First, we consider *the simple case that every element of* Ω *normalizes* K. Note that, in fact, for 'generic' reductive groups over local fields, we can take $\Omega = \{1\}$.

For representatives x_i of K_0/K, let $X_i \in \mathcal{H}(G,K)$ be the characteristic function of Kx_iK. For representatives y_i of $\Omega K_0/K$, let $Y_i \in \mathcal{H}(G,K)$ be the characteristic function of Ky_iK.

Corollary of Lemma: We have

$$\mathcal{H}(G,K) = \sum_{i,j} X_i \mathcal{A} Y_j$$

Proof: The level-K Hecke algebra $\mathcal{H}(G,K)$ has $\mathbb{C}$-basis consisting of the characteristic functions of sets KgK for $g \in G$. In light of the postulated decomposition of G, we need to show that the characteristic function of every set $Ka\omega zK$ is in the sum on the right-hand side. For $\alpha \in A^+$ and $z \in Z$, with x_i and y_j as just above, we have

$$Kx_iK \cdot K\alpha K \cdot KzK \cdot Ky_jK = Kx_i\alpha zy_jK$$

by the previous lemma.

To compare this with the outcome of convolution multiplication, note that the convolution of two elements of $\mathcal{H}(G,K)$ does indeed lie in $\mathcal{H}(G,K)$ again, so is a linear combination of characteristic functions of sets of the form KgK. If $g, h \in g$ have the special property that

$$KgK \cdot KhK = KghK$$

then the convolution of the characteristic functions of KgK and KhK is simply a constant multiple of the characteristic function of $KghK$. It is easy to check that it is not the zero multiple. Thus, we do obtain the whole level-K Hecke algebra as indicated. ♣

Now drop the assumption that Ω *normalizes* K. Let the y_j be representatives for $K\Omega K_0/K$, and let M be the functions in $\mathcal{H}(G,K)$ with support inside $KA^+ZK\Omega K_0$. Then M is a left $\mathcal{A}$-module and

$$\mathcal{H}(G,K) = \sum_i X_i M$$

Granting this, the following lemma directly implies that $\mathcal{H}(G,K)$ meets the hypotheses of the proposition, as desired.

Lemma: M as just above is a *finitely-generated* $\mathcal{A}$-module.

Proof: For $\alpha \in A^+$, let

$$\Gamma^+_\alpha = \alpha^{-1}\Gamma^+\alpha \subset \Gamma^+$$

As in the Lemma just above, for $\alpha, \beta \in A^+$ we compute

$$K\alpha K\beta y_jK = K\alpha\Gamma^-\Gamma^+\beta y_jK =$$

$$= K(\alpha\Gamma^-\alpha^{-1})\alpha\beta\beta^{-1}\Gamma^+\beta y_j K =$$
$$= K\alpha\beta K\Gamma^+_\beta y_j K$$

Similarly,

$$K\alpha\beta y_j K = K\Gamma^+\alpha\beta y_j K = K\alpha\beta\Gamma^+_{\alpha\beta} y_j K$$

Thus, if

$$\Gamma^+_{\alpha\beta} y_j K = \Gamma^+_\beta y_j K$$

then

$$K\alpha K \cdot K\beta y_j K = K\alpha\beta y_j K$$

As discussed in a different situation above, such an equality of sets implies an equality of convolutions of the characteristic functions, up to a non-zero constant.

For a subset Θ of K, let

$$\nu(\Theta) = \sum_j \mathrm{card}(\Theta y_j K/K)$$

If $\Theta' \subset \Theta$, then certainly

$$\nu(\Theta') \leq \nu(\Theta)$$

And if $\Theta' \subset \Theta$ and

$$\nu(\Theta') = \nu(\Theta)$$

then for all j we have

$$\Theta' y_j K = \Theta y_j K$$

Let $a_1, \ldots, a_\ell$ be the generators for $\mathcal{A}$. Let $D = D_\ell$ be the *'ℓ-dimensional integer orthant'*, i.e., the set of ℓ-tuples of non-negative integers. For $u = (u_1, \ldots, u_\ell) \in D$, put

$$a^u = \Pi_i \, a^{u_i}$$

Also, let

$$f(u) = \mathrm{card}(\Gamma^+_{a^u})$$

Write $u < v$ for $u, v \in D$ if $u \neq v$ and if $v - u \in D$.

If $v < u$ then $f(v) \geq f(u)$. If we further assume that $f(u) = f(v)$, then for every index j

$$\Gamma^+_{a^u} y_j K = \Gamma^+_{a^v} y_j K$$

That is, the characteristic functions of the sets $\Gamma^+_{a^u} y_j K$ lie in the $\mathcal{A}$-module generated by the characteristic functions of the sets $\Gamma^+_{a^v} y_j K$. The characteristic functions of the sets $\Gamma^+_{a^w} y_j K$ as j varies and $w \in D$ varies generate M as a $\mathcal{Z}$-module so we can choose as $\mathcal{A}$-module generators for M just the characteristic functions of the sets $\Gamma^+_{a^w} y_j K$ with w *'singular'*, in the sense that for all $v < w$ we have a *strict* inequality $f(v) > f(w)$.

Thus, we wish to prove:

Combinatorial Lemma: Let f be a non-negative integer valued function on D_ℓ. Show that there are only finitely-many singular points (in the sense just defined).

Proof: If w is singular, then $f(w) < f(0)$. By induction on $f(0)$, in $w + D$ there are only finitely-many singular points. Since $D - (w + D)$ is a finite union of copies of $(\ell - 1)$-dimensional integer orthants. By induction on ℓ, there are only finitely-many 'singular points' in each of these. ♣

Therefore, we can conclude that M is finitely-generated as an $\mathcal{A}$-module, so the level-K Hecke algebra $\mathcal{H}(G, K)$ satisfies the hypotheses of the proposition, and we see that there is an absolute bound on finite-dimensional irreducible representations of this Hecke algebra. This finishes the proof of Bernstein's estimate. ♣