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Pseudo-cuspforms, pseudo-Laplacians

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

This is an elaboration of parts of [Hejhal 1981][1] and [CdV 1983], with some example computations included for convenience. [2]

[0.1] Pseudo-cuspforms

Pseudo-cuspform is a $\mathbb{C}$-valued function f meeting *relaxed* versions of defining features or provable properties of genuine cuspforms. In the simplest case, namely, $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half-plane $\mathfrak{H}$, with invariant Laplacian $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$,

$$\begin{cases} \text{exponential decay:} & |f(x+iy)| \ll e^{-\varepsilon y} & (\text{for some } \varepsilon > 0, \text{ as } y \to +\infty) \\ \textit{eventually}\text{-vanishing constant term:} & \displaystyle\int_0^1 f(x+iy)\,dx = 0 & (\text{for all } y > a, \text{ for some } a > 0) \\ \textit{eventually } \Delta\text{-eigenfunction} & (\Delta - \lambda)f = 0 & (\text{for all } y > a, \text{ for some } a > 0) \end{cases}$$

The eventual-eigenfunction condition can be usefully recast as a differential equation

$$(\Delta - \lambda)f \;=\; T$$

for a Γ-invariant distribution T on $\mathfrak{H}$, compactly-supported on $\Gamma\backslash\mathfrak{H}$. The three main examples for T are described in detail below. That is, a slight *roughness* is tolerated in f at the support of the Γ-invariant distribution T. Three natural types of target distributions T are: (i) T is supported at a single point on $\Gamma\backslash\mathfrak{H}$, (ii) T is supported on a closed geodesic on $\Gamma\backslash\mathfrak{H}$, (iii) T evaluates the constant term at $y = a$. [3] Constructions are given below. Say that f is a *pseudo-eigenfunction* for Δ, and the corresponding λ is a *pseudo-eigenvalue*.

[0.2] Comparison to genuine cuspforms

By definition, cuspforms' constant terms vanish *entirely*, and the eigenfunction condition is *exact*, rather than tolerating any non-trivial leftover T. Exponential decay of genuine cuspforms is a provable consequence of the vanishing constant term, the eigenvector property, and L^2-ness.

As we see below, the most immediate deficiency in the definition of *pseudo*-cuspforms is the tolerated *roughness*, weakening the eigenfunction property, sabotaging arguments based on symmetry or self-adjointness, obstructing proof that the pseudo-eigenvalues are *real* and *non-positive*.

In larger groups, [Borcherds 1998] gave natural constructions of some automorphic forms with singularities.

[0.3] A flawed numerical procedure and visions of the Riemann Hypothesis

Pseudo-cuspforms made a dramatic appearance in an episode during which it briefly seemed that zeros s of zeta or of certain L-functions might give eigenvalues $\lambda = s(s-1)$ of Δ on $\Gamma\backslash\mathfrak{H}$, thus giving an approach

[1] As of June, 2001, the paper [Hejhal 1981] is not listed on MathSciNet.

[2] Thanks to E. Bombieri for some corrections, comments, and additions.

[3] These examples have a common group-theoretic structure, namely, the support of the distribution is essentially the orbit of an orthogonal group. The case of single points corresponds to *definite* groups $SO(2)$, and the closed geodesic case corresponds to $\mathbb{Q}$-anisotropic $SO(1,1)$. In both cases, arithmetic complications arise, as below.

to proving the Riemann Hypothesis: if $\lambda \in \mathbb{R}$ and $\lambda \leq 0$, then either $\mathrm{Re}(s) = \frac{1}{2}$ or $s \in [0,1]$. However, it turned out that these were *pseudo-cuspforms*, not cuspforms. We recall some points of this story recounted in [Hejhal 1981].

H. Haas, in his *diplomarbeit* [Haas 1977] under H. Neunhöffer at Heidelberg, numerically computed eigenvalues λ of the invariant Laplacian on $\Gamma\backslash\mathfrak{H}$. The eigenvalues were naturally parametrized as $\lambda = s(s-1)$, and Haas listed the s-values.

After some time, A. Terras in San Diego acquired a copy, and H. Stark observed several zeros s of $\zeta(s)$ in Haas' list. Soon after, D. Hejhal observed some zeros s of $L(s,\chi)$ in Haas' list, with χ the non-trivial Dirichlet character mod 3.

Naturally, people wondered whether all zeros s of zetas might appear among the s-parameters for eigenvalues, proving that $s(s-1)$ was real and non-positive, proving that s was either on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ or on the interval $[0,1]$, essentially proving the Riemann Hypothesis and more.

Hejhal attempted to reproduce Haas' numerical results, using more scrupulous numerical procedures. Strangely, the zeros of $\zeta(s)$ and $L(s,\chi)$ *failed* to appear on Hejhal's list, leaving the mystery of the appearance of these values in Haas' list. That is, flaws in numerical procedures do not routinely produce zeros of zeta as garbage outputs.

First, Hejhal found a specific flaw in Haas' procedure: Haas had inadvertently allowed some *non-smoothness* of purported eigenfunctions at the corners of the usual fundamental domain for Γ on $\mathfrak{H}$, that is, at the cube root and sixth root ω of unity, at the corners of the usual fundamental domain for $\Gamma = SL_2(\mathbb{Z})$. More details are in [Hejhal 1981].

Second, Hejhal accounted for the appearance of the spurious eigenvalues by observing that Haas' effectively solved the more tolerant equation $(\Delta-\lambda)u = \delta_\omega^{\mathrm{afc}}$, allowing an automorphic Dirac delta at ω on the right-hand side. This is the differential equation for an automorphic *fundamental solution* for $\Delta - \lambda$, or, equivalently, an automorphic *resolvent kernel* $G_s(z,w)$ evaluated at $w = \omega$, with $\lambda = s(s-1)$. By 1979, such fundamental solutions/resolvents were understood: [Neunhöffer 1973], [Niebur 1973], [Fay 1977]. In particular, for $y \gg 1$ the *constant term* of $G_s(z,\omega)$ contains a factor $\zeta(s)L(s,\chi)$, at whose zeros s_j the value $G_{s_j}(z,\omega)$ is a *pseudo-cuspform*. Thus, the spurious eigenvalues $\lambda = s_j(s_j-1)$ are *pseudo-eigenvalues*, corresponding to zeros of $\zeta(s)L(s,\chi)$, which is $\zeta_{\mathbb{Q}(\omega)}(s)$.

However, since pseudo-cuspforms are not *genuine* eigenfunctions for Δ, this does *not* necessarily imply that $\lambda = s_j(s_j-1)$ is real and non-positive.

Further, Hejhal noted examples in which vanishing of a constant term was equivalent to vanishing of *Epstein zeta functions* $E_s(\sqrt{-D})$ lacking Euler products, and *known* to have off-line zeros: [Potter-Titchmarsh 1935] [4] showed that, for example, $E_s(\sqrt{-5})$ has zeros off the critical line. For $0 < D \in \mathbb{Z}$ square-free, where the algebraic integers of $\mathbb{Q}(\sqrt{-D})$ have class number greater than 1, [Davenport-Heilbronn 1936] demonstrated the infinitude of zeros of $E_s(\sqrt{-D})$ in $\mathrm{Re}(s) > 1$ arbitrarily close to $\mathrm{Re}(s) = 1$. [Voronin 1976] demonstrates off-line zeros in $\frac{1}{2} < \mathrm{Re}(s) < 1$, as well.

Thus, as Hejhal commented, it would be unreasonable to try to prove that these demonstrably off-line zeros were on the critical line.

[0.4] L^2-ness implies exponential decay

Pseudo-cuspforms arise as functions u in $L^2(\Gamma\backslash\mathfrak{H})$ satisfying, on some set $y > a$, the eigenvalue property $(\Delta - \lambda)u = 0$ and constant-term vanishing $c_P u = 0$. These imply exponential decay, as follows.

[4] As of June, 2011, this paper of Potter and Titchmarsh is not listed on MathSciNet.

Let S be the Siegel set

$$S \;=\; \{x+iy \;:\; a<y,\ |x|\le \tfrac{1}{2}\}$$

Since S essentially injects to the quotient, square-integrability on $\Gamma\backslash\mathfrak{H}$ implies square-integrability on S. On S, there is the Fourier expansion

$$u(x+iy) \;=\; \sum_{n\in\mathbb{Z}} e^{2\pi inx}\, W_n(y)$$

and by separation of variables W_n satisfies

$$W_n'' - (4\pi^2n^2 + \frac{\lambda}{y^2})W_n \;=\; 0$$

Obviously $W_1(|n|y)$ is a solution of the n^{th} equation when W_1 is a solution of the first. For $n\neq 0$, there is a unique solution having asymptotic [5] $e^{-2\pi ny}(1+O(1/y))$. All other solutions apart from multiples of this have the asymptotic $e^{2\pi ny}(1+O(1/y))$. The square-integrability on S implies first that

$$u(x+iy) \;=\; \sum_{n\in\mathbb{Z}} e^{2\pi inx}\, c_n\, W_1(|n|y)$$

for $W_1(y)$ the solution asymptotic to $e^{-2\pi y}(1+O(1/y))$, with some constants c_n, and then

$$\sum_{n\neq 0} |c_n|^2 \cdot \int_a^\infty \big|W_1(|n|y)\big|^2\, \frac{dy}{y^2} \;<\; \infty$$

From $W_1(y) = e^{-2\pi y}(1+O(1/y)$, using the lower bound on y, this gives

$$\sum_{n\neq 0} |c_n|^2 \cdot \int_a^\infty e^{-4\pi ny}\Big(1-\frac{C}{|n|y}\Big)\, \frac{dy}{y^2} \;<\; \infty$$

for some C, and then

$$\sum_{n\neq 0} |c_n|^2 \cdot \int_a^\infty e^{-4\pi ny}\, \frac{dy}{y^2} \;<\; \infty$$

from which

$$\sum_{n\neq 0} |c_n|^2 \cdot e^{-4\pi na} \int_a^\infty \frac{dy}{y^2} \;<\; \infty$$

the easy bound $|c_n| = O(e^{2\pi a|n|})$. Then

$$|u(x+iy)| \;\le\; \sum_{n\neq 0} |c_n|\cdot\big|W_1(|n|y)\big| \;\ll\; \sum_{n\neq 0} e^{2\pi a|n|}\, e^{-2\pi|n|y} \;\ll\; e^{-2\pi y}$$

[0.5] Constant terms of solutions to automorphic differential equations

It is not obvious that solutions to automorphic differential equations $(\Delta-\lambda)u = T$ have intelligible constant terms. We give a heuristic for such a computation, and then explain how to make the discussion rigorous in the context of automorphic Sobolev spaces. [6] The main idea is to use the automorphic spectral expansion, and evaluate the continuous-spectrum part by residues.

[5] These functions are standard, classical special functions, In any case, this differential equation has an *irregular* singular point at infinity, of a sort admitting asymptotic expansions of solutions. See [Garrett 2011b].

[6] Nothing surprising happens in setting up global automorphic Sobolev spaces: see [Garrett 2010] for working-out of some of the details.

Let T be a Γ-invariant distribution on $\mathfrak{H}$, compactly supported on $\Gamma\backslash\mathfrak{H}$. In some $L^2(\Gamma\backslash\mathfrak{H})$-Sobolev space, it makes sense to write

$$T \;=\; \sum_F T\overline{F}\cdot F \;+\; \frac{T1\cdot 1}{\langle 1,1\rangle} \;+\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} TE_{1-s}\cdot E_s\; ds \qquad \text{(in a Sobolev space)}$$

where F ranges over an orthonormal basis for cuspforms and E_s is the usual Eisenstein series. Solving the differential equation $(\Delta-\lambda)u_w = T$ with $\lambda=\lambda_w = w(w-1)$ gives

$$u_w \;=\; \sum_F \frac{T\overline{F}\cdot F}{\lambda_F-\lambda_w} \;+\; \frac{T1\cdot 1}{-\lambda_w\cdot\langle 1,1\rangle} \;+\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{TE_{1-s}\cdot E_s}{\lambda_s-\lambda_w}\; ds \qquad \text{(in a Sobolev space)}$$

where λ_F is the eigenvalue of F. Taking the constant term gives

$$c_P u_w \;=\; \frac{T1\cdot 1}{-\lambda_w\cdot\langle 1,1\rangle} \;+\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{TE_{1-s}\cdot (y^s+c_s y^{1-s})}{\lambda_s-\lambda_w}\; ds \qquad \text{(in a Sobolev space)}$$

The functional equation $c_s E_{1-s}=E_s$ gives $TE_{1-s}\cdot c_s y^{1-s} = TE_s\cdot y^{1-s}$. Replacing s by $1-s$ in the second summand gives another copy of the first, so

$$c_P u_w \;=\; \frac{T1\cdot 1}{-\lambda_w\cdot\langle 1,1\rangle} \;+\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{TE_{1-s}\cdot y^s}{\lambda_s-\lambda_w}\; ds \qquad \text{(in a Sobolev space)}$$

Under various mild hypotheses on T, for $y>a$ with a large, depending on T, the contour can be moved to the left, capturing residues in the left half-plane. The pole of E_{1-s} at $s=0$ is exactly $-1/\langle 1,1\rangle$, so that residue of $TE_{1-s}/(\lambda_s-\lambda_w)$ cancels $T1/(-\lambda_w\cdot\langle 1,1\rangle)$, and we are left with the residue at $s=1-w$:

$$c_P u_w \;=\; \frac{TE_w\cdot y^{1-w}}{1-2w} \qquad \text{(for large } y\text{, under mild hypotheses)}$$

The solution u_w has poles at $\lambda_F=\lambda_w$, with residues $T\overline{F}\cdot F$, which may be 0. Away from such poles, if any, and for $y>a$ for large-enough a, the vanishing $TE_w=0$ gives vanishing of the constant term. We treat three examples.

[0.6] Example: constant term of fundamental solutions

The automorphic distribution $Tf = f(\omega)$ as in [Hejhal 1981] gives $TE_s = 3(\sqrt{3}/2)^s\zeta_k(s)/\zeta(2s)$, where $k=\mathbb{Q}(\omega)$, and

$$c_P u_w \;=\; \frac{1}{-\lambda_w\cdot\langle 1,1\rangle} \;+\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{3(\sqrt{3}/2)^{1-s}\zeta_k(1-s)\cdot y^s}{\zeta(2-2s)\,(\lambda_s-\lambda_w)}\; ds$$

For $y>\sqrt{3}/2$, the contour can be shifted to the left, with auxiliary estimates provided by the convexity bound on $\zeta_k(s)$. In the half-plane $\mathrm{Re}(s)<\frac{1}{2}$, the zeta function $\zeta_k(1-s)$ has a pole at $s=0$, and the resulting residue is cancelled by the constant term of $1/(-\lambda_w\cdot\langle 1,1\rangle)$. For $\mathrm{Re}(w)>\frac{1}{2}$ the denominator $\lambda_s-\lambda_w = (s-w)(s-(1-w))$ gives a pole at $s=1-w$, so

$$c_P u_w \;=\; \frac{3(\sqrt{3}/2)^w\cdot\zeta_k(w)\cdot y^{1-w}}{\zeta(2w)\cdot(1-2w)} \qquad \text{(for } y>\sqrt{3}/2\text{)}$$

This obviously *meromorphically continues* in w, to include the line $\mathrm{Re}(w)=\frac{1}{2}$. The (meromorphically-continued) function u_w has constant term vanishing in $y>\sqrt{3}/2$ exactly when $\zeta_k(w)=0$. Any such connection with zeros of zetas is provocative.

[0.7] Example: constant term related to closed geodesics

With

$$H \;=\; \{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \;:\; a^2 - 2b^2 = 1\}$$

the presence of non-trivial units in $\mathbb{Z}[\sqrt{2}]$ means that orbits

$$(\Gamma \cap H)\backslash H \cdot z_o \;\subset\; \Gamma\backslash\mathfrak{H}$$

are *closed* (geodesics). The functional on automorphic forms computing *periods* along this closed geodesic is an automorphic distribution

$$Tf \;=\; \int_{(\Gamma\cap H)\backslash H} f(h\cdot i)\, dh$$

with compact support. [7] Letting $k = \mathbb{Q}(\sqrt{2})$, for some positive constant

$$TE_s \;=\; (\mathrm{const})^s \cdot \frac{\xi_k(s)}{\xi(2s)}$$

where $\xi_k(s)$ is the completed zeta function. [8] Using the duplication formula,

$$TE_s \;=\; C^s \cdot \frac{\Gamma(\frac{s}{2})\,\zeta_k(s)}{\Gamma(\frac{s+1}{2})\,\zeta(2s)}$$

for an elementary constant C. Recall [9] $\Gamma(z)/\Gamma(z+\frac{1}{2}) \sim 1/\sqrt{z}$. The solution u_w to

$$(\Delta - \lambda_w)u_w \;=\; T$$

has constant term computed as outlined above:

$$c_P u_w \;=\; \frac{TE_w \cdot y^{1-w}}{1-2w} \;=\; C^{-w} \cdot \frac{\Gamma(\frac{w}{2})\,\zeta_k(w)}{\Gamma(\frac{w+1}{2})\,\zeta(2w)} \qquad\qquad (\text{for } y > C \text{ and } \mathrm{Re}(w) > \tfrac{1}{2})$$

Again, this has the obvious meromorphic continuation to include $\mathrm{Re}(w) = \frac{1}{2}$. Away from poles caused by equalities $\lambda_F = \lambda_w$ with cuspforms F, the constant term vanishes at zeros of $\zeta_k(w)$.

[0.8] Example: constant term related to evaluation of constant term (!)

The automorphic distribution

$$Tf \;=\; (c_P f)(ia)$$

was significant in [CdV 1981,82,83]. The solution u_w to

$$(\Delta - \lambda_w)u_w \;=\; T$$

[7] In addition to values at CM-points, Hecke and Maaß had computed the periods of Eisenstein series along these closed geodesics. [Garrett 2009], for example, recalls these and some other well-known period computations, executing them in the context of Iwasawa-Tate theory.

[8] The same result applied to $k = \mathbb{Q}(\omega)$, but in the latter case the gamma factors cancelled, due to the duplication formula $\Gamma(s) = \pi^{-1/2}2^{2s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$.

[9] This asymptotic follows from the Laplace-Stirling formula, but can also be obtained simply, via Watson's lemma, as in [Garrett 2007].

has constant term computed by a small variation of the earlier outline, as follows. The spectral expansion is

$$T \;=\; \sum_F TF\cdot F + \frac{T1\cdot 1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} TE_{1-s}\cdot E_s\;ds \;=\; \frac{1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (a^{1-s}+c_{1-s}a^s)\cdot E_s\;ds$$

Thus, as in general,

$$u_w \;=\; \frac{1}{\langle 1,1\rangle\cdot(\lambda_1-\lambda_w)} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(a^{1-s}+c_{1-s}a^s)\cdot E_s\;ds}{\lambda_s-\lambda_w}$$

and

$$Tu_w \;=\; \frac{1}{\langle 1,1\rangle\cdot(\lambda_1-\lambda_w)} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(a^{1-s}+c_{1-s}a^s)\cdot(y^s+c_sy^{1-s})\;ds}{\lambda_s-\lambda_w}$$

A potential problem is the fact that c_s has many poles in $\mathrm{Re}(s)<\frac{1}{2}$, and the integrand has both c_s and c_{1-s}. Fortunately, $c_sc_{1-s}=1$. Further, we can move the contour for *one* part of the integral to the *left*, and the contour for the *other* part of the integral to the *right*:

$$c_Pu_w \;=\; \frac{1}{\langle 1,1\rangle\cdot(\lambda_1-\lambda_w)} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{(a^{1-s}+c_{1-s}a^s)\cdot(y^s+c_sy^{1-s})\;ds}{\lambda_s-\lambda_w}$$

$$=\; \frac{1}{\langle 1,1\rangle\cdot(\lambda_1-\lambda_w)} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{a^{1-s}y^s+c_{1-s}a^sy^s+c_sa^{1-s}y^{1-s}+a^sy^{1-s}\;ds}{\lambda_s-\lambda_w}$$

$$=\; \frac{(a^w+c_wa^{1-w})y^{1-w})}{2((1-w)-w)} - \frac{(c_wa^{1-w}+a^w)y^{1-w}}{2(w-(1-w))} \;=\; \frac{(a^w+c_wa^{1-w})y^{1-w}}{1-2w}$$

This is for $y>a$, $y>a^{-1}$ and $\mathrm{Re}(w)>\frac{1}{2}$.

Again, u_w has a meromorphic continuation to include $\mathrm{Re}(w)=\frac{1}{2}$, now with no concern for poles due to cuspforms, since T annihilates them. Given a, when w is chosen such that $a^w+c_wa^{1-w}=0$, the constant term of u_w vanishes for $y>a$ and $y>a^{-1}$. Although $c_w=\xi(2w-1)/\xi(2w)$, this vanishing condition is not as provocative as those more obviously connected to vanishing of zeta functions.

[0.9] If only pseudo-eigenvalues were genuine eigenvalues

Thus, pseudo-cuspforms with pseudo-eigenvalues $\lambda=\lambda_w=w(w-1)$ exist for w a zero of $\zeta_k(w)$, with $k=\mathbb{Q}(\omega)$, for example. Thus, again, *if* we could prove that $\lambda\in\mathbb{R}$ and $\lambda\le 0$, as though pseudo-cuspforms were *genuine* eigenvectors, *then* we would conclude that w lies on the critical line $\mathrm{Re}(w)=\frac{1}{2}$, or on the interval $[0,1]$. [10]

However, the slight roughness sabotages straightforward arguments applicable to genuine eigenfunctions of symmetric or self-adjoint operators. We *cannot* immediately conclude that $\lambda\in\mathbb{R}$ and $\lambda\le 0$. With $\lambda=w(w-1)$, we *cannot* readily conclude that w is on the union of the critical line $\mathrm{Re}(w)=\frac{1}{2}$ or interval $[0,1]$. This is another obstacle to using pseudo-cuspforms to try to prove the Riemann Hypothesis.

[0.10] Another objection: off-line zeros of $s\to E_s(\sqrt{-D})$

In the family of cases $(\Delta-\lambda)u=\delta_{z_o}^{\mathrm{afc}}$, that is, when the tolerated roughness is at a single point z_o on $\Gamma\backslash\mathfrak{H}$, [Hejhal 1981] notes that it is *unreasonable* to expect to prove that all pseudo-eigenvalues $\lambda=s(s-1)$ of pseudo-cuspforms correspond to s on the critical line, because for most $z_o\in\mathfrak{H}$ the $s\to E_s(z_o)$ is *not*

[10] Even if this device were to succeed, it would not exclude *Siegel zeros*, that is, zeros on $(\frac{1}{2},1]$.

expected to satisfy a Riemann Hypothesis, and in many examples, these functions are *known* to have zeros off the critical line, (and off $[0,1]$, as well).

We review some facts from [Stark 1967]. [Potter-Titchmarsh 1935] showed that, for example, $E_s(\sqrt{-5})$ has zeros off the critical line. For $0 < D \in \mathbb{Z}$ square-free, where the algebraic integers of $\mathbb{Q}(\sqrt{-D})$ have class number greater than 1, [Davenport-Heilbronn 1936] demonstrated the infinitude of zeros of $s \to E_s(\sqrt{-D})$ in $\mathrm{Re}(s) > 1$ arbitrarily close to $\mathrm{Re}(s) = 1$.

A weak counter-objection is that $E_s(z_o)$ is not *natural* unless z_o is quadratic over $\mathbb{Q}$ and $\mathbb{Z}[z_o]$ has class number one. Rather, one should take the *linear combinations* of values $E_s(w_i)$ corresponding to an adelic period of E_s over a subgroup given by a rationally imbedded copy of $\mathbb{Q}(\sqrt{-D})^\times$, giving $\zeta_{\mathbb{Q}(\sqrt{-D})}(s)/\zeta(2s)$. However, there is no palpable compulsion to do so, and off-line zeros of $E_s(\sqrt{-D})$ exist.

[0.11] Pseudo-Laplacians

The objection that pseudo-eigenvalues are not genuine eigenvalues of self-adjoint operators can be partly overcome, by consideration of variants of the usual (graph-closure of) the Laplacian. These are Friedrichs' self-adjoint extensions $\tilde{\Delta}_a$ of *restrictions* Δ_a of the Laplacian Δ to subspaces $L^2(\Gamma\backslash\mathfrak{H})_a$ of $L^2(\Gamma\backslash\mathfrak{H})$ consisting of automorphic forms with constant terms vanishing above $y = a$.

These *pseudo-Laplacians* $\tilde{\Delta}_a$ have more (genuine) eigenfunctions than Δ. This is not obvious. Specifically, [CdV 1981,82,83] shows that $\tilde{\Delta}_a$ has *compact resolvent*, so has *discrete* spectrum. [11] Certain *truncated* Eisenstein series, definitely not eigenfunctions for Δ, nor for its self-adjoint [12] closure, are genuine eigenfunctions for $\tilde{\Delta}_a$. Specifically, truncated Eisenstein series $\wedge^a E_s$ with $a^s + c_s a^{1-s} = 0$ are genuine eigenfunctions for $\tilde{\Delta}_a$, despite the fact that such truncations are not smooth.

That is, non-smooth functions *can* be *genuine* eigenfunctions for a self-adjoint extension of (a restriction of) a differential operator. Proof depends on specifics of the Friedrichs self-adjoint extension: see [CdV 1983] or [Garrett 2011].

In other words, letting T_a be the automorphic distribution evaluating the constant term $c_P f$ of a smooth automorphic form at $y = a$,

$$T_a(f) \;=\; (c_P f)(ia)$$

truncated Eisenstein series $\wedge^a E_s$ with $T_a E_s \;=\; a^s + c_s a^{1-s} \;=\; 0$ are pseudo-cuspforms: letting H_a be the Heaviside function taking value 1 for $y > a$ and 0 for $y < a$, with $\lambda = s(s-1)$,

$$(\Delta - \lambda)(\wedge^a E_s) \;=\; \Big(y^2 \frac{\partial^2}{\partial y^2} - s(s-1)\Big)\Big((1 - H_a)\cdot(y^s + c_s y^{1-s})\Big)$$

$$=\; (s a^s + c_s(1-s)a^{1-s})\cdot\delta_a \;=\; (2s-1)a^s\cdot\delta_a \qquad \text{(for } a^s + c_s a^{1-s} = 0\text{)}$$

Since eigenvalues of non-positive self-adjoint operators $\tilde{\Delta}_a$ are non-positive real, the values s for which $\wedge^a E_s$ has $a^s + c_s a^{1-s} = 0$ are either on $\mathrm{Re}(s) = \frac{1}{2}$ or $[0,1]$. However, disappointingly, this has no direct bearing on the location of zeros of zeta functions.

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[11] The apparent immediate goal of [CdV 1981] is meromorphic continuation of Eisenstein series. [CdV 1982,83] provide details and a larger context.

[12] Starting with the domain of smooth $L^2(\Gamma\backslash\mathfrak{H})$ functions with all derivatives in $L^2(\Gamma\backslash\mathfrak{H})$, it is not trivial to prove that Δ's (graph-) closure is self-adjoint. Nevertheless, it is so. By elementary arguments about adjoints, this is the *unique* self-adjoint extension. However, this does not preclude the existence of self-adjoint extensions of *restrictions* of Δ to subspaces of $L^2(\Gamma\backslash\mathfrak{H})$.

[0.12] New eigenfunctions of pseudo-Laplacians are mostly truncated Eisenstein series

In the context of the pseudo-cuspforms, [CdV 1981,82,83]'s pseudo-Laplacians do offer a hope of realizing pseudo-cuspforms as *genuine* eigenfunctions of self-adjoint operators, if not of Δ itself.

However, [CdV 1983] recalls that [Lax-Phillips 1976] pp. 202-204 proves that for cut-offs $a > 1$ the *only* discrete spectrum of $\tilde{\Delta}_a$ with $|\lambda| > \frac{1}{4}$ consists of cuspforms and suitable truncated Eisenstein series, the latter unfortunately having no impact on location of zeros of zeta. Thus, to have a chance of obtaining *other* eigenfunctions, take a lower, for example, $a = \sqrt{3}/2$.

[0.13] Rekindling hope of legitimizing pseudo-cuspforms

There is also a technical hazard: Friedrichs' construction demonstrably puts all eigenfunctions inside a +1-index Sobolev space, limiting the non-smoothness that any Friedrichs extension of a second-order elliptic operator can tolerate. Indeed, the Dirac delta on a two-dimensional manifold is in Sobolev spaces with index $-1-\varepsilon$ for all $\varepsilon > 0$, so a fundamental solution is in $+1-\varepsilon$, which just misses the +1 Sobolev space. That is, the fundamental solution itself could not possibly be an eigenfunction for *any* Friedrichs extension.

The evaluate-constant-term distribution *is* in the $-\frac{1}{2}-\varepsilon$ Sobolev space, but this has no immediate bearing on zeros of zetas.

The closed geodesic period distributions are better, in that they *are* connected to zeros of zeta, and they are in the $-\frac{1}{2}-\varepsilon$ Sobolev space

[CdV 1983] attempts to surmount this technical issue by projecting a solution u_w for $(\Delta-\lambda_w)u_w = \delta_\omega^{\rm afc}$ to the orthogonal complement of the discrete spectrum. That is, for a pseudo-cuspform u satisfying $(\Delta-\lambda)u = \delta_\omega^{\rm afc}$, consider the projection $\pi_c u$ to the orthogonal complement of the discrete spectrum. [CdV 1983] cites a Lindelöf-on-average bound [Motohashi 1970] for second moments of Dedekind zeta of quadratic fields to demonstrate that $\pi_c u$ is in the +1 Sobolev space, so is not *a priori* prohibited from being an eigenfunction of the Friedrichs extension. In fact, as Bombieri notes, the classic [Hardy-Littlewood 1918] suffices for this estimate.

For that matter, [CdV 1983] notes that all we really want is the conclusion that the apparent eigenvalue is *real*. To this end, it suffices to have

$$0 \;=\; \langle (\tilde{\Delta}_a - \lambda)\pi_c u, \pi_c u\rangle \qquad\qquad (\text{with } a = \sqrt{3}/2)$$

since then the usual argument for real-ness of λ succeeds, using the self-adjointness of $\tilde{\Delta}_a$. Since $(\tilde{\Delta}_a - \lambda)u = \delta_\omega^{\rm afc}$, we want $\pi_c u(\omega) = 0$.

We might express $\pi_c u(\omega) = 0$ in terms of the spectral decomposition of $\pi_c u$ in $L^2(\Gamma\backslash\mathfrak{H})$: as above, the s^{th} component is given by the usual

$$\widehat{\pi_c u}(s) \;=\; \int_{\Gamma\backslash\mathfrak{H}} \pi_c u \cdot E_{1-s} \;=\; \int_{\Gamma\backslash\mathfrak{H}} u \cdot E_{1-s} \;=\; \frac{E_{1-s}(\omega)}{\lambda_{1-s}-\lambda_w} \;=\; \frac{\zeta_k(1-s)}{\zeta(2-2s)\cdot(\lambda_{1-s}-\lambda_w)}$$

with $k = \mathbb{Q}(\omega)$. Then we would *hope* that the spectral synthesis

$$\pi_c u \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \widehat{\pi_c u}(s)\cdot E_s\, ds \qquad\qquad (\text{in } L^2(\Gamma\backslash\mathfrak{H}))$$

really does converge uniformly pointwise on compacts. Sobolev's inequality implies that $+1+\varepsilon$ Sobolev norms dominate C^o, so we want to see that $\pi_c u$ is not merely in the +1 Sobolev space, but in the $+1+\varepsilon$ Sobolev space. Letting $k = \mathbb{Q}(\omega)$, this requirement is

$$\int_0^\infty \Big|\frac{\widehat{\pi_c u}(\frac{1}{2}+it)}{\lambda_{\frac{1}{2}+it}-\lambda}\Big|^2 dt \;=\; \int_0^\infty \Big|\frac{\zeta_k(\frac{1}{2}+it)}{\zeta(1+2it)\cdot(\lambda_{\frac{1}{2}+it}-\lambda)}\Big|^2 dt \;<\; \infty$$

Since $\zeta(1+2it)^{-1} = O(t^\varepsilon)$, by summation by parts, it suffices to show that

$$\int_0^T \left|\zeta_k(\tfrac{1}{2}+it)\right|^2 dt \;\ll\; T^{2-\delta} \qquad\qquad \text{(for some } \delta > 0\text{)}$$

This needs anything better than convexity, so [Hardy-Littlewood 1918] suffices.

Thus, to prove that the pseudo-eigenvalue $\lambda = s_j(s_j-1)$ for the pseudo-cuspform u is *real*, it suffices to prove that

$$0 \;=\; \int_0^\infty \frac{|\zeta_k(\frac{1}{2}+it)|^2 \;\; dt}{|\zeta(1+2it)|^2 \cdot (-\frac{1}{4} - t^2 - s_j(s_j-1))} \qquad\qquad \text{(with } \zeta_k(s_j) = 0\text{)}$$

[CdV 1983] notes both the difficulty of numerically evaluating such integrals, and, yet, some compatibility with other numerical results.

Bombieri notes [13] that the numerical conjecture made (in effect) by Colin de Verdière does not hold. Namely, that specific pseudo-Laplacian's eigenvalues are sufficiently precisely computable (for those who know how) so that the s-parameters for eigenvalues can be distinguished from the (known) roots of the relevant zeta and L-functions.

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