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Siegel's integral

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I first saw this sketched in [Siegel 1939]. [1] The off-hand manner with which Siegel invokes it suggests that such mechanisms were well-known at that time.

Let C_n be the cone of positive-definite symmetric real n-by-n matrices, and V the real vector space of symmetric real n-by-n matrices. The standard pairing $\langle\rangle : V \times V \to \mathbb{R}$ is $\langle v, w\rangle = \mathrm{tr}\,(vw)$. We claim that, for $y \in C_n$,

$$\int_V e^{i\langle x,\xi\rangle}\,\det(y-ix)^{-s}\,dx \;=\; \begin{cases} c(s)\cdot e^{-\langle y,\xi\rangle}\,(\det\xi)^{s-\frac{n+1}{2}} & (\text{for } \xi \in C_n) \\ 0 & (\text{for } \xi \notin C_n) \end{cases}$$

where dx is the product of usual Lebesgue measures on the coordinates x_{ij} with $i \le j$, with

$$c(s) \;=\; \frac{1}{\Gamma(s)\,\Gamma(s-\frac{1}{2})\,\Gamma(s-\frac{2}{2})\,\Gamma(s-\frac{3}{2})\ldots\Gamma(s-\frac{n-1}{2})\;(2\pi)^n\;\pi^{n(n-1)}}$$

Proof: The *gamma function* $\Gamma_n(s)$ attached to the cone C_n is [2]

$$\Gamma_n(s) \;=\; \int_{C_n} e^{-\mathrm{tr}\;\xi}\,(\det\xi)^s\,\frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}$$

with $d\xi$ the product of the usual Lebesgue measure on the usual coordinates ξ_{ij} with $i \le j$. The measure $d\xi/(\det\xi)^{\frac{n+1}{2}}$ is invariant under the action $\xi \to A\xi A^\top$ of $A \in GL_n(\mathbb{R})$ on C_n. An element $y \in C_n$ has a unique square root $\sqrt{y}$ in C_n. Note that

$$\mathrm{tr}\,(\sqrt{y}\xi\sqrt{y}) \;=\; \mathrm{tr}\,(y\xi) \;=\; \langle y,\xi\rangle$$

Replacing ξ by $\sqrt{y}\xi\sqrt{y}$ in the integral defining $\Gamma_n(s)$, using the invariance of the measure $d\xi/(\det\xi)^{\frac{n+1}{2}}$,

$$\Gamma_n(s) \;=\; (\det y)^s \int_{C_n} e^{-\langle y,\xi\rangle}\,(\det\xi)^s\,\frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}$$

By analytic continuation, [3] for $x \in V$

$$\Gamma_n(s) \;=\; (\det y - ix)^s \int_{C_n} e^{-\langle y-ix,\xi\rangle}\,(\det\xi)^s\,\frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}$$

[1] Siegel used this device to compute the archimedean factor in the Euler factorization of the big Bruhat cell contributions to Fourier coefficients of Eisenstein series. By the 1970s the device was well-known in the theory of Siegel modular forms and holomorphic automorphic forms of other sorts. However, this simple special idea is often lost among weightier issues, motivating the present recollection of it. The review [Gross 1998] of [Farault-Koranyi 1994] gives a quick survey of related ideas, and the latter book-length treatment contains much more.

[2] As in [Farault-Koranyi 1994], a completely parallel treatment applies to the other homogeneous cones: positive-definite n-by-n hermitian matrices, positive-definite n-by-n quaternion matrices, the light-cone $x_0^2 > x_1^2 + \ldots + x_n^2$, and the exceptional cone. The discussion can be made intrinsic by expressing cones in terms of *Jordan algebras*, as do various papers of M. Köcher, E.B. Vinberg, and others. However, the list of cones is very short, and the overhead involved in using Jordan algebras is considerable. Nevertheless, one should compare [Farault-Koranyi 1994].

[3] The analytic continuation is in $z = x + iy$ in the Siegel upper half-space, with $y \in C_n$ and $x \in V$. The analogous analytic continuation to the corresponding *tube domains* formed from the other cones play corresponding roles in the computations for those cones.

That is,

$$\frac{\Gamma_n(s)}{(\det(y-ix))^s} \;=\; \int_{C_n} e^{i\langle x,\xi\rangle}\cdot e^{-\langle y,\xi\rangle}\,(\det\xi)^s\,\frac{d\xi}{(\det\xi)^{\frac{n+1}{2}}}$$

View the integral as an inverse Fourier transform on V of the function

$$\varphi_y(\xi) \;=\; \begin{cases} e^{-\langle y,\xi\rangle}\,(\det\xi)^{s-\frac{n+1}{2}} & (\text{for } \xi\in C_n) \\ 0 & (\text{for } \xi\notin C_n)\end{cases}$$

With Fourier transform on V normalized to

$$\widehat{f}(\xi) \;=\; \int_V e^{-i\langle x,\xi\rangle}\,f(x)\,dx$$

and inverse transform

$$f^\vee(x) \;=\; \int_V e^{i\langle x,\xi\rangle}\,f(\xi)\,d\xi$$

The constant in Fourier inversion is given by

$$(2\pi)^{-n}\pi^{-\frac{n(n-1)}{2}}\cdot f(x) \;=\; \int_V e^{i\langle x,\xi\rangle}\,\widehat{f}(\xi)\,d\xi \;=\; \int_V e^{-i\langle x,\xi\rangle}\,f^\vee(x)\,dx$$

Thus, since

$$\phi_y^\vee(x) \;=\; \frac{\Gamma_n(s)}{\det(y-ix)^s}$$

by Fourier inversion

$$\Big(\frac{\Gamma_n(s)}{\det(y-ix)^s}\Big)^\wedge(\xi) \;=\; (2\pi)^{-n}\pi^{-\frac{n(n-1)}{2}}\cdot\varphi_y(\xi)$$

That is,

$$\int_V e^{-i\langle x,\xi\rangle}\,(\det(y-ix))^{-s}\,dx \;=\; \frac{1}{\Gamma_n(s)\,(2\pi)^n\pi^{\frac{n(n-1)}{2}}}\cdot\varphi_y(\xi)$$

Further, this gamma function $\Gamma_n(s)$ is expressible in terms of the classical gamma function

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}$$

as follows. Let

$$f \;:\; C_{n-1}\times\mathbb{R}^{n-1}\times C_1 \;\longrightarrow\; C_n$$

by

$$f(y,v,t) \;=\; \begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & t\end{pmatrix}\begin{pmatrix}1 & 0\\ v^\top & 1\end{pmatrix} \;=\; \begin{pmatrix}y+vtv^\top & tv\\ tv^\top & t\end{pmatrix} \qquad (\text{with } \mathbb{R}^{n-1} \text{ as column vectors})$$

Thus,

$$\Gamma_n(s) \;=\; \int_{C_{n-1}\times\mathbb{R}^{n-1}\times C_1} e^{-\mathrm{tr}\,(y+vtv^\top+t)}\,(\det y)^s\,t^s\,\frac{dy\,t^{n-1}\,dv\,dt}{(\det y)^{\frac{n+1}{2}}\,t^{\frac{n+1}{2}}}$$

$$= \int_{\mathbb{R}^{n-1}} e^{-v^\top v}\,dv\cdot\int_{C_{n-1}} e^{-\mathrm{tr}\,y}\,(\det y)^{s-\frac12}\,\frac{dy}{(\det y)^{\frac{(n-1)+1}{2}}}\cdot\int_0^\infty e^{-t}\,t^{s+(n-1)-\frac{n+1}{2}-\frac{n-1}{2}+1}\,\frac{dt}{t}$$

$$= \pi^{(n-1)/2}\cdot\Gamma(s)\cdot\Gamma_{n-1}(s-\tfrac12)$$

By induction,

$$\Gamma_n(s) \;=\; \pi^{n(n-1)/2}\,\Gamma(s)\,\Gamma(s-\tfrac{1}{2})\,\Gamma(s-\tfrac{2}{2})\,\Gamma(s-\tfrac{3}{2})\ldots\Gamma(s-\tfrac{n-2}{2})\,\Gamma(s-\tfrac{n-1}{2})$$

Thus, we have determined the constant $c(s)$, and have computed the asserted Fourier transform

$$\int_V e^{-i\langle x,\xi\rangle}\,(\det(y-ix))^{-s}\,dx \;=\; \frac{1}{\Gamma(s)\Gamma(s-\frac{1}{2})\ldots\Gamma(s-\frac{n-1}{2})\,(2\pi)^n\pi^{n(n-1)}}\,\varphi_y(\xi)$$

[Faraut-Koranyi 1994] J. Faraut, A. Koranyi, *Analysis on symmetric cones*, Oxford University Press, 1994.

[Gross 1998] K.I. Gross, *Review of* Analysis on symmetric cones*, by Farault and Koranyi*, Bulletin of AMS **35** no. 1 (1998), 77-86.

[Siegel 1939] C.L. Siegel, *Einfuhrung in die Theorie Modulfunktionen n-ten Grades*, Math. Ann. **116** (1939), 617-657.
