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# *Smooth Representations of Totally Disconnected Groups*

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The basic representation theory of totally disconnected groups here incorporates some novelties. We consider smooth representations on vectorspaces over arbitrary fields in characteristic zero. Integrals are replaced with invariant or equivariant functionals, and no infinite sums appear unless all but finitely-many summands are zero.

In particular, the fact that ideas regarding *Jacquet modules* and *the double coset method* (regarding *intertwining operators*) can be developed in this generality is useful in applications. Since we are interested in such things as Jacquet modules, we more generally consider the notions of *isotype* and *co-isotype*. *Spherical representations* and *admissibility*, at the end, may seem mysterious at this point, but these are essential later.

# 1. *Algebraic concepts regarding representation theory*

All vectorspaces are over a field $k$. Let $G$ be a group.

Let $V$ be a $k$-vectorspace and $\pi$ a group homomorphism $\pi : G \to GL(V)$ where $GL(V) = GL_k(V) = \mathrm{Aut}_k(V)$ is the group of $k$-linear automorphisms of $V$. Such $(\pi, V)$ is a **representation** of $G$ (over the field $k$). Or, say that $V$ is a **representation space** for $G$, or that $V$ is a representation of $G$ (with $\pi$ merely implied), or that $\pi$ is a representation of $G$ on $V$, etc. We may write $gv$ for $\pi(g)(v)$.

The **trivial representation** of $G$ (over $k$) is the one-dimensional $k$-vectorspace $k$ itself with the action $gv = v$ for all $g \in G$ and $v \in V$. This representation will be denoted by $k$ or 1.

Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be two representations of G. A $k$-linear map $f : V_1 \to V_2$ is a $G$-**morphism** (or **intertwining operator** or $G$-**homomorphism**) if

$$f(\pi_1(g)(v)) = \pi_2(g)(f(v))$$

for all $v \in V_1$ and $g \in G$. Suppressing the $\pi$'s, this condition is

$$f(gv) \;=\; gf(v)$$

A **quotient representation** is a $G$-morphism so that the underlying vectorspace map is surjective. A **subrepresentation** is a $G$-morphism so that the underlying vectorspace map is injective. As usual, identify subrepresentations and quotient representations with their images.

A representation $(\pi, V)$ of $G$ is **irreducible** if it contains no proper subrepresentation, i.e., contains no subrepresentation other than $\{0\}$ and the whole $V$. This condition is equivalent to the non-existence of a proper quotient.

Let $E$ be a field extension of $k$. A representation $(\pi, V)$ of $G$ on a $k$-vectorspace $V$ naturally gives rise to a representation $\pi \otimes_k E$ of $G$ on $V \otimes_k E$) defined by **extension of scalars**

$$(\pi \otimes_k E)(g)(v \otimes 1) \;=\; \pi(g)v \otimes 1$$

The representation $(\pi, V)$ is **irreducible over** $E$ when the extended representation $(\pi \otimes_k E, V \otimes_k E)$ of $G$ is irreducible, as a representation over $E$.

A representation $(\pi, V)$ on a $k$-vectorspace $k$ is **absolutely irreducible** if it is irreducible over an algebraic closure $\bar{k}$ of $k$.

The **direct sum** $\pi_1 \oplus \pi_2$ of two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$ has vectorspace $V_1 \oplus V_2$ with $g \in G$ acting by

$$g(v_1 \oplus v_2) = gv_1 \oplus gv_2 \;=\; \pi_1(g)(v_1) \oplus \pi_2(g)(v_2)$$

The **(internal) tensor product** $\pi_1 \otimes \pi_2$ of two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$ has vectorspace $V_1 \otimes V_2 = V_1 \otimes_k V_2$ with $g \in G$ acting by

$$g(v_1 \otimes v_2) \;=\; gv_1 \otimes gv_2 \;=\; \pi_1(g)(v_1) \otimes \pi_2(g)(v_2)$$

The **(external) tensor product** $\pi_1 \otimes \pi_2$ of two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of two groups $G_1, G_2$ has vectorspace $V_1 \otimes V_2$ with $g_1 \times g_2 \in G_1 \times G_2$ acting by

$$(g_1 \times g_2)(v_1 \otimes v_2) = g_1 v_1 \otimes g_2 v_2 = \pi_1(g_1)(v_1) \otimes \pi_2(g_2)(v_2)$$

Let $V^*$ be the $k$-linear dual of $V$, i.e., the space of $k$-linear maps $V \to k$. The **linear dual** or **linear contragredient** representation $(\pi^*, V^*)$ of $G$ on $V^*$ is defined by

$$(\pi^*(g)\lambda)(v) \;=\; \lambda(\pi(g^{-1})v)$$

Often the natural bilinear map

$$V \times V^* \longrightarrow k$$

will be denoted by angular brackets

$$v \times \lambda \longrightarrow \langle v, \lambda \rangle$$

Given a $G$-homomorphism

$$\varphi : (\pi_1, V_1) \longrightarrow (\pi_2, V_2)$$

the **adjoint map**

$$\varphi^* : (\pi_2^*, V_2^*) \longrightarrow (\pi_1^*, V_1^*)$$

is

$$\varphi^*(\lambda_2)(v_1) = \lambda_2(\varphi(v_1))$$

The (matrix-) **coefficient function**

$$c_{v\lambda} = c^\pi_{v,\lambda}$$

of a vector $v \in V$ and $\lambda \in V^*$ is a $k$-valued function on $G$ defined as

$$c_{v\lambda}(g) = \langle \pi(g)v, \lambda \rangle$$

We have the simple properties

$$R_g c_{v\lambda} = c_{\pi(g)v,\lambda} \qquad\qquad L_g c_{v\lambda} = c_{v,\pi^*(g)\lambda}$$

Let $(\pi, V)$ be a representation of a group $G$, and let $K$ be a subgroup of $G$. The set of **$K$-fixed vectors in $V$** is

$$V^K = \{v \in V : \pi(\theta)v = v, \ \forall \theta \in K\}$$

The **isotropy group** of a vector $v \in V$ is

$$G_v = \{g \in G : \pi(g)(v) = v\}$$

Let $H$ be a subgroup of a group $G$, and let $(\pi, V)$ be a representation of $G$. The **restriction** representation

$$(\mathrm{Res}^G_H \pi, V) = \mathrm{Res}^G_H (\pi, V)$$

is the representation of $H$ on the $k$-vectorspace $V$ obtained by letting

$$(\mathrm{Res}^G_H \pi)(h)(v) = (\pi h)(v)$$

Let $(\pi, V)$ be a representation of $G$, and $K$ a subgroup of $G$. A vector $v \in V$ is **$K$-finite** when the $k$-span of the vectors $\pi(\theta)v$ (for $\theta \in K$) is finite-dimensional.

A representation $(\pi, V)$ of $G$ is **finitely-generated** when there is a finite subset $X$ of $V$ so that every element of $V$ can be written in the form

$$\sum_i c_i \, \pi(g_i) x_i \qquad\qquad \text{(for some } c_i \in k,\ g_i \in G, \text{ and } x_i \in X)$$

We claim that *a finitely-generated representation has an irreducible quotient*, from Zorn's Lemma. We claim that there exist maximal elements among the set of $G$-subrepresentations, ordered by inclusion. To prove this, show that for an ascending chain

$$V_1 \subset V_2 \subset \ldots$$

of *proper* submodules the union is still a *proper* submodule. If not, then each $x$ in a finite set $X$ of generators for $V$ lies in some $V_{i(x)}$. Let $j$ be the maximum of the finite set of $i(x)$, $x \in X$. Then $X \subset V_j$, so $V = V_j$, contradiction. ///

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# 2. *Totally disconnected spaces and groups*

A topological space $X$ is **totally disconnected** when, for every $x \neq y$ in $X$ there are open sets $U, V$ so that $U \cap V = \emptyset$, $U \cup V = X$, and $x \in U, y \in V$.

In particular, a totally disconnected space is *Hausdorff*. The sets $U, V$ in the definition are not only open but also *closed*.

We claim that *at every point $x$ of a locally compact totally disconnected space $X$ there is a local basis consisting of compact open sets.* To see this, take an open set $V$ containing $x$ and so that the closure $\bar{V}$ is compact. The boundary

$$\partial V \; = \; \bar{V} \cap (X - V)$$

is closed, so is compact. For $y \in \partial V$, there are open (and closed) sets $U_y$ and $V_y$ so that $U_y \cap V_y = \emptyset$ and $U_y \cup V_y = X$, and $y \in V_y$ and $x \in U_y$. Take a finite subcover $V_{y_1}, \ldots, V_{y_n}$ of $\partial V$. The set

$$V - (\bigcup_i \bar{V}_{y_i}) \; = \; \bar{V} - (\bigcup_i V_{y_i})$$

is both open and closed, and, being a closed subset of the compact set $\bar{V}$ in a Hausdorff space, is compact. ///

Next, we claim that *a locally compact totally disconnected topological group $G$ has a basis at $1 = 1_G$ consisting of compact open subgroups.* To prove this, let $V$ be a compact open subset of $G$ containing 1, by the previous paragraph. Let

$$K \; = \; \{x \in G : xV \subset V \& x^{-1}V \subset V\}$$

It is clear that $K$ is a subgroup of $G$, and

$$K \; = \; (\bigcap_{v \in V} \; Vv^{-1}) \; \cap \; (\bigcap_{v \in V} \; Vv^{-1})^{-1}$$

shows that $K$ is the continuous image of compact sets, so is compact. What remains to be shown is that $K$ is open.

To the latter end, it certainly suffices to show that the compact-open topology on $G$ constructed from the 'original' topology on $G$ is the original topology on $G$. That is, show that, for compact $C$ in $G$ and for open $V$ in $G$, the set

$$U \; = \; U_{C,V} \; = \; \{x \in G : xC \subset V\}$$

is open in $G$. Take $U$ is non-empty, and $x \in U$. For all points $xy \in xC$ for $y \in C$, there is a small-enough open neighborhood $U_y$ of 1 so that the open neighborhood $xU_y y$ of $xy$ is contained in $V$. By continuity of the multiplication in $G$, there is an open neighborhood $W_y$ of 1 so that $W_y W_y \subset U_y$. The sets $xW_y y$ cover $xC$; let $xW_{y_1}y_1, \ldots, xW_{y_n}y_n$ be a finite subcover. Put $W = \bigcap_i W_{y_i}$. Then $xW$ is a neighborhood of $x$ and

$$xW \cdot C \; \subset \; xW \cdot \bigcup_i W_{y_i} y_i$$

and

$$xWW_{y_i}y_i \; \subset \; xW_{y_i}W_{y_i}y_i \subset xU_{y_i}y_i$$

Thus, $U$ is open. ///

# 3. *Smooth representations of totally disconnected groups*

Let $G$ be a locally compact, Hausdorff topological group with a countable basis, and *totally disconnected*. Take this to mean that $G$ has a local basis at the identity consisting of compact open subgroups.

Consider representations of $G$ on vectorspaces over a field $k$ *of characteristic zero*. For many purposes, the precise nature of $k$ is irrelevant. On the other hand, some more refined results will require $k = \mathbb{C}$ or $\mathbb{R}$. Nevertheless, for certain applications, e.g., to *families* of representations, it is useful to have general groundfields. Thus, in the sequel, speak of *measures and integrals* in situations more general than those condoned by the conservative criterion that demand our groundfield be $\mathbb{R}$ or $\mathbb{C}$.

A representation $(\pi, V)$ of $G$ is a **smooth representation** when, for all $v \in V$, the isotropy group $G_v$ is *open*. Because of the total-disconnectedness, this condition is equivalent to

$$V = \bigcup_K V^K$$

where $K$ runs over compact open subgroups of $G$ and $V^K$ is the subspace of $K$-fixed vectors in $V$.

From the definitions, *any $G$-subrepresentation of a smooth representation is again smooth.* Therefore, a **$G$-homomorphism** of *smooth* representations is defined to be any $G$-homomorphism of (smooth) $G$-representations. That is, the *smoothness* is not a property directly possessed by morphisms, but by the representations.

Generally, given an arbitrary representation $\pi$ of $G$ on a vectorspace $V$, the subspace

$$V^\infty = \{v \in V : G_v \text{ is open}\}$$

is the subspace of **smooth vectors**. Clearly $V^\infty$ is $G$-stable, so the restriction $\pi^\infty$ of $\pi$ to

$$\pi^\infty : G \longrightarrow GL(V^\infty)$$

is a $G$-subrepresentation of $V$.

For a smooth representation $(\pi, V)$ of $G$, the **(smooth) dual** or **(smooth) contragredient** $(\check{\pi}, \check{V})$ of $\pi$ is the representation of $G$ on the smooth vectors in the linear dual $V^*$. In other words,

$$(\check{\pi}, \check{V}) = ((\pi^*)^\infty, (V^*)^\infty)$$

There is the usual natural map

$$(\pi, V) \longrightarrow (\check{\check{\pi}}, \check{\check{V}}) \qquad\qquad (\text{by } v(\lambda) = \lambda(v))$$

When this map is surjective, $\pi$ is said to be **reflexive**.

A smooth representation $(\pi, V)$ of $G$ is **irreducible** if it contains no proper subrepresentation, i.e., contains no subrepresentation other than $\{0\}$ and the whole $V$. (Again, a $G$-stable subspace is *necessarily* a smooth representation of $G$). That is, irreducibility here is no more than the *algebraic* irreducibility mentioned previously.

# 4. *Test functions and distributions*

Let $X$ be a **totally disconnected** space. That is, given $x \neq y$ in $X$ there are open sets $U, V$ in $X$ so that $x \in U, y \in V$, and $X = U \cup V$. For present purposes suppose that $X$ is locally compact and has a countable basis.

Fix a field $k$ of *characteristic zero*. Let $W$ be a $k$-vectorspace. A $W$-valued function $f$ on $X$ is **locally constant** when for all $x \in X$ there is an open neighborhood $U$ of $x$ so that for $y \in U$ $f(y) = f(x)$. This condition would be that of *continuity* if $W$ had the discrete topology. However, we do *not* give $W$ the discrete topology, nor any other topology.

The space
$$\mathcal{D}(X, W) \;=\; C_c^\infty(X, W)$$
of **$W$-valued test functions** on $X$ is the $k$-vectorspace of compactly-supported, locally constant $W$-valued functions on $X$. In particular, the test function space (over $k$), $\mathcal{D}(X) = \mathcal{D}(X, k)$, has a $k$-basis consisting of the characteristic functions of compact open subsets of $X$.

Observe the natural isomorphism of $k$-vectorspaces
$$\mathcal{D}(X) \otimes_k W \longrightarrow \mathcal{D}(X, W) \qquad\qquad (\text{given by } (f \otimes w)(x) = f(x)w)$$

Let $W^*$ be the $k$-linear dual $\mathrm{Hom}_k(W, k)$ of $W$. The space $\mathcal{D}^*(X, W^*)$ of **$W^*$-valued distributions** on $X$ is the $k$-linear dual to the space $\mathcal{D}(X, W)$ of $W$-valued test functions on $X$. That is, it is the space of *all* $k$-linear maps from $\mathcal{D}(X, W)$ to $k$.

More generally, extending the previous notation and ideas, refer to
$$\mathrm{Hom}_k(\mathcal{D}(X, W), W')$$
as the space of **$\mathrm{Hom}_k(W, W')$-valued distributions** on $X$. Write
$$\mathcal{D}^*(X, \mathrm{Hom}_k(W, W')) \;=\; \mathrm{Hom}_k(\mathcal{D}(X, W), W')$$
for this space, justified by the natural isomorphism
$$\mathrm{Hom}_k(V \otimes W, W') \;\approx\; \mathrm{Hom}_k(V, \mathrm{Hom}_k(W, W'))$$

The lack of topological requirements is appropriate, for the following reason. In many applications, the $k$-vectorspace $W$ is a union $W = \bigcup_i W_i$ of *finite-dimensional* $k$-vectorspaces $W_i$, where $W_i \subset W_{i+1}$. For each finite list $\mathcal{U} = U_1, \ldots, U_n$ of mutually disjoint open sets in $X$ each having compact closure, and for each index $i$, let $F(\mathcal{U}, i)$ be the collection of $W_i$-valued functions which are 0 off $U_1 \cup \ldots \cup U_n$, and are constant on each $U_i$. This $F(\mathcal{U}, i)$ is a finite-dimensional $k$-subspace of $\mathcal{D}(X, W)$. The assumption that $X$ is totally disconnected implies that $\mathcal{D}(X, W)$ is the union of all such $F(\mathcal{U}, i)$.

The **support** of a distribution $u \in \mathcal{D}^*(X, \mathrm{Hom}_k(W, W'))$ is the smallest closed subset $C = \mathrm{spt}(u)$ of $X$ so that if $f \in \mathcal{D}(X, W)$ has support *not* meeting $C$, then $u(f) = 0 \in W'$. That is, if $\mathrm{spt}(u) \cap \mathrm{spt}(f) = \emptyset$, then $u(f) = 0$.

For example, the $k$-vectorspace of distributions $u \in \mathcal{D}^*(X)$ with $\mathrm{spt}(u)$ a single point $\{x_0\}$ consists of scalar multiples of the functional $f \to u_0(f) = f(x_0)$. Indeed, given $f$ in $\mathcal{D}(X, k) = \mathcal{D}(X)$, let $U$ be any small-enough compact neighborhood of $x_0$ so that $f$ is constant on $U$. Let $\mathrm{ch}_U$ be the characteristic function of $U$. Then $f - f(x_0)\mathrm{ch}_U$ is 0 on a *neighborhood* of $x_0$, so $u(f - f(x_0)\mathrm{ch}_U) = 0$. That is,
$$u(f) = u(f(x_0)\mathrm{ch}_U) = f(x_0)\, u(\mathrm{ch}_U)$$
This equality holds for *any* small-enough $U$ (depending upon $f$), giving the desired result.

For two totally disconnected spaces $X, Y$, it is easy to exhibit a natural isomorphism
$$C_c^\infty(X) \otimes_k C_c^\infty(Y) \;\approx\; C_c^\infty(X \times Y) \qquad\qquad (\text{by } f \otimes g \to F_{f\otimes g} \text{ with } F_{f\otimes g}(x \times y) = f(x)g(y))$$

# 5. *Integration on totally disconnected groups*

Let $G$ be a totally disconnected, locally compact topological group $G$. Construct a *Haar integral* for test functions on $G$ taking values in an arbitrary characteristic-zero field $k$. More precisely, construct *invariant distributions.*

For certain applications (e.g., to study of *parametrized families of representations*), it is necessary to be able to consider more general groundfields. Write invariant distributions as *integrals* because the notation is suggestive.

By now we know that there is a local basis at $1 \in G$ consisting of compact open *subgroups.*

For $f \in C_c^\infty(G) = C_c^\infty(G,k) = \mathcal{D}(G)$, the local constancy assures that for every $x \in \mathrm{spt}(f)$ there is a compact open subgroup $K_x$ so that $f(x') = f(x)$ for $x' \in xK_x$. Since $\mathrm{spt}(f)$ is compact, it is covered by finitely-many of these $K_x$, say $x_1K_{x_1}, \ldots, x_nK_{x_n}$. And if $y \in x_iK_{x_i}$ then

$$yK_{x_i} \subset xK_{x_i} \cdot K_{x_i} = xK_{x_i}$$

Thus, $f$ is **uniformly locally constant**: that is, letting $K = \bigcap_i K_{x_i}$, for any $x$, for $x' \in xK$ we have $f(x') = f(x)$. In other words, $f$ is **right $K$-invariant**.

Thus, given $f \in C_c^\infty(G)$, for sufficiently small compact open subgroup $K$ it is true that $f$ is right $K$-invariant. A symmetrical argument also shows that a given test function $f$ is *left* $K$-invariant for small-enough $K$.

Therefore, for small-enough $K$, there are group elements $x_i$ and $c_i \in k$ so that

$$f(g) = \sum_i c_i \,\mathrm{ch}_K(gx_i)$$

where $\mathrm{ch}_X$ is the characteristic function of a subset $X$ of $G$.

We want to make a **right-invariant integral** on $\mathcal{D}(G) = C_c^\infty(G)$. That is, we want $u \in \mathcal{D}^*(G)$ *right translation invariant* in the following sense. For a function $f$ on $G$ and for $g, h \in G$, the **right-translation action** of $g \in G$ on $f$ is

$$R_g f(h) = f(hg)$$

The **dual** or **contragredient right translation** action of $g \in G$ on an element $u \in \mathcal{D}^*(G)$ is

$$(R_g^* u)(f) = u(R_{g^{-1}} f) = u(R_g^{-1} f)$$

The $g^{-1}$ occurs to have associativity

$$R_{gh}^* = R_g^* R_h^*$$

The requirement of right translation invariance is that, for all $g \in G$,

$$R_g^* u = u$$

The previous observations show that the values of $u$ on all $\mathrm{ch}_K$ (with $K$ a compact open subgroup) completely determine $u$, if $u$ exists. Further, for $K' \subset K$ are two compact open subgroups,

$$\mathrm{ch}_K(g) = \sum_{x \in K' \backslash K} \mathrm{ch}_{K'}(gx)$$

Thus,

$$u(ch_K) = [K : K']\, u(\mathrm{ch}_{K'}) \qquad \text{(where } [K : K'] \text{ is the index)}$$

Since the intersection of any two compact open subgroups is again such, *the $k$-vectorspace of all such distributions $u$ is at most one-dimensional.*

The assumption that $k$ is of characteristic zero is used to prove *existence* of a non-zero functional. Fix a compact open subgroup $K_0$ of $G$. Take $f \in \mathcal{D}(G)$ and let $X = \mathrm{spt}(f)$. For a compact open subgroup $K$ of $G$ sufficiently small so that $f$ is left $K$-invariant and $K \subset K_0$, and define

$$u_K(f) \;=\; [K_0 : K]^{-1} \sum_{x \in K\backslash X} f(x)$$

As in the uniqueness discussion, the value $u_K(f)$ does not depend upon $K$ for $K$ sufficiently small. Therefore, put

$$u(f) \;=\; \lim_K u_K(f) \qquad\qquad (K \text{ compact open subgroup shrinking to } \{1\})$$

The sense of this limit is the following reasonable one. For a $k$-valued function $K \to c_K$ on compact open subgroups, define $\lim_K c_K$ to be the element $c \in k$ so that, for some compact open subgroup $K_1$, $K \subset K_1$ implies $c_K = c$.

To check the right $G$-invariance:

$$u_K(R_g f) \;=\; [K_0 : K]^{-1} \sum_{x \in K\backslash X g^{-1}} f(xg) \;=\; [K_0 : K]^{-1} \sum_{x \in K\backslash X} f(x) \qquad\qquad (\text{replacing } x \text{ by } xg^{-1})$$

The assumption on the characteristic allows division by the index $[K_0 : K]$.

*There is a choice of right $G$-invariant distribution $u$ so that for a compact open subgroup $U$ of $G$ the value $u(\mathrm{ch}_U)$ is in $\mathbb{Q}$.* Indeed, the construction gives some fixed compact open subgroup $K_0$ measure 1, the smaller compact open subgroup $K = K_0 \cap U$ has measure $1/[K_0 : K]$, and so $U$ has measure

$$\mathrm{meas}\,(U) \;=\; [U : U \cap K_0]/[K_0 : U \cap K_0] \in \mathbb{Q}$$

Write

$$u(f) \;=\; \int_G f(g)\, dg$$

and refer to *right Haar measure* (i.e., *right translation invariant measure*), without specifying $u$ from the one-dimensional space of invariant distributions, not to mention that we have in no way indicated how to integrate more general types of functions.

Given a (right translation) invariant $u \in \mathcal{D}^*(G)$, we can *compatibly* integrate **vector-valued functions** $f \in \mathcal{D}(G, W)$ for any $k$-vectorspace $W$, as follows. Recall the isomorphism

$$\mathcal{D}(G) \otimes W \longrightarrow \mathcal{D}(G, W) \qquad\qquad (\text{given by } (f \otimes w)(g) = f(g)w)$$

Define

$$u(f \otimes w) \;=\; u(f)w$$

Writing this as integrals, it is

$$\int_G (f \otimes w)(g)\, dg \;=\; \Big(\int_G f\, dg\Big)\, w$$

This gives $W$-valued integrals of $W$-valued test functions.

Symmetrically, make a *left*-invariant 'integral', i.e., construct $u \in \mathcal{D}^*(G)$ *left translation invariant* in the following sense. For a function $f$ on $G$ and for $g, h \in G$, define the **left-translation action** of $g \in G$ on $f$ by

$$L_g f(h) \;=\; f(g^{-1}h)$$

The **dual** or **contragredient left translation** action of $g \in G$ on an element $u \in \mathcal{D}^*(G)$ is

$$(L_g^* u)(f) \;=\; u(L_{g^{-1}} f) = u(L_g^{-1} f)$$

The $g^{-1}$ occurs to have the associativity

$$L_{gh}^* \;=\; L_g^* L_h^*$$

A symmetrical argument to that above shows that the $k$-vectorspace of *left*-invariant distributions on $C_c^\infty(G)$ is one-dimensional.

When a left-invariant distribution is also right-invariant, the group $G$ is **unimodular (relative to the groundfield $k$).** Certainly abelian groups are unimodular. The choice of groundfield can affect unimodularity of a group, although many groups in applications will be unimodular (or not) for all groundfields. The classical notion of unimodularity for literal Haar measures is an instance of our present one, in effect with $k = \mathbb{R}$.

Let $u$ be a non-zero right translation invariant distribution on $\mathcal{D}(G)$. Since left translations commute with right translations, $L_g^* u$ is again a right invariant distribution. By the uniqueness shown above, this distribution is a scalar multiple of $u$. The **modular function** $\delta = \delta_G$ of the group is the $k^\times$-valued function defined on $G$ first by the heuristic

$$\delta(g) \;=\; \frac{d(gx)}{dx} \quad \text{(where } dx \text{ is right Haar measure)}$$

and then precisely by the formula

$$\delta(g) \cdot u \;=\; L_{g^{-1}}^* u$$

It is immediate that $\delta$ is a group homomorphism $G \to k^\times$. Further, $\delta$ is *locally constant* on $G$: take any $f \in \mathcal{D}(G)$ with $uf \neq 0$, and let $K$ be a compact open subgroup of $G$ so that $L_h f = f$ for $h \in K$. That there is such $K$ follows from the local constancy and compact support of $f$. Then

$$\delta(gh)\, u(f) \;=\; (L_{(gh)^{-1}}^* u)(f) = (L_{h^{-1}}^* L_{g^{-1}}^* u)(f) \;=\; (L_{g^{-1}}^* u)(L_h f) = \delta(g)\, u(f) \qquad \text{(for } g \in G,\, h \in K)$$

Again, in integral notation, the previous definition of the modular function gives the following: for $h, g \in G$

$$\delta(h) \int_G f(g)\, dg \;=\; \delta(h)\, u(f) \;=\; (L_{h^{-1}}^* u)(f) \;=\; u(L_h f) \;=\; \int f(h^{-1} g)\, dg \;=\; \int f(g) d(hg)$$

Thus,

$$d(hg)/dg \;=\; \delta(h)$$

We claim that, when $u$ is a *right*-invariant distribution, the distribution

$$v(f) \;=\; u(\delta_G^{-1}\, f)$$

is *left*-invariant. That is, as an integral,

$$f \;\longrightarrow\; \int_G f(x)\, \frac{1}{\delta(x)}\, dx$$

is left-invariant. The heuristic is that

$$\frac{dx}{\delta(x)} \;=\; \text{left Haar measure}$$

Indeed, letting $\delta = \delta_G$,

$$v(L_g f) \;=\; u(\delta^{-1} L_g f) = u(\delta^{-1}(g)\, L_g(\delta^{-1} f)) \;=\; \delta^{-1}(g)\, u(L_g(\delta^{-1} f)) \;=\; \delta^{-1}(g)\, \delta(g)\, u(\delta^{-1} f) \;=\; v(f)$$

Since $\delta$ is locally constant, $\delta f$ is again a test function.

For $G$ is *compact*, $f(g) = 1$ is in $C_c^\infty(G)$, and for all $g \in G$ (with right $G$-invariant distribution $u$)

$$\delta(g)u(f) \;=\; (L_g^* u)(f) \;=\; u(L_g^{-1} f) \;=\; u(f)$$

since $L_g f = f$ for all $g \in G$. Thus, for compact $G$ the modular function $\delta$ is trivial. Thus, *compact groups are unimodular.*

Now consider the effect that the anti-automorphism $g \to g^{-1}$ of the group $G$ has on a (right) invariant distribution. Specifically, for $f \in C_c^\infty(G)$, let

$$\check{f}(g) \;=\; f(g^{-1})$$

Claim that, for a right $G$-invariant distribution $u$ on $C_c^\infty(G)$,

$$u(f) \;=\; u(\delta_G^{-1} \check{f})$$

To see this, first note that

$$u((L_g f)\check{}) \;=\; u(R_g(\check{f})) \;=\; u(\check{f})$$

Thus, $f \to u(\check{f})$ is a left-invariant distribution. On the other hand, $f \to u(\delta_G^{-1} f)$ is a left-invariant distribution, and unique up to a scalar. Up to a constant, this proves the asserted identity. For the constant, let $f$ be the characteristic function of a compact open subgroup $K$ of $G$. Then $\check{f} = f$, and $\delta_G$ is identically 1 on $K$, so the constant is 1, proving the desired equality.

Let $H, G$ be two groups. It is easy to verify the natural isomorphism

$$\Phi : C_c^\infty(G) \otimes C_c^\infty(H) \;\approx\; C_c^\infty(G \times H) \qquad\qquad (\text{by } \Phi(f \otimes \varphi)(g \times h) = f(g)\varphi(h))$$

A similar isomorphism exists more generally:

$$\mathcal{D}(G, V) \otimes \mathcal{D}(H, W) \;\approx\; \mathcal{D}(G \times H, V \otimes W)$$

In particular, every right $G \times H$-invariant distribution on $\mathcal{D}(G \times H)$ is $u \otimes v$, where $u, v$ are right invariant distributions on $G, H$. We have a trivial *Fubini Reciprocity*

$$(u \otimes v)(f \otimes \varphi) \;=\; uf \otimes v\varphi \qquad\qquad (\text{for } f, \varphi \text{ in } \mathcal{D}(G,V), \mathcal{D}(H,W)\ )$$

---

# 6. *Averaging maps and test functions on quotients*

Let $X$ be a locally compact totally disconnected space. Again, a function $f$ on $X$ is **locally constant** if, for all $x$ in the support of the function $f$, there is a neighborhood $U$ of $x$ so that for $y \in U$ we have $f(x) = f(y)$.

Let $H$ be a totally disconnected group acting continuously on the left on $X$. A function $f$ on $X$ is **compactly-supported left modulo** $H$ when $H\backslash(H \cdot \mathrm{spt}(f))$ is compact in $H\backslash X$.

Let $(\sigma, V)$ be a smooth representation of $H$. Define a space $C_c^\infty(H\backslash X, \sigma)$ of $V$**-valued equivariant test functions** by taking the locally constant, $V$-valued functions $f$ compactly supported left modulo $H$, and so that for all $h \in H$ and $x \in X$

$$f(hx) \;=\; \sigma(h) f(x)$$

When $X = X_1 \times X_2$ and $H = H_1 \times H_2$ with $H_i$ acting on $X_i$ (and $H_1$ acting trivially upon $X_2$ and $H_2$ acting trivially upon $X_1$), there is the natural product action

$$(h_1, h_2)(x_1, x_2) \;=\; (h_1 x_1, h_2 x_2)$$

Let $\sigma_i$ be a smooth representation of $H_i$, and let

$$\sigma \;=\; \sigma_1 \otimes \sigma_2$$

be the external tensor product representation of $H_1 \times H_2$. We have a natural isomorphism

$$C_c^\infty(H_1\backslash X_1, \sigma_1) \otimes C_c^\infty(H_2\backslash X_2, \sigma_2) \;\approx\; C_c^\infty(H\backslash X, \sigma) \qquad\qquad (\text{by } f_1 \otimes f_2 \to F_{f_1\otimes f_2}\ )$$

with the latter defined by

$$F_{f_1\otimes f_2}(x_1, x_2) \;=\; f_1(x_1) \otimes f_2(x_2)$$

The proof is easy:

Let $Y$ be a closed subset of a locally compact totally disconnected topological group $\Omega$ in which $H$ is a closed subgroup. Let $H$ act on $Y$ by left multiplication. Let $Z$ be a totally disconnected locally compact space upon which $H$ acts trivially. Take $X$ of the special form $X = Y \times Z$. Fix a non-zero right $H$-invariant distribution $u_H$ on $\mathcal{D}(H)$, and as usual write

$$u(f) \;=\; \int_H f(h)\, dh$$

Define an **averaging map**

$$\alpha : C_c^\infty(X) \otimes V \;\longrightarrow\; C_c^\infty(H\backslash X, \sigma)$$

by

$$\alpha(f \otimes v)(x) = \int_H f(hx)\, \sigma(h)^{-1} v\; dh$$

By the lemma on the topology of quotients, for all $x \in X$ and $f \in C_c^\infty(X)$ the function on $H$ given by $h \to f(hx)$ is in $C_c^\infty(H)$, so this makes sense.

[6.0.1] Lemma: With $X = Y \times Z$ as above, the averaging map $\alpha$ from $C_c^\infty(X) \otimes V$ to $C_c^\infty(H\backslash X, \sigma)$ is a surjection.

*Proof:* Since

$$C_c^\infty(H\backslash Y \times Z, \sigma) \;\approx\; C_c^\infty(H\backslash Y, \sigma) \otimes C_c^\infty(Z)$$

we can restrict our attention to the simpler case that $X = Y$ is a closed subset of a totally disconnected locally compact topological group $\Omega$, of which $H$ is a closed subgroup.

Given $F \in C_c^\infty(H\backslash X, \sigma)$, for each $x$ in the support of $F$ take a compact open subgroup $K(x)$ of $\Omega$ so that $F(x') = F(x)$ for all $x' \in xK(x)$. Since $\sigma$ is *smooth* this is possible. Since $F$ has compact support left modulo $H$, there are finitely-many $K(x_i)$ so that the open sets $HK(x_i)$ cover the support of $F$. Let $K$ be the compact open subgroup of $\Omega$ which is intersection of the $K(x_i)$, and let $\Xi$ be a set of representatives for the *finite* set $H\backslash \mathrm{spt}(F)/K$. One computes that

$$\alpha(ch_{\xi K} F)(x) \;=\; u_H(ch_{H\cap \xi K \xi^{-1}})\, F(x) \cdot \mathrm{ch}_{H\xi K}(x)$$

Let

$$\mu_\xi \;=\; u_H(ch_{H\cap \xi K \xi^{-1}})$$

Since the groundfield $k$ has characteristic zero, and since each $H \cap \xi K \xi^{-1}$ is a non-empty (compact) open subgroup of $H$, no $\mu_\xi$ vanishes. By construction,

$$\alpha \sum_{\xi\in\Xi} \mu_\xi^{-1} ch_{\xi K}\, F \;=\; \sum_{\xi\in\Xi} \chi_{H\xi K} \cdot F \;=\; F$$

///

Now describe a very general construction of smooth representations. Given a totally disconnected locally compact group $G$ acting on the right on $X$, with $H$ acting on the left (i.e., $G \times H$ acts, so the two actions commute). Let $G$ act on functions on $X$ by *right translations.* Then *$C_c^\infty(H\backslash X, \sigma)$ is a smooth representation space for $G$.* Such representations play a fundamental role in the sequel.

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# 7. *Invariant distributions on quotients: a mock-Fubini theorem*

The more traditional discussion of $G$-invariant *measures* on quotients $H\backslash G$ with $H$ a closed subgroup is supplanted by discussion of $G$-invariant *distributions* on $\mathcal{D}(H\backslash G, \sigma)$ with $\sigma$ an *irreducible smooth representation* of $H$.

As in the simpler case treated earlier, define a space of $\sigma^*$-valued **equivariant distributions** as the $k$-linear dual to the test function space $\mathcal{D}(H\backslash G, \sigma)$:

$$\mathcal{D}^*(H\backslash G, \sigma^*) \;=\; \mathrm{Hom}_k(\mathcal{D}(H\backslash G, \sigma), k)$$

The group $G$ acts on $\mathcal{D}(H\backslash G, \sigma)$ by right translations

$$(R_g f)(g') \;=\; f(g'g)$$

The dual action of $G$ on distributions is

$$(R_g^* u)(f) \;=\; u(R_g^{-1} f)$$

We are interested in $G$-invariant integrals for functions in $\mathcal{D}(H\backslash G, \sigma)$. This refers to distributions rather than measures and integrals.

[7.0.1] **Proposition:** For $(\sigma, W)$ be an irreducible smooth representation of $H$, $\mathcal{D}^*(H\backslash G, \sigma^*)$ has a $G$-invariant element if and only if $\sigma$ is the one-dimensional (hence irreducible) smooth representation

$$\sigma \;=\; \frac{\delta_H}{\delta_G\big|_H}$$

where the $\delta$'s are the modular functions. Suppose this condition is met. Let $w$ be a right $G$-invariant distribution on $\mathcal{D}(G)$, $v$ a right $H$-invariant distribution on $\mathcal{D}(H)$, and let $\alpha$ be an averaging map

$$\alpha : \mathcal{D}(G) \otimes \sigma \;\longrightarrow\; \mathcal{D}(H\backslash G, \sigma)$$

given via $v$ as in the previous section. There is a unique right $G$-invariant distribution $u$ on $\mathcal{D}(H\backslash G, \sigma)$ normalized by the condition $u \circ \alpha = w$.

*Proof:* We have already shown that, up to scalar multiples, there is a unique right $G$-invariant distribution

$$w : \mathcal{D}(G) \;\longrightarrow\; k$$

Let

$$\alpha : \mathcal{D}(G) \otimes \sigma \;\longrightarrow\; \mathcal{D}(H\backslash G, \sigma)$$

be the averaging map

$$\alpha(f \otimes w)(g) \;=\; \int_H f(hg)\, \sigma(h)^{-1}(w)\, dh$$

as above.

For every right $G$-invariant $u \in \mathcal{D}^*(H\backslash G, \sigma^*)$, $u \circ \alpha$ is a right $G$-invariant distribution in $\mathcal{D}^*(G)$. The map $u \to u \circ \alpha$ is a $k$-linear map which, because of the surjectivity of the averaging map $\alpha$, has trivial kernel. Thus, the space of right $G$-invariant distributions on $\mathcal{D}(H\backslash G, \sigma)$ is at most one-dimensional.

Use $L_g$ notation for left translations, $R_g$ for right. One computes directly that

$$\alpha(L_h f \otimes w) \;=\; (\delta_H \sigma^{-1})(h)\, \alpha(f \otimes w)$$

Recall that $f \to v(\delta_H^{-1} f)$ is *left* invariant, by definition of the modular function.

For $f \in \mathcal{D}(G)$ and $h \in H$

$$(u \circ \alpha)(L_h f) \;=\; u(\alpha(L_h f)) = (\delta_H \sigma^{-1})(h)\, u(\alpha(f))$$

On the other hand, as just recalled, the right $G$-invariance of $u \circ \alpha$ implies the left $G$-invariance of

$$f \;\longrightarrow\; (u \circ \alpha)(\delta_G^{-1} f)$$

Therefore, by a similar computation,

$$(u \circ \alpha)(L_h f) \;=\; (u \circ \alpha)(\delta_G^{-1}(h)\, L_h(\delta_G^{-1} f)) \;=\; \delta_G(h)(u \circ \alpha)(f)$$

Applying the $k$-linear averaging map $\alpha$ to the analogous identity from the previous paragraph,

$$(u \circ \alpha)(L_h f) \;=\; (\delta_H \sigma^{-1})(h)\, (u(\alpha f))$$

Combining these two equalities, if $u$ is not 0 (as a distribution),

$$\delta_H \cdot \sigma^{-1} \;=\; \delta_G$$

That is, the asserted condition is *necessary.*

Now verify sufficiency of the condition. Let $w$ be a (non-zero) right $G$-invariant distribution on $\mathcal{D}(G)$. Given $f \in \mathcal{D}(H\backslash G, \sigma)$ choose $f_0 \in \mathcal{D}(G)$ so that $\alpha f_0 = f$, invoking the surjectivity of the averaging map $\alpha$. Define $uf = wf_0$. We must show that this does not depend upon choice of $f_0$.

One way of doing this is to define a family of maps

$$s_U : \mathcal{D}(H\backslash G, \sigma) \;\longrightarrow\; \mathcal{D}(G)$$

which are *nearly* one-sided inverses to $\alpha$, i.e., so that for given $f$ in $\mathcal{D}(H\backslash G, \sigma)$

$$(\alpha \circ s_U) f \;=\; f$$

for $U$ sufficiently large, and then define

$$uf \;=\; \lim_U\, (w \circ s_U) f$$

Specifically, let $U$ vary over (non-empty) compact open subsets of $G$, and define

$$s_U f \;=\; v(ch_{U \cap H})^{-1}\, ch_U\, f$$

The normalizing constant $v(ch_{U \cap H})$ appears because

$$\alpha(ch_U\, f) \;=\; v(ch_{U \cap H})\, f$$

when $U$ is sufficiently large so that $\mathrm{spt}(f) \subset H \cdot U$.

Let $V$ be a compact open subset of $G$ containing a compact open subset $V$. Keep in mind the fact noted above that

$$v(f) \;=\; v(\delta_H^{-1}\check{f}) \qquad\qquad (\text{where } \check{f}(h) = f(h^{-1}))$$

For fixed $f \in \mathcal{D}(H\backslash G, \sigma)$ and for $V$ large-enough depending upon the support of $f$,

$$w \circ s_U\, f \;=\; w \circ s_U \circ \alpha \circ s_V\, f \;=\; \int_G \int_H ch_U(hg)\, ch_V(g)\, f(g)\, dh\, dg$$

Changing the order of integration, replacing $g$ by $h^{-1}g$ and $h$ by $h^{-1}$, turns this into

$$\int\!\!\int ch_U(g)\, ch_V(hg)\, f(g)\, \delta_G^{-1}(h)\delta_H(h)\sigma(h)\, dh\, dg \;=\; w(s_V \alpha s_U(f))$$

*if the condition relating $\sigma$ and the modular functions is met.* That is, if this condition is met, then

$$ws_U f \;=\; ws_U\alpha s_V f = ws_V \alpha s_U f = ws_V f$$

That is, the limit exists. ///

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# 8. *Hecke (convolution) algebras*

The test function space $C_c^\infty(G) = C_c^\infty(G, k)$ has a **convolution product** given by

$$(f_1 * f_2)(g) \;=\; \int_G f_1(gh^{-1}) f_2(h)\; dh$$

with *right* Haar measure. For a compact open subgroup $K$ of $G$, the **Hecke algebra of level** $K$ is

$$\mathcal{H}(G,K) \;=\; \mathcal{H}(G,K)_k \;=\; \text{compactly-supported left and right } K\text{-invariant}$$

$$k\text{-valued functions on } G$$

The full **Hecke algebra** of $G$ is

$$\mathcal{H}_G \;=\; \mathcal{H}(G) \;=\; \mathcal{H}(G)_k \;=\; \bigcup_K \mathcal{H}(G,K)$$

Fix a *right Haar measure* $dg$ on $G$. Each $\mathcal{H}(G,K)$ is a *convolution algebra* with the convolution multiplication in $C_c^\infty(G)$. Define

$$e_K \;=\; \mathrm{ch}_K / \operatorname{meas}(K)$$

It is easy to check that $e_K$ is the unit in $\mathcal{H}(G,K)$.

For a smooth representation $(\pi, V)$ of $G$ on a $k$-vectorspace $V$, $\eta \in \mathcal{H}_G$ acts on $v \in V$ by

$$\eta v \;=\; \pi(\eta)(v) \;=\; \int_G \eta(g)\; \pi(g)(v)\; dg$$

Because of the compact support and local constancy the previous integral is actually a finite sum

$$\sum_i c_i\; \pi(g_i)(v) \qquad\qquad (\text{with } c_i \in k \text{ and } g_i \in G)$$

Indeed, fix $v \in V$ and take a small-enough compact open subgroup $K$ such that $v \in V^K$ and $\eta$ is right $K$-invariant. By integration, letting $X$ be the support $\mathrm{spt}(\eta)$ of a function $\eta$,

$$\pi(\eta)v \;=\; \int_G \eta(g)\,\pi(g)v\,dg \;=\; \sum_{x\in X/K}\int_K \eta(x\theta)\,\pi(x\theta)v\,d\theta \;=\; \sum_x \eta(x)\,\pi(x)v\ \mathrm{meas}\,(xK)$$

Since $X = \mathrm{spt}(\eta)$ is compact, the sum over $x$ is finite.

The map $\eta \to \pi(\eta)$ is a $k$-algebra homomorphism from $\mathcal{H}_G$ (with convolution) to endomorphisms of $V$ (with composition), proven by direct computation: using right Haar measures,

$$\pi(f*F)v \;=\; \int\!\!\int f(gh^{-1})\,F(h)\,\pi(g)v\,dh\,dg \;=\; \int\!\!\int f(gh^{-1})\,F(h)\,\pi(g)v\,dg\,dh$$

$$=\; \int\!\!\int f(g)\,F(h)\,\pi(gh)v\,dg\,dh \;=\; \int f(g)\,\pi(g)F(h)\,\pi(h)v\,dg\,dh = \pi(f)\,\pi(F)v$$

by Fubini's theorem (these are finite sums) and changing variables replacing $g$ by $gh$. ///

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# 9. *Smooth $\mathcal{H}$-modules versus smooth $G$-representations*

Let $K$ be a compact open subgroup of $G$, and define an element of the Hecke algebra $\mathcal{H}$ by

$$e_K \;=\; \mathrm{ch}_K/\,\mathrm{meas}\,(K) \qquad\qquad \text{(with right Haar measure)}$$

The property of smoothness of a $G$-representation $(\pi, V)$ assures that for all $v \in V$ there is a small-enough $K$ so that $\pi(e_K)v = v$. In this spirit, a module $V$ over the ring $\mathcal{H}_G$ is **smooth** when for every finite subset $X$ of $V$ there is a small-enough compact open subgroup $K$ so that $e_K x = x$ for all $x \in X$.

The elements $e_K$ in $\mathcal{H}$ are *idempotents* in $\mathcal{H}$, and for $K' \subset K$

$$e_K * e_{K'} \;=\; e_{K'} * e_K \;=\; e_K$$

This follows by direct computation: in

$$(e_K * e_{K'})(g) \;=\; \int_G e_K(gh^{-1})\,e_{K'}(h)dh$$

the integrand is zero unless $h \in K'$ and $gh \in K$, i.e., unless $h \in K'$ and $g \in K$ since $K' \subset K$, in which case it is

$$\mathrm{meas}\,(K')^{-1}\,\mathrm{meas}\,(K)^{-1}$$

The integral of this over $K'$ is $\mathrm{meas}\,(K)^{-1}$. Thus, $e_K$, as claimed.

Now we claim that *smooth $\mathcal{H}$-modules are in bijection with smooth $G$-representations, by*

$$(\pi, V) \;\longrightarrow\; \mathcal{H}\text{-module with } \eta v = \pi(\eta)v$$

We have already seen how to get smooth $\mathcal{H}$-modules from smooth $G$-representations (as indicated). We need to recover the $G$-representation from the $\mathcal{H}$-module.

Generalizing previous notation, for a compact open subset $X$ of $G$, let

$$e_X \;=\; \mathrm{meas}\,(X)^{-1}\,\mathrm{ch}_X \qquad\qquad \text{(with } \textit{right} \text{ Haar measure)}$$

For $v$ in an $\mathcal{H}$-module $V$ with $K$ a small-enough compact open subgroup so that $e_K v = v$, and for $g \in G$, put $\pi_V(g)v = e_{gK}v$.

First, check that $K$ may be made smaller without altering $e_{gK}(v)$. It suffices to show that, for $K' \subset K$,

$$e_{gK'} * e_K \;=\; e_{gK}$$

This is a direct computation. Second, check that $\pi_V$ is a group homomorphism: this too is a direct computation.

Next, it is easy to check that $G$-homomorphisms and $\mathcal{H}_G$-module homomorphisms are interchanged under this bijection between smooth $G$-representations and smooth $\mathcal{H}_G$ modules (both over a field $k$).

In particular, under this bijection, *irreducible $G$-representations are simple $\mathcal{H}_G$-modules, and vice-versa.*

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# 10. *Central characters and relative Hecke algebras*

Let $Z$ be a closed subgroup of the center of $G$. In applications, it may be appropriate to take $Z$ smaller than the whole center.

For present purposes, a **(one-dimensional) character** $\omega$ on $Z$ is a locally constant group homomorphism $\omega : Z \to k^\times$.

A smooth representation $(\pi, V)$ has **central character** $\omega$ when

$$\pi(z)v \;=\; \omega(z)\cdot v \qquad\qquad (\text{for } z \in Z \text{ and } v \in V)$$

Fix a character $\omega$ of the closed subgroup $Z$ of the center of $G$. Let $\mathcal{H}_\omega(Z\backslash G, K)$ be the collection of left and right $K$-invariant $k$-valued functions $f$ on $G$ so that $f(zg) = \omega(z)f(g)$ for all $z \in Z$ and $g \in G$, and so that $f$ is compactly-supported modulo $Z$. Define

$$\mathcal{H}_\omega(Z\backslash G) \;=\; \bigcup_K \mathcal{H}_\omega(Z\backslash G, K)$$

This is the **relative Hecke algebra** for the central character $\omega$.

There is a *relative convolution* $\star$ on $\mathcal{H}_\omega(Z\backslash G)$,

$$(f \star \varphi)(g) \;=\; \int_{Z\backslash G} f(gh^{-1})\,\varphi(h)dh \qquad\qquad (\text{right Haar measure on } Z\backslash G)$$

Since

$$h \;\longrightarrow\; f(gh^{-1})\,\varphi(h)$$

is $Z$-invariant, compactly-supported on $Z\backslash G$, and locally constant on $Z\backslash G$, the integral is a finite sum.

It is straightforward that the averaging map $\alpha : \mathcal{H}(G) \to \mathcal{H}_\omega(G)$ given by

$$(\alpha f)(g) = \int_Z \omega(z)\, f(z^{-1}g)\, dz$$

is not only surjective (as proven above more generally), but is also a *convolution algebra homomorphism.*

The images $\alpha e_K$ of the idempotents

$$e_K \;=\; \mathrm{ch}_K/\,\mathrm{meas}\,(K)$$

in the Hecke algebra $\mathcal{H}(G)$ are easily seen to be idempotents in the relative Hecke algebra $\mathcal{H}_\omega(G)$. A module $V$ over the ring $\mathcal{H}_\omega(G)$ is **smooth** if, for every finite subset $X$ of $V$ there is a compact open subgroup $K$ of $G$ so that $(\alpha e_K)x = x$ for all $x \in X$.

A smooth $G$-representation $(\pi, V)$ *with central character* $\omega^{-1}$ gives rise to a smooth $\mathcal{H}_\omega(G)$-module, as follows. define

$$\pi(f)v \;=\; \int_{Z\backslash G} f(g)\,\pi(g)v\,dg \qquad\qquad (\text{for } f \in \mathcal{H}_\omega(G) \text{ and } v \in V)$$

Note that the $V$-valued function

$$g \;\longrightarrow\; f(g)\,\pi(g)v$$

is indeed $Z$-invariant.

As in the earlier simple case where $Z = \{1\}$, there is converse: every smooth $\mathcal{H}_\omega(G)$-module gives rise to a smooth $G$-representation with central character $\omega^{-1}$. The smooth $G$-representations with central character $\omega^{-1}$ are in bijection with smooth $\mathcal{H}(G)$-modules. The proof is analogous to that given above for $Z = \{1\}$.

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# 11. *Schur's Lemma*

We use the hypothesis that $G$ *has a countable basis*. Further, assume that the groundfield $k$ is *uncountable and algebraically closed*. This includes the case $k = \mathbb{C}$. The important corollary here, that irreducibles have central characters, is not obviously true without some such hypothesis. This is already demonstrable for finite-dimensional representations of finite abelian groups, for example.

[11.0.1] Theorem: Let $(\pi, V)$ be an irreducible smooth representation of $G$ on a $k$-vectorspace $V$, where $k$ is an uncountable and algebraically-closed field. Let $T \in \mathrm{End}_k(V)$ be a $k$-linear map commuting with all maps $\pi(g)$ with $g \in G$. Then $T$ is a *scalar*, that is, multiplication by an element of $k$.

*Proof:* (Jacquet) Since $G$ has a countable basis, $\mathcal{H}$ has countable dimension over $k$. Irreducibility implies that, for $v \neq 0$ in $V$, $\mathcal{H} \cdot v = V$, so $V$ is of countable $k$-dimension. Further, an $\mathcal{H}$-endomorphism $T$ is completely determined by $Tv$ for one $v \neq 0$, since $T(\eta v) = \eta T(v)$ for $\eta \in \mathcal{H}$. Thus, the ring $D$ of $\mathcal{H}$-endomorphisms of $V$ has countable $k$-dimension. As $V$ is irreducible,for all $T \in D$ both the kernel and image of $T$ are $\mathcal{H}$-submodules, so can be only 0 or $V$. Thus, $D$ is a division ring with $k$ in its center.

Since $k$ is algebraically closed, non-scalar $T \in \mathrm{End}_G(V)$ is necessarily *transcendental* over $k$. Therefore, for $T \in D$ *not* a scalar the elements

$$S_\lambda \;=\; (T - \lambda)^{-1}$$

in $D$ (with $\lambda$ varying over $k$) are *linearly independent* over $k$, by uniqueness of partial fraction expansions in $k(T)$. As $k$ is uncountable, this would yield an uncountable set of linearly-independent elements of $D$, contradiction. ///

[11.0.2] Corollary: With uncountable and algebraically closed groundfield $k$, an irreducible smooth representation of a group $G$ with countable basis has a central character, necessarily a smooth $k^\times$-valued representation.

*Proof:* By Schur's Lemma, each $\pi(z)$ with $z \in Z$ is a scalar. Since $\pi$ is a group homomorphism, so is $\omega$. For continuity, fix $v \neq 0$ in $V^K$ with $K$ a compact open subgroup of $G$. For $h \in K \cap Z$ and $z \in Z$

$$\omega(zh)v \;=\; \omega(z)\omega(h)v \;=\; \omega(z)\pi(h)v \;=\; \omega(z)v$$

Thus, $\omega$ is locally constant, and is continuous. ///

## 12. *Left, right, and biregular representations*

The **right regular representation** of $G$ on $V = C_c^\infty(G)$ is

$$(R_g f)(h) \;=\; f(hg)$$

The **left regular representation** of $G$ on $V = C_c^\infty(G)$ is

$$(L_g f)(h) \;=\; f(g^{-1}h)$$

The $g^{-1}$ appears, rather than just $g$, so that $g \to L_g$ is a group homomorphism (rather than anti-homomorphism).

The **biregular representation** of $G \times G$ is the action of $G \times G$ on $V = C_c^\infty(G)$ by

$$(\pi_{bi}(g \times g')f)(h) \;=\; f(g'^{-1}hg) \qquad \text{or} \qquad (\pi_{bi}(g \times g')f)(h) = f(g^{-1}hg')$$

In practice, the left translation $L_g$ and right translation $R_g$ operators are applied to all functions on $G$. Certainly these translations make sense with no particular hypotheses on the functions involved. Concommitantly, the terminology is often used in an imprecise way, referring to any $f \to R_g f$, $f \to L_g f$ as right and left regular representations, etc., without concern for the nature of the function $f$.

## 13. *An elementary dualization identity*

Let $\pi$ and $\sigma$ be two smooth representations of $G$. Let $k$ denote the trivial representation of $G$.

[13.0.1] **Proposition:** There is a natural $k$-isomorphism

$$\mathrm{Hom}_G(\sigma \otimes \pi, k) \;\approx\; \mathrm{Hom}_G(\sigma, \check{\pi})$$

given by $\varphi \to \Phi_\varphi$ where

$$\Phi_\varphi(v)(w) \;=\; \varphi(v \otimes w) \qquad (\text{for } v \in \sigma \text{ and } w \in \pi)$$

The reverse map is given by $\varphi_\Phi \leftarrow \Phi$ where

$$\varphi_\Phi(v \otimes w) \;=\; \Phi(v)(w)$$

*Proof:* Once the maps are given, only some small details remain to be checked. ///

## 14. *Induced representations c-Ind$_H^G\,\sigma$ and duals* $\mathrm{Ind}_H^G\,\check{\sigma}\delta_H\delta_G^{-1}$

Let $(\sigma, W)$ be a smooth representation of a closed subgroup $H$ of $G$. As before, let $C_c^\infty(H\backslash G, \sigma)$ be the space of $W$-valued functions $f$ on $G$ compactly-supported left-modulo $H$, locally constant, and so that

$$f(hg) \;=\; \sigma(h)\,f(g) \qquad (\text{for } h \in H \text{ and } g \in G)$$

The **compact-induced representation** $(\pi, V) = \mathrm{ind}_H^G\,\sigma$ has representation space

$$V \;=\; C_c^\infty(H\backslash G, \sigma)$$

and has the right translation action of $G$

$$((\pi g)f)(g') \;=\; f(g'g) \;=\; (R_g f)(g)$$

It is easy to check that this is a smooth representation of $G$.

The **induced representation** $(\pi, V) = \mathrm{Ind}_H^G\,\sigma$ has representation space $V$ consisting of *uniformly* locally constant $W$-valued functions $f$ on $G$ satisfying

$$f(hg) \;=\; \sigma(h)\,f(g) \qquad\qquad (\text{for } h \in H \text{ and } g \in G)$$

This has the right translation action

$$((\pi g)f)(g') = f(g'g) = (R_g f)(g') \qquad\qquad (\text{for } g \in G)$$

It is easy to check that this is a smooth representation of $G$. The *uniform* condition is that for $f$ in this space of functions there be a compact open subgroup $\Theta$ so that

$$f(g\theta) \;=\; f(g) \qquad\qquad (\text{for all } g \in G \text{ and for all } \theta \in \Theta)$$

Let $u$ be a right $G$-invariant distribution on $\mathrm{ind}_H^G\,\delta_H/\delta_G$, unique up to scalar multiples, whose existence and uniqueness noted earlier, in discussion of $G$-invariant distributions on spaces

$$\mathcal{D}(H\backslash G, \sigma) \;=\; \mathrm{ind}_H^G \sigma$$

Let

$$\alpha : \mathcal{D}(G) \otimes \sigma \;\longrightarrow\; \mathcal{D}(H\backslash G, \sigma)$$

be an averaging map as before.

**[14.0.1] Proposition:** Let $\delta_H, \delta_G$ be the modular functions of $H, G$. The induced representation $\mathrm{Ind}_H^G\,\check{\sigma}\delta_H\delta_G^{-1}$ is naturally isomorphic to the smooth dual of the compact-induced representation $\mathrm{ind}_H^G\,\sigma$, by the map described as follows. For $f \in \mathrm{ind}_H^G\,\sigma$ and $F \in \mathrm{Ind}_H^G\,\check{\sigma}\delta_H\delta_G^{-1}$, put

$$\varphi(g) \;=\; F(g)(f(g))$$

and define the pairing

$$\langle f, F\rangle \;=\; u(g \;\longrightarrow\; F(g)(f(g)))$$

Then $f \to \langle f, F\rangle$ is the smooth linear functional attached to $F$, and all smooth linear functionals on $\mathrm{ind}_H{}^G\,\sigma$ are given by such an expression.

*Proof:* First, claim that the function $\varphi$ above is in $\mathrm{ind}_H^G \delta_H\delta_G^{-1}$. Since $F$ is uniformly locally constant and $f$ is locally constant and compactly-supported left modulo $H$, $\varphi$ is locally constant and compactly-supported left modulo $H$. And, $F$ is $\check{\sigma}$-valued and $f$ is $\sigma$-valued, so $\varphi(g) \in k$. Further,

$$\varphi(hg) \;=\; F(hg)(f(hg)) \;=\; (\delta_H\delta_G^{-1})(h)\,\sigma^*(h)F(g)\,(\sigma(h)f(g))$$

$$=\; (\delta_H\delta_G^{-1})(h)\,F(g)\,\sigma(h^{-1})\,\sigma(h)f(g) \;=\; (\delta_H\delta_G^{-1})(h)\,F(g)f(g) \;=\; (\delta_H\delta_G^{-1})(h)\,\varphi(g)$$

It is easy to check that

$$\lambda_F : f \;\longrightarrow\; \langle f, F\rangle$$

is a *smooth* functional. Therefore, the functionals $\lambda_F$ form a $k$-linear subspace of the smooth dual of $\mathrm{ind}_H^G\,\sigma$. That $F \to \lambda_F$ is a $G$-homomorphism is also apparent.

To check that $F \to \lambda_F$ is *injective*, choose $x \in G$ so that $F(x) \neq 0$ and $K$ a compact open subgroup so that $F$ is right $K$-invariant. Let $v \in \sigma$ so that $F(x)v \neq 0$, and define $f \in \mathrm{ind}_H^G \sigma$ by

$$f(g) \;=\; \alpha(\mathrm{ch}_{xK} \otimes v) \qquad\qquad (\alpha \text{ the averaging map})$$

Then

$$\langle f, F\rangle \;=\; u(ch_x K) \;=\; \mathrm{meas}\,(xK) \;\neq\; 0 \qquad\qquad (\mathrm{ch}_X \text{ characteristic function of } X)$$

For *surjectivity*, prove surjectivity to each $(\mathrm{ind}_H^G\sigma)^K$, for compact open subgroups $K$. Fix a set of representatives $x_i$ for $H\backslash G/K$ and let $f_i$ be the characteristic function of $x_i K$. Take $\lambda \in (\mathrm{ind}_H^G\sigma)^K$. Define $\mu_i \in \check{V}$ by

$$\mu_i(v) \;=\; \lambda(\alpha(f_i \otimes v)) \qquad\qquad (\text{for } v \in V)$$

It is easy to check that this is a smooth functional on $V$. Define

$$F(hx_i\theta) \;=\; (\check{\sigma}\delta_H\delta_G^{-1})(h)\mu_i$$

It is routine that under the pairing $F$ recovers the given functional $\lambda$. ///

# 15. *Frobenius Reciprocity*

Let $H$ be a closed subgroup of $G$. Let $\sigma$ be a smooth representation of $H$ and let $\pi$ be a smooth representation of $G$.

[15.0.1] Theorem: **Frobenius Reciprocity:** There is a natural $k$-vectorspace isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) \longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G\pi, \sigma) \qquad\qquad (\text{by } \Phi \to \varphi_\Phi)$$

where

$$\varphi_\Phi(v) \;=\; \Phi(v)(1_G)$$

and the inverse is $\Phi_\varphi \leftarrow \varphi$ where

$$\Phi_\varphi(v)(g) \;=\; \varphi(R_g v)$$

*Proof:* Once the formulas are written the proof is easy. One should check that the $\Phi$'s are $G$-homomorphisms and that the $\varphi$'s are $H$-homomorphisms, and that they do map to the indicated spaces. ///

[15.0.2] Remark: There is the hazard that, for $\pi$ given as a collection of *functions* with smoothness in terms of compact open subgroups of $G$, the restriction $\mathrm{Res}_H^G\,\pi$ is probably *not* describable in terms of smoothness conditions from $H$.

# 16. *Compact induction as a tensor product*

We describe compactly-induced representations as **tensor products**.

Let $\sigma$ be a smooth representation of a closed subgroup $H$ of $G$. Give $\mathcal{H}_H = C_c^\infty(G)$ the natural *right* $\mathcal{H}_H$-module structure

$$(\eta\,\zeta)(g) \;=\; \int_H \eta(gh^{-1})\zeta(h)\,dh$$

Form the tensor product

$$\mathcal{H}_G \otimes_{\mathcal{H}_H} \sigma \qquad\qquad (\text{a } \mathcal{H}_G\text{-module by } \eta(f\otimes v) = (\eta\cdot f)\otimes v)$$

The comparison of this module-theoretic induction with the group-theoretic is not entirely trivial, as left-right issues and normalization-of-measures issues arise.

Let $\delta_H, \delta_G$ be the modular functions on $H, G$, respectively. That is, in terms of the right $G$-invariant functional

$$f \longrightarrow \int_G f(g)\, dg$$

written as an integral

$$\int_G f(x^{-1}g)\, dg = \delta_G(x) \int_G f(g)\, dg$$

and similarly for $H$. For a function $f$ on $G$, let

$$\check{f}(g) \;=\; f(g^{-1})$$

The needed map is not quite the obvious averaging map, but

$$\beta(f \otimes v)(g) \;=\; \int_H \frac{\check{f}(hg)}{\delta_G(hg)}\, \frac{\delta_G(h)}{\delta_H(h)}\, dh \;=\; \frac{1}{\delta_G(g)} \int_H \check{f}(hg)\, \frac{dh}{\delta_H(h)}$$

While the latter expression is simpler, some aspects of the structure are better revealed in the former.

**[16.0.1] Proposition:** The map $\alpha$ induces an isomorphism

$$\beta : \mathcal{H}_G \otimes_{\mathcal{H}_H} V \longrightarrow \mathrm{ind}_H^G\,(\sigma \otimes \frac{\delta_H}{\delta_G})$$

*Proof:* It is a matter of changing variables in the integral to see that the image $\beta(f \otimes v)$ lies in the indicated compact-induced representation space.

*Surjectivity* is proven in the same manner as earlier for simpler averaging maps. It remains to show that the map factors through the smaller tensor product over $\mathcal{H}_H$ (i.e., not merely over the field $k$), i.e., that

$$\beta(f\zeta \otimes v) \;=\; \beta(f \otimes \zeta v)$$

We also must show that the induced map respects the $\mathcal{H}_G$-module structure. And, finally, show that this induced map is a vectorspace isomorphism.

To prove that the induced map respects the $\mathcal{H}_G$-module structure it suffices to take $H = \{1\}$ and $\sigma$ trivial, since the $\mathcal{H}_H$-action and $\mathcal{H}_G$-actions commute in any case, one being on the left and the other on the right. This simplifies the appearance of the formulas considerably. Take $\eta, f \in \mathcal{H}_H$. For $g \in G$ the definition of $\beta$ simplifies to

$$\beta(f)(g) \;=\; \check{f}(g)/\delta_G(g)$$

Then

$$\beta(\eta * f)(g) \;=\; \frac{1}{\delta_G(g)} \int_G \eta(g^{-1}h^{-1}) f(h)\, dh$$

$$= \frac{1}{\delta_G(g)} \int_G \check{f}(h^{-1}) \eta(g^{-1}h^{-1})\, dh \;=\; \frac{1}{\delta_G(g)} \int_G \check{f}(gh^{-1}) \eta(h^{-1})\, dh$$

by replacing $h$ by $hg^{-1}$. Recalling that sending $h$ to $h^{-1}$ turns a right-invariant integral into a left-invariant one, replacing $h$ by $h^{-1}$ turns the above into

$$\frac{1}{\delta_G(g)} \int_G \check{f}(gh) \eta(h)\, \frac{dh}{\delta_G(h)} = \int_G \eta(h)\, \frac{\check{f}(gh)}{\delta_G(hg)}\, dh$$

$$= \int_G \eta(h)\ (\beta f)(gh)\ dh$$

confirming that the $\mathcal{H}_G$-module structure is respected. The appearance of $\delta_G$ is unavoidable.

Now verify that $\beta$ factors through the smaller tensor product $\mathcal{H}_H \otimes_{\mathcal{H}_H} \sigma$. Take $\zeta \in \mathcal{H}_H$ and $f \in \mathcal{H}_G$. Then

$$\beta(f *_H \zeta \otimes v)(g) \;=\; \int_H \frac{(f *_H \zeta)\check{}(yg)}{\delta_H(y)}\sigma(y^{-1})v\ dy \;=\; \int_H \int_H \frac{f(g^{-1}y^{-1}x^{-1})}{\delta_H(y)}\ \zeta(x)\sigma(y^{-1})v\ dy\ dx$$

$$= \int_H \int_H \frac{\check{f}(xyg)}{\delta_H(y)}\ \zeta(x)\sigma(y^{-1})v\ dy\ dx \;=\; \int_H \int_H \frac{\check{f}(yg)}{\delta_H(x^{-1}y)}\ \zeta(x)\sigma(y^{-1}x)v\ \delta_H(x^{-1})\ dy\ dx$$

by replacing $y$ by $x^{-1}y$, using the definition of $\delta_H$. This is

$$\int_H \int_H \frac{\check{f}(yg)}{\delta_H(y)}\sigma(y^{-1})\ \zeta(x)\sigma(x)v\ dy\ dx \;=\; \int_H \frac{\check{f}(yg)}{\delta_H(y)}\sigma(y^{-1})\ \zeta v\ dy = \beta(f \otimes (\zeta v))$$

where $\zeta v = \sigma(\zeta)v$. Thus, the map factors through the tensor product $\mathcal{H}_G \otimes_{\mathcal{H}_H} \sigma$ as asserted.

It remains to prove *injectivity* of the induced map $\beta$, i.e., to show that the kernel of the averaging map $\beta$ is spanned by the collection of differences

$$(f\zeta \otimes v) - (f \otimes \zeta v) \qquad\qquad (\text{in } C_c^\infty(G) \otimes_k V)$$

Fix a compact open subgroup $K$ of $G$. Fix a set of representatives $x_i$ for $H\backslash G/K$, and let $f_i$ be the characteristic function of $x_i K$. For $f$ a left-and-right $K$-invariant element of $C_c^\infty(G)$, the difference

$$(f\zeta \otimes v) - (f \otimes \zeta v)$$

is a finite linear combination of elements of the form

$$(f \otimes (\delta_H(\sigma h)v)) - (\delta_H(h)^{-1} L_h^{-1} f \otimes v) \qquad\qquad (\text{for } h \in H)$$

Thus, every element of the image of $C_c^\infty(G)^K \otimes V$ in $C_c^\infty(G) \otimes_{\mathcal{H}_H} V$ can be written as a finite sum

$$\sum_i f_i \otimes v_i \qquad\qquad (\text{for some } 0 \neq v_i \in V)$$

Since the supports of the functions $\beta(f_i \otimes v_i)$ are *disjoint*, the averages $\beta(f_i \otimes v_i)$ must be linearly independent. (One may check directly that each is not identically zero.) This proves injectivity. ///

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# 17. *Iterated induction*

Let $Q$ be a closed subgroup of $H$ and $H$ a closed subgroup of $G$. Let $(\sigma, V)$ be a smooth representation of $Q$. We claim that *compact-induction and induction of representations are compatible with iteration,* i.e., that we have natural isomorphisms

$$\mathrm{ind}_H^G(\mathrm{ind}_Q^H \sigma) \;\approx\; \mathrm{ind}_Q^G \sigma \qquad\qquad \mathrm{Ind}_H^G(\mathrm{Ind}_Q^H \sigma) \;\approx\; \mathrm{Ind}_Q^G \sigma$$

The proof of the second immediately reduces to that of the first, granting the fact, already proven, that

$$(\mathrm{ind}_Q^G \sigma)\check{} \;\approx\; \mathrm{Ind}_Q^G\, \check{\sigma}\delta_Q/\delta_G$$

From the previous section, compact induction is the same as taking a tensor product, so, with suitable module structures,

$$\mathrm{ind}_H^G(\mathrm{ind}_Q^H\sigma) \;\approx\; \mathcal{H}(G)\otimes_{\mathcal{H}(H)}(\mathcal{H}(H)\otimes_{\mathcal{H}(Q)}\sigma) \;\approx\; \mathcal{H}(G)\otimes_{\mathcal{H}(Q)}\sigma \;\approx\; \mathrm{ind}_Q^G\sigma$$

We are granting that *even for certain rings without units*

$$A\otimes_B(B\otimes_C V) \;\approx\; A\otimes_C V$$

This holds for **idempotented** rings $A, B, C$: a ring $R$ not necessarily possessing a *unit* is *idempotented* when, for every finite subset $X$ of $R$, there is an idempotent element $e$ of $R$ so that $ex = x = xe$ for all $x \in X$. This property holds for these Hecke algebras.

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# 18. *Isotypes and co-isotypes (Jacquet modules)*

Let $(\pi, V)$ be an irreducible smooth representation of $G$, and let $(\sigma, W)$ be an arbitrary smooth representation of $G$.

The $\pi$**-isotype** $\sigma^\pi$ in $\sigma$ is the smallest $G$-subrepresentation of $\sigma$ so that every $G$-homomorphism $\pi \to \sigma$ factors through it. Because of the arrow-theoretic nature of this definition, uniqueness is clear.

Existence follows from a different description, as

$$\sigma^\pi \;=\; \sum_\varphi \varphi(V_\pi)$$

where $\varphi$ is summed over $\mathrm{Hom}_G(\pi,\sigma)$. The **multiplicity** of $\pi$ in $\sigma$ is $\dim_k \mathrm{Hom}_G(\pi,\sigma)$.

The $\pi$**-co-isotype** $\sigma_\pi$ of $\sigma$ is the smallest quotient of $\sigma$ so that every $G$-homomorphism $\varphi : \sigma \to \pi$ factors through $\sigma_\pi$. This definition yields uniqueness of the co-isotype.

We give another description to prove existence of the co-isotype. Let $Q$ be the intersection of the kernels of all $G$-homomorphisms $\varphi : \sigma \to \pi$. Then

$$W_\pi \;=\; W/Q \qquad\qquad \text{(with the obvious quotient map)}$$

The **co-multiplicity** of $\pi$ in $\sigma$ is $\dim_k \mathrm{Hom}_G(\sigma,\pi)$.

Writing $\sigma^\pi$ for the isotype and $\sigma_\pi$ for the co-isotype is not so standard, but is consistent and reasonable. When $\pi$ is the trivial representation (i.e., on a one-dimensional $k$-space), use the earlier notation $W^G = \pi^G$ for the isotype of the trivial representation.

Consider the special case that $\pi$ is one-dimensional, especially that $\pi$ is the trivial representation. As usual, write $\chi$ instead of $\pi$ for a one-dimensional smooth representation, and view $\chi$ as $k^\times$-valued:

$$\chi : G \longrightarrow k^\times$$

In this case, we can explicitly *construct the $\chi$-co-isotype of a representation* $(\sigma, V)$, as follows. Let $Q$ be the $k$-subspace of $\sigma$ spanned by differences $\sigma(g)w - \chi(g)w$ for $w \in W$ and $g \in G$. Note that $Q$ is a $G$-subspace. We *claim that the quotient $W/Q$ is the $\chi$-co-isotype.* Certainly $G$ acts trivially on this quotient: for $g \in G$ and $w \in W$

$$\sigma(g)w + Q \;=\; \chi(g)w + (\sigma(g)w - \chi(g)w) + Q \;=\; \chi(g)w + Q$$

On the other hand, by construction, for $\varphi$ a $G$-homomorphism to $\chi$, certainly all the elements spanning $Q$ map to 0.

Consider a totally disconnected group $P$ with a *normal* subgroup $N$. Fix a one-dimensional smooth representation

$$\chi : N \longrightarrow k^\times$$

of $N$, and assume that the action of $P$ stabilizes $\chi$, i.e.,

$$\chi(pnp^{-1}) \;=\; \chi(n) \qquad\qquad (\text{for all } p \in P \text{ and } n \in N)$$

Given a smooth representation $(\sigma, W)$ of $P$, again let $Q$ be the $k$-subspace of $V$ spanned by differences $\chi(n)v - \sigma(n)v$ for $n \in N$ and $v \in V$. Then $Q$ is $N$-stable and $P$-stable, so the co-isotype $W/Q$ *is a $P$-representation*, as well as being acted-upon trivially by $N$.

With the further hypothesis that $N$ *is the union of an ascending chain of compact open subgroups*, viewed as a representation of $P$, the trivial $N$-co-isotype is a (generalized) **Jacquet module**. The implications of these hypotheses on $N$ are discussed below.

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# 19. *Representations of compact* $G/Z$

Let $Z$ be a closed subgroup of $G$ inside the center of $G$, and suppose that $G/Z$ is *compact*. Consider representations $(\pi, V)$ with central character $\omega : Z \to k^\times$, i.e., so that

$$\pi(z)v \;=\; \omega(z)v \qquad\qquad (\text{for all } v \in V \text{ and } z \in Z)$$

This generality is useful, but the simpler situation that $G$ itself is compact and $Z$ is trivial might be contemplated to see more clearly what's going on.

[19.0.1] Proposition: Every finitely-generated smooth representation $(\pi, V)$ of $G$ with central character $\omega$ (for $Z$) is finite-dimensional.

*Proof:* Take a compact open subgroup $K$ small enough so that a (finite) set $X$ of generators for $V$ lies inside $V^K$. Let $Y$ be a choice of a set of representatives for $G/ZK$; since $G/Z$ is compact, $Y$ is finite. The set of all vectors $\pi(g)v$ with $v \in X$ and $g \in G$ is contained in the span of the *finite* set of vectors $\pi(y)x$ for $y \in Y$ and $x \in X$. ///

[19.0.2] Corollary: Every irreducible smooth representation of $G$ having a central character for $Z$ is finite-dimensional ///

[19.0.3] Proposition: Let $f : M \to N$ be a surjective $G$-homomorphism of two $G$-representation spaces, both with central character $\omega$ (for $Z$). Suppose there is a compact open subgroup $K$ of $G$ so that $M^K = M$ and $N^K = N$ (as is the case if $M, N$ are finitely-generated). There is a $G$-homomorphism $\varphi : N \to M$ so that $f \circ \varphi$ is the identity map $\mathrm{id}_N$ on $N$.

pr Let $n$ be the cardinality of $G/ZK$. Let $\psi : N \to M$ be any $k$-vectorspace map so that $f \circ \psi = \mathrm{id}_N$: take any $k$-vectorspace $N_1$ in $M$ complementary to the kernel of $f$, and let $\psi$ be the inverse of the restriction of $f$ to $N_1$. Define

$$\varphi v \;=\; \frac{1}{n} \sum_{h \in G/ZK} h^{-1}\psi h v$$

The hypotheses assure that this $\varphi$ is independent of the choice of representatives for $G/ZK$, and it is immediate (by changing variables in the sum) that this averaged-out version of $\psi$ is a $G$-homomorphism providing a one-sided inverse to $f$. ///

[19.0.4] Corollary: Let $f : M \to N$ be an injective $G$-homomorphism of two $G$-representation spaces, both with central character $\omega$ (for $Z$). Suppose that there is a compact open subgroup $K$ of $G$ so that $M^K = M$

and $N^K = N$ (as is the case if $M, N$ are finitely-generated). There is a $G$-homomorphism $\varphi : M \to N$ so that $\varphi \circ f$ is the identity map $\mathrm{id}_M$ on $M$. In particular, every $G$-submodule of $N$ has a complementary submodule.

*Proof:* Let $Q = N/fM$ be the quotient, and $q : N \to Q$ the quotient map. The previous proposition yields $\psi : Q \to N$ so that $q \circ \psi = \mathrm{id}_Q$. Since $N = fM \oplus \psi Q$ and $fM \approx M$, $N/\psi Q$ is naturally isomorphic to $M$, and the composition

$$N \longrightarrow N/\psi \approx M$$

is the desired $\varphi$. ///

[19.0.5] Corollary: (Complete Reducibility) Every smooth representation of $G$ with central character $\omega$ (for $Z$) is a direct sum of irreducible smooth representations (each with central character $\omega$ for $Z$).

*Proof:* This will follow from the previous and from Zorn's Lemma.

First, show that a finite-dimensional smooth representation $M$ contains a non-zero irreducible. Since $M$ is finite-dimensional it is finitely-generated, so has an irreducible quotient $q : M \to Q$. By the above discussion, there is a $G$-subspace $M'$ of $M$ so that as $G$-spaces $M \approx M' \oplus Q$. Thus, $M$ contains the irreducible $Q$.

Let $M = \oplus_\alpha M_\alpha$ be a maximal direct sum of (necessarily finite-dimensional) irreducibles inside $N$, and suppose that $M \neq N$. Take $x \in N$ not lying in $M$, and let $X$ be the $G$-subspace of $N$ generated by $x$. Then $X$ is finitely-generated, so is finite-dimensional, and has a non-zero irreducible quotient $Q$. From above, there is a copy $Q'$ of $Q$ inside $X$ and $X = Q' \oplus X'$ for some $X'$. By the maximality of $M$, $Q$ must be inside $M$ already. Apply the same argument to $X'$, so by induction on dimension conclude that $X$ was 0. ///

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# 20. *Exactness of isotype, co-isotype (Jacquet) functors*

The first result here is a slight generalization of the exactness of both isotype and co-isotype functors for *compact* groups. The second result is related but somewhat different, showing that certain other special co-isotype functors *(Jacquet functors)* are also exact. The methods of proof are related, although this may not be superficially visible.

Fix a totally disconnected group $G$ and let $\pi$ be an irreducible smooth representation of $G$. Let $Z$ be a closed subgroup of $G$ contained in the center of $G$, and suppose $G/Z$ is *compact.* Suppose $\pi$ has central character $\omega$, i.e., when restricted to $Z$ is equal to its $\omega$-isotype.

The case that $G$ is *compact* and $Z = \{1\}$ is of most interest, but the slightly greater generality is inexpensive and useful.

Consider the two functors

$$F^\pi : W \to W^\pi \qquad\qquad F_\pi : W \longrightarrow W_\pi$$

taking a smooth representation $(\sigma, W)$ of $G$ to (respectively) the $\pi$-isotype $W^\pi$ and $\pi$-co-isotype $W_\pi$ of $W$.

Claim that *for $G/Z$ compact and for $\pi$ having a central character $\omega$ (for $Z$)* the isotype functors $F^\pi$ and $F_\pi$ take short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of $G$-representations *having central character* $\omega$ to exact sequences

$$0 \longrightarrow M'^\pi \longrightarrow M^\pi \longrightarrow M''^\pi \longrightarrow 0$$

$$0 \longrightarrow M'_\pi \longrightarrow M_\pi \longrightarrow M''_\pi \longrightarrow 0$$

(respectively.) That is, $F^\pi$ and $F_\pi$ *are exact* on the category of $G$-representations with central character $\omega$ for $Z$.

When $G$ itself is compact we may take $Z = \{1\}$, and consider the whole category of smooth representations of $G$. The assertion follows directly from the fact that in this situation all submodules admit complementary submodules.

For the special but important case of *Jacquet modules*, use notation conforming to anticipated applications: $N$ is a locally compact totally disconnected group assumed to be the *union of an ascending chain of compact open subgroups*, and $\psi$ is a smooth one-dimensional representation $\psi : N \to k^\times$. The first hypothesis implies that *every compact subset of $N$ is contained in some compact open subgroup.* With these hypotheses on $N$ and $\psi$ call $F_\psi$ a **(generalized) Jacquet (co-isotype) functor**.

*Proof:* Jacquet co-isotype functors $F_\psi$ are exact.

*Proof:* Fix a smooth representation $(\sigma, W)$ of $N$. To prove the assertion we need an alternative characterization of the $\psi$-co-isotype, as follows.

Let $u$ be a fixed right $N$-invariant distribution on $C_c^\infty(N)$. First, check that $u$ is also *left* invariant, i.e., that $N$ is *unimodular*. Given $f \in C_c^\infty(N)$ and $n \in N$, take a compact open subgroup $K$ of $N$ large enough to contain both $n$ and the support of $f$. As $K$ is open in $N$, $u$ restricted to $C_c^\infty(K)$ is a right $K$-invariant distribution, unique up to constants. As earlier, $K$ is unimodular, so

$$u(L_n f) \;=\; u(f)$$

This proves the unimodularity of $N$.

Having fixed a right Haar integral, as usual there is an associated action of the Hecke algebra $\mathcal{H}(N)$ on $W$ via 'integrating' $\sigma$

$$\sigma(\varphi)w \;=\; \int_G \varphi(g)\sigma(g)w\,dg \qquad\qquad (\text{for } \varphi \in C_c^\infty(N))$$

As before, define

$$e_K \;=\; \mathrm{ch}_K/u(\mathrm{ch}_K) = \mathrm{ch}_K/\,\mathrm{meas}\,(K) \qquad\qquad (\text{for compact open subgroup } K \text{ of } N)$$

For $K \subset K'$,

$$e_K * e_{K'} \;=\; e_{K'} * e_K \;=\; e_{K'}$$

Even though $\psi$ itself is not in $C_c^\infty(N)$, any product $\psi e_K$ *is* in $C_c^\infty(N)$, and for $K \subset K'$ an easy computation shows that

$$(\psi e_K) * (\psi e_{K'}) \;=\; (\psi e_{K'}) * (\psi e_K) = \psi e_{K'}$$

Therefore,

$$\ker\, \sigma(\psi^{-1} e_K) \;\subset\; \ker\, \sigma(\psi^{-1} e_{K'})$$

Claim that the kernel $Q$ of the map $W \to W_\psi$ to the $\psi$-co-isotype $W_\psi$ is

$$Q \;=\; \bigcup \ker\, \sigma(\psi^{-1} e_K) \;=\; \bigcup \ker\, \sigma(\psi^{-1} \mathrm{ch}_K)$$

On one hand, given an element $\psi(n)w - \sigma(n)w$, take a compact open subgroup $K$ large enough so that $n \in K$. Then

$$\sigma(\psi^{-1} ch_K)(\psi(n)w - \sigma(n)w) \;=\; \psi(n)\int_K \psi(h^{-1})\sigma(h)w\,dh - \int_K \psi(h^{-1})\sigma(hn)w\,dh$$

$$=\; \psi(n)\int_K \psi(h^{-1})\sigma(h)w\,dh - \psi(n)\int_K \psi(h^{-1})\sigma(h)w\,dh \;=\; 0$$

replacing $h$ by $hn^{-1}$ in the second integral. Since the variant definition of the trivial co-isotype is that it is the quotient of $W$ by the span of such elements $\psi(n)w - \sigma(n)w$,

$$Q \subset \bigcup \ker \sigma(\psi^{-1} ch_K)$$

On the other hand, suppose that for some compact open subgroup $K$

$$\int_K \psi(h^{-1})\sigma(h)w\,dh \;=\; 0$$

Let $K'$ be a small-enough compact open subgroup inside $K$ so that $w \in W^{K'}$ and $\psi(K') = \{1\}$. Then

$$0 \;=\; \int_K \psi(h^{-1})\sigma(h)w\,dh \;=\; \sum_{x \in K/K'} \psi(x^{-1})\mathrm{meas}\,(xK')\,\sigma(x)w$$

As $N$ is *unimodular*, this is

$$0 \;=\; \int_K \psi(h^{-1})\,\sigma(h)w\,dh \;=\; \mathrm{meas}\,(K') \sum_{x \in K/K'} \psi(x^{-1})\sigma(x)w$$

Therefore, with $n = [K : K']$,

$$w \;=\; w - 0 \;=\; w - \frac{1}{n}\,\mathrm{meas}\,(K')^{-1} \int_K \psi(h^{-1})\sigma(h)w\,dh \;=\; w - \frac{1}{n} \sum_{x \in K/K'} \psi(x^{-1})\sigma(x)w$$

$$= \frac{1}{n} \sum_{x \in K/K'} w - \psi(x^{-1})\sigma(x)w \;=\; \frac{1}{n} \sum_{x \in K/K'} \psi(x^{-1}) \cdot (\psi(x)w - \sigma(x)w)$$

This shows the opposite inclusion.

*This completes the alternative characterization of the Jacquet $\psi$-co-isotype.* Now prove the exactness of the Jacquet co-isotype functor, denoted by $\mathcal{J}$.

Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence, and $Q, Q', Q''$ the kernels of the maps of $M, M', M''$ to their respective co-isotypes $M_\psi, M'_\psi, M''_\psi$. Each such kernel is spanned by elements of the form $\psi(n)x - nx$, where $n \times x \to nx$ is the action of $n \in N$ upon $x$ in $M, M'$, or $M''$.

Thus, $fQ' \subset Q$ and $gQ \subset Q''$. That is, this co-isotype functor really is a functor, in that we have well-defined maps

$$\mathcal{J}f : \mathcal{J}M' \longrightarrow \mathcal{J}M \qquad \text{and} \qquad \mathcal{J}g : \mathcal{J}M \longrightarrow \mathcal{J}M''$$

Further, the surjectivity of $\mathcal{J}g$ is immediate from this and from the surjectivity of $g$.

For the injectivity of $\mathcal{J}f$, suppose that $\mathcal{J}fm' = 0 \in \mathcal{J}M$. Invoke the above variant characterization of the co-isotype: for some large-enough compact open subgroup $K$

$$0 \;=\; \int_K \psi(n^{-1})n(fm')\,dn$$

Since $f$ is an $N$-homomorphism,

$$f \int_K \psi(n^{-1})nm'\,dn \;=\; 0$$

Therefore,

$$\int_K \psi(n^{-1})nm'\,dn$$

is in the kernel of $f$, so is 0. Thus, $m' \in Q'$, giving injectivity of $\mathcal{J}f$.

Finally, prove exactness at the middle joint. By remarks just above, from the definition (or *by construction*, for one-dimensional $\psi$),

$$\mathcal{J}g \circ \mathcal{J}f \;=\; 0$$

On the other hand, suppose $\mathcal{J}gm = 0$. For some large enough compact open subgroup $K$ of $N$

$$\int_K \psi(n^{-1})n\,(gm)\,dn \;=\; 0$$

Since $g$ is an $N$-homomorphism,

$$g\int_K \psi(n^{-1})nm\,dn \;=\; 0$$

By the exactness of the original sequence, there is $m' \in M'$ so that

$$fm' \;=\; \int_K \psi(n^{-1})nm\,dn$$

Changing by the measure of $K$, and writing more economically,

$$fm' \;=\; (\psi^{-1}e_K)m$$

Therefore,

$$(\psi^{-1}e_K)fm' \;=\; (\psi^{-1}e_K) * (\psi^{-1}e_K)m \;=\; (\psi^{-1}e_K)m$$

by an elementary and direct computation used before. In other words,

$$(\psi^{-1}e_K)(fm' - m) \;=\; 0$$

That is, $fm' - m \in Q$. This proves that the kernel of $\mathcal{J}g$ is contained in the image of $\mathcal{J}f$. ///

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# 21. *Spherical representations: elementary results*

These are the most important representations for applications. However, the present general discussion is insufficient for finer study of spherical representations of totally disconnected groups $G$. Indeed, we can do nothing further with spherical representations without assuming $G$ is **p**-adic reductive.

Suppose for this section that $G$ is *unimodular*, and fix a compact open subgroup $K$ in $G$. The **$K$-spherical Hecke algebra** is $\mathcal{H}(G,K)$. Consider $k$-algebra homomorphisms

$$\Lambda : \mathcal{H}(G,K) \;\longrightarrow\; k$$

where $\mathcal{H}(G,K)$ has the convolution algebra structure. In an $\mathcal{H}(G,K)$-module $M$, say that $m \in M$ is an **$\mathcal{H}(G,K)$-eigenvector with eigenvalue** $\Lambda$ when there is a $k$-algebra homomorphisms

$$\Lambda : \mathcal{H}(G,K) \to k$$

so that

$$\eta m \;=\; \Lambda(\eta)m \qquad\qquad (\text{for } \eta \in \mathcal{H}(G,K))$$

A $K$-**spherical vector** in a smooth representation $\pi$ of $G$ is a vector $0 \neq v \in \pi^K$ which is an eigenvector for $\mathcal{H}(G,K)$ with *some* eigenvalue $\Lambda$.

A $K$-**spherical function** is a $k$-valued function on $G$ which is left and right $K$-invariant and which is an eigenvector for $\mathcal{H}(G,K)$ under the right-translation action of $G$ on $k$-valued functions on $G$. Usually it is further required that *a spherical function assume the value* 1 *at* $1_G$. To emphasize this normalization say that $\varphi$ is a **normalized spherical function**.

A $K$-**spherical representation** of $G$ is an *irreducible* (smooth) representation $\pi$ of $G$ with a (non-zero) $k$-spherical vector.

**[21.0.1] Lemma:** Let $v \neq 0$ be a $K$-spherical vector in a $K$-spherical representation $\pi$ of $G$. Then

$$\pi^K = k \cdot v$$

*Proof:* As usual, let

$$e_K \;=\; \mathrm{ch}_K / \mathrm{meas}\,(K)$$

The irreducibility of $\pi$ implies that

$$\mathcal{H}(G)\cdot v \;=\; \pi \qquad\qquad \text{(with } \mathcal{H}(G) \text{ is the full Hecke algebra)}$$

Therefore, for any $w \in \pi^K$ there is $\eta \in \mathcal{H}(G)$ so that $\eta v = w$. Then

$$w \;=\; e_K w \;=\; e_K * \eta v \;=\; e_K * \eta * e_K v \;=\; \Lambda(e_K * \eta * e_K) v \in k \cdot v$$

This is the desired result. ///

Let

$$\langle , \rangle : \pi \times \check{\pi} \;\longrightarrow\; k$$

be the canonical $k$-bilinear pairing, and let

$$c_{v\lambda}(g) = c(v,\lambda)(g) \;=\; \langle \pi(g)v, \lambda \rangle$$

be the coefficient function, as usual.

**[21.0.2] Lemma:** Let $\pi$ be a smooth representation of $G$ with a $K$-spherical vector $v \neq 0$. Let $\lambda \in (\check{\pi})^K$ be a smooth functional such that $\lambda v \neq 0$. The $k$-valued function $f$ on $G$ defined by

$$\varphi(g) \;=\; c_{v\lambda}(g) \;=\; \langle \pi(g)v, \lambda \rangle$$

is a $K$-spherical function.

*Proof:* Let $\Lambda$ be the eigenvalue of $v$. The left and right $K$-invariance follows from the $K$-invariance of $v$ and $\lambda$, by elementary properties of the coefficient functions. From $R_g\varphi = c(\pi(g)v, \lambda)$, by integrating, for $\eta \in \mathcal{H}(G,K)$,

$$R_\eta \varphi \;=\; c(\pi(\eta)v, \lambda) \;=\; c(\Lambda(\eta)\cdot v, \lambda) \;=\; \Lambda(\eta)\cdot c(v,\lambda) = \Lambda(\eta)\varphi$$

Note that there is *some* $\lambda' \in \pi^*$ with $\lambda' v \neq 0$. Then

$$(\pi^*(e_K)\lambda')(v) \;=\; \lambda'(e_K v) = \lambda' v \;\neq\; 0$$

and $\pi^*(e_K)\lambda'$ is certainly in $\check{\pi}$. ///

**[21.0.3] Lemma:** Assume a given $G$ Haar measure so that $\mathrm{meas}\,(K) = 1$. Given a $k$-algebra homomorphism $\Lambda : \mathcal{H}(G,K) \to k$, there is at most one $K$-spherical function $\varphi$ such that $\varphi(1) = 1$ and

$$R_\eta \varphi = \Lambda(\eta)\varphi$$

for all $\eta \in \mathcal{H}(G,K)$. In particular,

$$\varphi(g) \;=\; \Lambda(e_{KgK}) \qquad\qquad (\mathrm{for} e_{KgK} = \mathrm{ch}_{KgK} / \,\mathrm{meas}\,(KgK))$$

*Proof:* We have

$$\varphi(g) \;=\; (R_g\varphi)(1) \;=\; (R_{e_{KgK}}\varphi)(1) \;=\; \Lambda(e_{KgK})\varphi(1)$$

which proves uniqueness and the asserted formula. ///

Recall the notation $\check{\eta}(g) = \eta(g^{-1})$.

**[21.0.4] Corollary:** Let $\varphi$ be a spherical function with eigenvalue $\Lambda$ and with $\varphi(1) = 1$. For $\eta \in \mathcal{H}(G,K)$,

$$\eta * \varphi = \varphi * \eta = \Lambda(\check{\eta})\varphi$$

*Proof:* Half of this is a direct computation:

$$(\varphi * \eta)(g) \;=\; \int_G \varphi(gh^{-1})\eta(h)\,dh \;=\; \int_G \varphi(gh)\eta(h^{-1})\,dh \;=\; \int_G \varphi(gh)\check{\eta}(h)\,dh = R_{\check{\eta}}\varphi$$

using unimodularity in the change of variables $h \to h^{-1}$. From the definition of the spherical function it follows that

$$R_{\check{\eta}}\varphi \;=\; \Lambda(\check{\eta})\varphi$$

For the other half of the proof, note that

$$(\eta * \varphi)(g) \;=\; \int \eta(gh^{-1})\varphi(h)\,dh \;=\; \int \eta(h^{-1})\varphi(hg)\,dh$$

so that $\eta * \varphi$ is a finite sum of left translates of $\varphi$. By the *uniqueness* of $K$-spherical functions above,

$$(\eta * \varphi) \;=\; (\eta * \varphi)(1) \times \varphi$$

and

$$(\eta * \varphi)(1) \;=\; \int \eta(h^{-1})\varphi(h)\,dh \;=\; \int \eta(h)\varphi(h^{-1})\,dh \;=\; (\varphi * \eta)(1) \;=\; \Lambda(\check{\eta})$$

by the first computation. ///

**[21.0.5] Proposition:** (Godement) Let $\Lambda : \mathcal{H}(G,K) \to k$ be a $k$-vectorspace map. Define a $k$-valued function $\varphi$ on $G$ by

$$\varphi(g) = \Lambda(e_{KgK})$$

Then $\Lambda$ is a $k$-algebra map if and only if the functional equation

$$\int_K \varphi(g_1\theta g_2)\,d\theta \;=\; \varphi(g_1)\,\varphi(g_2) \;\times\; \mathrm{meas}\,(K)$$

holds. If this does hold, the $\varphi$ is the unique normalized $K$-spherical function associated to $\Lambda$.

*Proof:* Suppose the functional equation holds. The functions $e_{KgK}$ certainly span $\mathcal{H}(G,K)$ over $k$. We have

$$(e_{KgK}\varphi)(h) \;=\; \int_{KgK} \varphi(hx)\,dx \,/\,\mathrm{meas}\,(KgK) \;=\; \int_K\int_K \varphi(h\theta_1 g\theta_2)\,d\theta_1\,d\theta_2\,/\,\mathrm{meas}\,(K)^2 \;=\; \frac{\int_K \varphi(h\theta g)\,d\theta}{\mathrm{meas}\,(K)}$$

since $\varphi$ is right $K$-invariant. This is

$$\varphi(h)\varphi(g) \;=\; \Lambda(e_{KgK})\,\varphi(h)$$

by definition of $\varphi$ and by the functional equation. Thus, $R_\eta\varphi \;=\; \Lambda(\eta)\varphi$. Then

$$\Lambda(\eta_1 * \eta_2)\varphi \;=\; R_{\eta_1*\eta_2}\varphi \;=\; R_{\eta_1}R_{\eta_2}\varphi = \Lambda(\eta_1)\Lambda(\eta_2)\varphi$$

Thus, $\varphi$ is an eigenfunction and $\Lambda$ is a $k$-algebra homomorphism, and then also $\varphi(1) = 1$.

On the other hand, suppose that $\Lambda$ is a $k$-algebra homomorphism and $\varphi$ a normalized eigenvector. Reversing the above computation,

$$\int_K \varphi(g_1\theta g_2)\,d\theta/\,\mathrm{meas}\,(K) \;=\; \int_K\int_K \varphi(g_1\theta_1 g_2\theta_2)\,d\theta_1\,d\theta_2\,/\,\mathrm{meas}\,(K)^2$$

$$= \int_K \varphi(g_1 x)\,dx = (R_{e_{Kg_2K}}\varphi)(g_1) \;=\; (R_{e_{Kg_1K}}R_{e_{Kg_2K}}\varphi)(1) \;=\; (R_{e_{Kg_1K}*e_{Kg_2K}}\varphi)(1)$$

$$= \Lambda(e_{Kg_1K} * e_{Kg_2K})\,\varphi(1) \;=\; \Lambda(e_{Kg_1K} * e_{Kg_2K}) \;=\; \Lambda(e_{Kg_1K})\,\Lambda(e_{Kg_2K}) \;=\; \varphi(g_1)\,\varphi(g_2)$$

by definition of $\varphi$. ///

The preceding proposition proves the bijection of normalized spherical functions and $k$-algebra homomorphisms of $\mathcal{H}(G,K)$ to $k$. Finally, prove that spherical representations and normalized spherical functions are in bijection. Fix an algebra homomorphism

$$\Lambda : \mathcal{H}(G,K) \;\longrightarrow\; k$$

and let $\varphi(g) = \Lambda(e_{KgK})$ be the associated normalized spherical function. Let

$$V \;=\; \{\text{ finite sums } \sum_i c_i R_{g_i}\varphi : g_i \in G,\; c_i \in k\}$$

Let $G$ act on $V$ by right translations, denoted by $\pi$.

[21.0.6] Proposition: The representation $(\pi, V)$ is a $K$-spherical representation with associated eigenvalue given simply by $\Lambda$. Further, any $K$-spherical representation with eigenvalue $\Lambda$ is isomorphic to $\pi$.

pr By construction, $\pi$ is smooth. Given $0 \neq f \in V$, take $g_0 \in G$ so that $f(g_0) \neq 0$, and write

$$f(g) \;=\; \sum_i c_i\,\varphi(gg_i)$$

Computing as in the previous proposition,

$$(\pi(e_{Kg_0K})f)(g) \;=\; \int_K f(g\theta g_0)\,d\theta \;=\; \sum_i c_i \int_K \varphi(g\theta g_0 g_i)\,d\theta \;=\; \sum_i c_i\,\varphi(g)\,\varphi(g_0 g_i)\;\mathrm{meas}\,(K)$$

by the functional equation of $\varphi$. This is

$$\varphi(g)\,f(g_0)\;\mathrm{meas}\,(K)$$

Thus, $\varphi \in \mathcal{H}(G)f$. That is, every non-zero $\mathcal{H}(G)$-submodule of $\pi$ contains $\varphi$, so must be all of $\pi$. Therefore, $\pi$ is irreducible. It is easy to see that $\Lambda$ is the eigenvalue associated to $\pi$.

Let $\sigma$ be another $K$-spherical representation with eigenvalue $\Lambda$. Let $0 \neq v$ be a $K$-spherical vector, and take $\lambda \in \check{\sigma}^K$ so that $\lambda v = 1$. As above, $c_{v\lambda}$ is a normalized spherical function. By the *uniqueness of normalized spherical functions*, $c_{v\lambda} = \varphi$, *the* normalized spherical function attached to $\Lambda$.

For $\eta \in \mathcal{H}(G)$ and $w \in \sigma(\eta)v$,

$$c_{w\lambda} = c_{\sigma(\eta)v,\lambda} = \pi(\eta)\, c_{v\lambda}$$

by elementary properties of coefficient functions. Therefore, $w \to c_{w\lambda}$ gives a non-zero $G$-homomorphism $\sigma \to \pi$. Since these two representations are irreducible, this must be an isomorphism. ///

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# 22. *Admissibility*

A smooth representation $(\pi, K)$ of $G$ is **admissible** when, for every compact open subgroup $K$ of $G$, the $K$-fixed vector space $V^K$ is *finite-dimensional*. All further assertions in this section, follow directly from the *complete reducibility* of smooth representations of compact totally disconnected groups.

**[22.0.1] Proposition:** An equivalent characterization of admissibility is that, for a fixed compact open subgroup $K$ of $G$, every irreducible smooth representation $\delta$ of $K$ has finite multiplicity in $\mathrm{Res}^G_K \pi$.

*Proof:* Suppose $\pi^{K'}$ is finite-dimensional for all compact open subgroups $K'$. Let $K$ be a fixed compact open subgroup and $\delta$ an irreducible (smooth) representation of $K$. We know $\delta$ is finite-dimensional. Let $v_1, \ldots, v_n$ be a $k$-basis. Since $\delta$ is smooth, each isotropy group $K_{v_i}$ is open, and the finite intersection $K' = \bigcap_i K_{v_i}$ is a compact open subgroup inside $K$. Then $K'$ contains another compact open subgroup $K''$ which is *normal* in $K$: let $K$ act (on the left) on the space $K/K'$ of cosets $kK'$, and take $K''$ to be the subgroup of $K$ fixing every coset $kK'$. Then $K''$ acts trivially on the representation space of $\delta$, so $\pi^\delta$ is contained in the set $\pi^{K''}$ of $K''$-fixed vectors.

For the converse, suppose that $K$ is a fixed compact open subgroup and that for every irreducible $\delta$ of $K$ the $\delta$-isotype in $\pi$ is of finite multiplicity. Given a smaller compact open subgroup $K'$ , without loss of generality shrink it to be inside $K$, and (as in the previous paragraph) to be *normal* in $K$. The $K'$-fixed vectors $\pi^{K'}$ of $\pi$ all lie inside the sum of the $\delta$-isotypes, summed over irreducibles $\delta$ so that $\delta|_{K'}$ contains a $K'$-fixed vector. By complete reducibility, the trivial representation 1 of $K'$ occurs inside $\delta|_{K'}$ if and only if $\mathrm{Hom}_{K'}(\delta|_{K'}, 1)$ is non-trivial. By Frobenius Reciprocity,

$$\mathrm{Hom}'_K(\delta|_{K'}, 1) \;\approx\; \mathrm{Hom}_K(\delta, Ind^K_{K'} 1)$$

Since $K'\backslash K$ is finite, $Ind^K_{K'}1$ is finite-dimensional, so there are only finitely-many (isomorphism classes of) irreducibles $\delta$ occuring in $Ind^K_{K'}1$. Thus, the space $\pi^{K'}$ is finite-dimensional. ///

Since an irreducible $\delta$ of $K$ is necessarily finite-dimensional, admissibility of a representation $\pi$ is equivalent to the assertion that *the $\delta$-isotypic subspace in $\pi$ is finite-dimensional, for all irreducibles $\delta$ of $K$.*

Proofs of the following are left to the reader, as they are but further exercises in application of the smooth representation theory of compact (totally disconnected) groups.

*Subrepresentations and quotient representations of admissible representations are admissible.*

*The (smooth) dual (i.e., (smooth) contragredient) of an admissible representation is admissible.*

*Admissible $\pi$ is reflexive: the inclusion of $\pi$ into its (smooth) double dual $\check{\check{\pi}}$ is an isomorphism.*

*An admissible representation is irreducible if and only if its smooth dual is irreducible.*

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