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# The Siegel-Weil formula in the convergent range

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# [Draft]

We give a very simple, mostly local, argument for the equality of certain linear combinations of theta series and certain Eisenstein series, a *Siegel-Weil formula*.

As Michael Harris has pointed out, the germ of this sort of argument is present in [Andrianov 1979], though of course in a different language. Not surprisingly, with the benefit of sufficient hindsight, the proof of Siegel-Weil far into the convergent region can be made a straightforward consequence of by-now-standard constructions and ideas.

By its nature, this argument succeeds only for weights ($K_\infty$-types) sufficiently far into the regions of convergence of Eisenstein series, in contrast to vastly more delicate work of Kudla-Rallis and of Jiang concerning Siegel-Weil formulas and related matters outside the region of convergence.

In accidentally rediscovering this argument, I was reminded of the idea of comparing sizes of Hecke eigenvalues of holomorphic Eisenstein series versus holomorphic cuspforms by [Harris 1981], where such a comparison was used to give a very brief argument for rationality properties of holomorphic Eisenstein series. And Harris has informed me that the idea of comparison of sizes of eigenvalues came to him from from Andrianov's use [Andrianov 1979] of such comparisons in his argument for Siegel-Weil.

With regard to evolution of styles and viwpoints: already in a note added in proof, in [Harris 1981], Harris observed that a very general argument about boundedness of spherical functions, using [MacDonald 1971], could replace baroque (needlessly detailed) arguments using Hecke operators. We use such ideas here.

[Andrianov 1979] A.N. Andrianov, *The multiplicative arithmetic of Siegel modular forms*, Russian Math Surveys **34** (1979), pp. 75-148.

[Harris 1981] M. Harris, *The rationality of holomorphic Eisenstein series*, Inv. Math. **63** (1981), pp. 305-310.

[Harris 1984] M. Harris, *Eisenstein series of Shimura varieties*, Ann. Math. **119** (1984), pp. 59-94.

[Klingen 1968] H. Klingen, *Zum darstellungssatz für Siegelsche Modulformen*, Math. Z. **102** (1967), pp. 399-400

[MacDonald 1971] I.G. MacDonald, *Spherical Functions on a Group of p-adic Type*, Ramanujan Institute, Univ. of Madras Publications, 1971.



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# 1. Statement of the theorem

Let $Q$ be a quadratic form of dimension $2m$ over a totally real field $k$ and suppose that $Q$ is anisotropic at every real place of $k$. This implies (by a part of *reduction theory* often called *Mahler's lemma*) that the quotient $O(Q)_k\backslash O(Q)_{\mathbf{A}}$ is *compact*. In particular, we suppose that $Q$ is *positive-definite* at every real place.

The *Weil* or *oscillator* representation of the adele group $O(Q) \times Sp(n)$ is a representation on Schwartz functions $f$ on $(Q \otimes k^n)_{\mathbf{A}}$ given by

$$h \cdot f(x) = f(h^{-1}x) \quad (\text{for } h \in O(Q)_{\mathbf{A}})$$

The representation is more complicated to define for $g \in Sp(n)$, and depends upon a character $\psi$ on $\mathbf{A}/k$.

For $\Phi$ in the Schwartz space on the adelized $Q \otimes k^n$, the corresponding **theta kernel** $\Theta_\Phi$ is the function on $(Q \otimes k^n)_{\mathbf{A}}$

$$\Theta_f(x) = \sum_{y \in Q \otimes k^n} \Phi(x+y)$$

For a function $f$ on $O(Q)_k \backslash O(Q)_{\mathbf{A}}$, the integral

$$\theta_{\Phi,f} = \int_{O(Q)_k \backslash O(Q)_{\mathbf{A}}} f(h)\ \Theta_{h\cdot\Phi}(0)\, dh$$

is the **theta lift** of $f$ (via the kernel made from $\Phi$). Letting $g \in Sp(n)_{\mathbf{A}}$ act, we have a function $\theta_{\Phi,f}$ given by

$$\theta_{\Phi,f}(g) = \int_{O(Q)_k \backslash O(Q)_{\mathbf{A}}} f(h)\ \Theta_{gh\cdot\Phi}(0)\, dh$$

For $f$ continuous, since the quotient is compact, the integral converges nicely, and yields a left $Sp(n)_k$-invariant function $\theta_{\Phi,f}$ on $Sp(n)_{\mathbf{A}}$. In particular, taking $f = 1$, we have

$$\theta_\Phi(g) = \int_{O(Q)_k \backslash O(Q)_{\mathbf{A}}} \Theta_{gh\cdot\Phi}(0)\, dh$$

On the other hand, the function

$$\varphi(g) = (g \cdot \Phi)(0)$$

is in the *degenerate principal series*

$$I_\chi = \{f \text{ on } Sp(n)_{\mathbf{A}} : f(pg) = \chi(p) \cdot f(g)\}$$

where

$$\chi \begin{pmatrix} A & * \\ 0 & {}^\top A^{-1} \end{pmatrix} = (\det A)^m \cdot \chi(\det A)$$

where $\chi$ is the quadratic character attached to the discriminant of $Q$. We can form the *Siegel-type (degenerate) Eisenstein series*

$$E_\Phi(g) = \sum_{\gamma \in P_k \backslash Sp(n)_k} (\gamma g \cdot \Phi)(0) = \sum_{\gamma \in P_k \backslash Sp(n)_k} \varphi(\gamma g)$$

This is nicely convergent for $m > n+1$, that is, for $\dim Q > 2n+2$. We will see that the whole family of more general *Klingen-type* holomorphic Eisenstein series arising in the proof need $m > 2n$ for convergence, or $\dim Q > 4n$.

**Theorem:** *(Siegel-Weil)* Let $\Phi$ be in the Schwartz space on the adelized $Q \otimes k^n$, with $\dim Q > 4n$. Then

$$\theta_\Phi = E_\Phi$$

(Of course, it is quite possible that both sides are 0.)

**Remark:** When translated into classical terms, the left-hand side is a linear combination of (holomorphic) theta series on $Sp(n)$.

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# 2. Ingredients of the proof

There are two *global* parts of the argument. First, there is the Klingen structure theorem which expresses holomorphic automorphic forms as *sums* (rather than something more complicated, such as *integrals*) of holomorphic cuspforms and holomorphic Eisenstein series. There is also the global part of the construction of the Weil/oscillator representation, in effect proving that a global quadratic character is a Hecke character (rather than merely a random product of local characters), and Poisson summation (which can be used to show directly that the $\chi$ is a Hecke character).

There are several *local* parts of the argument. First, there is the specific archimedean computation to see that the $\infty$-type of the theta lift $\theta_\Phi$ is a holomorphic discrete series of the same type as the Siegel Eisenstein series. This is well-known, though perhaps by now so apocryphal that it deserves repeating. Second, there is the Jacquet-module computation at a finite prime (at which $\Phi$ is sufficiently well behaved), to see that (almost everywhere, locally) the representation generated by the theta lift matches that of the Siegel Eisenstein series.

The last local issue consists of some relatively easy estimates to prove that the degenerate principal series (generated locally almost everywhere by both the theta series and the Siegel Eisenstein series) cannot possibly occur as the local representations generated by holomorphic cuspforms or the (non-Siegel) holomorphic Klingen Eisenstein series. At heart, the essential point concerns boundedness of spherical functions, as in [MacDonald 1971]. Again, it is not difficult to redo the parts of this we need.

If we were simply looking at $SL(2)$, the Klingen idea is more elementary and classical, namely

$$\theta_\Phi - E_\Phi = \text{holomorphic cuspform}$$

But the local representations at almost all finite primes by the two terms on the left are the same, and are *irreducible* unramified principal series, because the parameter is far away from the strip in which any reducibility occurs. And these (irreducible) unramified principal series are *not unitary* (by simple local estimates, after MacDonald), so could not arise as local representations generated by a cuspform, which would be $L^2$, and, thus, unitary.

# 3. Klingen's structure theorem

The fact that holomorphic automorphic forms of sufficiently high weight are expressible as sums of cuspforms and Eisenstein series is a peculiarity of the holomorphic case, and also does depend on having sufficiently high weight ($K_\infty$-type).

The Eisenstein series

$$E(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)$$

with

$$P = \text{Siegel parabolic} = \text{popular parabolic} = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \}$$

$$\varphi \begin{pmatrix} A & * \\ 0 & {}^\top\! A^{-1} \end{pmatrix} = |\det A|^s$$

on $G = Sp(n)$ converges for $\mathrm{Re}(s) > n+1$.

This follows, for example, from Godement's criterion [].

This estimate yields the convergence of the *holomorphic* Siegel-type Eisenstein series of (scalar) weights $k > n+1$.

Also, note that the holomorphic discrete series of lowest $K_\infty$-type

$$\rho : \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longrightarrow \det(A+iB)^k \quad (\text{with } A+iB \in U(n))$$

is a subrepresentation of the degenerate principal series (at real places)

$$I_s = \{ f : f(pg) = \chi_s(p) \cdot f(g) \}$$

where

$$\chi_s \begin{pmatrix} A & * \\ 0 & {}^\top A^{-1} \end{pmatrix} = (\det A)^s$$

A holomorphic modular form for $SL(2) = Sp(1)$ can be uniquely expressed as the sum of a holomorphic cuspform and a holomorphic Eisenstein series. For larger groups the situation is similar, though somewhat more complicated.

**Theorem:** *(Andrianov, Klingen)* Any holomorphic Siegel modular form on $Sp(n)$ of scalar $K_\infty$-type is uniquely expressible as the sum of a holomorphic cuspform, a holomorphic Siegel-type Eisenstein series, and for each $0 < i < n$ an intermediate (now called *Klingen-type*) Eisenstein series, defined as follows.

First the classical definition illustrates the intent best. For $Z = \begin{pmatrix} z & u \\ u^\top & t \end{pmatrix}$ in the Siegel upper half space $H_n$ of degree $n$, with $t$ $r$-by-$r$, etc., let

$$\mathrm{pr}_r(Z) = \mathrm{pr}_r \begin{pmatrix} z & u \\ u^\top & t \end{pmatrix} = t \in H_r$$

As usual, let

$$j(g, Z) = \det(cZ + d) \qquad \text{for } \; g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in Sp(n)$$

The corresponding maximal parabolic is

$$P^r = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \end{pmatrix} \in Sp(n)$$

where the bottom right block is $r$-by-$r$, etc. For a holomorphic cuspform $f$ on $H_r$ of weight $k$, define the Klingen-type Eisenstein series

$$E_f(Z) = \sum_{\gamma \in P^r_{\mathbf{Z}} \backslash G_{\mathbf{Z}}} j(\gamma, Z)^{-k} \, f(\mathrm{pr}_r Z)$$

A crucial point about *holomorphic* Eisenstein series is the simplicity of their constant terms. For example, given a holomorphic automorphic form $F$ of weight $k > 2$ for $SL(2, \mathbf{Z})$, with Fourier expansion

$$F(z) = \sum_{n \geq 0} c_n \, e^{2\pi i n z}$$

the difference

$$F(z) - c_0 \cdot \sum_{\gamma \in P_{\mathbf{Z}} \backslash SL(2, \mathbf{Z})} j(cz + d)^{-k}$$

is a *cuspform*. This is true only because in the Bruhat cell decomposition of the constant term of the Eisenstein series only the big-cell term contributes anything non-zero.

In the Fourier expansions of Siegel modular forms

$$F(Z) = \sum_{T \in \Lambda} c_T \, e^{2\pi i \mathrm{tr}\, TZ}$$

it is useful to talk about the *rank* of the indices $T$. The analogous and essential fact more generally is

**Lemma:** In a Klingen Eisenstein series $E_f$ made from a holomorphic cuspform $f$ on $H_r$, Fourier coefficients $c_T$ of $E_f$ with $\mathrm{rk} T < r$ are all 0. And a generalized form of *constant term* exactly recaptures $f$, namely

$$\lim_{y \longrightarrow +\infty} E_f \begin{pmatrix} iy \cdot 1_{n-r} & * \\ * & t \end{pmatrix} = f(t)$$

Granting this, we can see how to successively subtract Eisenstein series to leave a cuspform. Given a holomorphic Siegel modular form $F$ with Fourier coefficients $C_T$, first

$$c_0 = \lim_{y \longrightarrow +\infty} F\,(\,iy \cdot 1_n\,)$$

The Siegel-type Eisenstein series $E$ of the same weight as $F$ is, after all, an extreme case of Klingen's, and

$$F^{n-1} = F - c_0 \cdot E$$

has $0^{\mathrm{th}}$ Fourier coefficient 0. Continuing,

$$f_1(t) = \lim_{y \longrightarrow +\infty} (F - c_0 \cdot E) \begin{pmatrix} iy \cdot 1_{n-1} & * \\ * & t \end{pmatrix}$$

is a holomorphic cuspform on $H_1$. Next,

$$f_2(t) = \lim_{y \longrightarrow +\infty} (F - c_0 \cdot E - E_{f_1}) \begin{pmatrix} iy \cdot 1_{n-2} & * \\ * & t \end{pmatrix}$$

is a holomorphic cuspform on $H_2$. Eventually,

$$f_n(Z) = F - c_0 \cdot E - E_{f_1} - \ldots - E_{f_{n-1}}$$

is a holomorphic cuspform on $H_n$.