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Spectral Theory for $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$

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- Pseudo-Eisenstein series
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We will restrict out attention to the simplest possible case, namely $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, and right $K = SO(2)$-invariant functions on $\Gamma\backslash G$. That is, we neglect the finite primes and holomorphic automorphic forms. Let N be the subgroup of G consisting of upper-triangular unipotent matrices, and P the parabolic subgroup consisting of all upper-triangular matrices. For simplicity we give K total measure 1, rather than 2π. This will account for some discrepancies between formulas below and their analogues in other sources.

For a locally integrable function f on $\Gamma\backslash G$ the **constant term** $c_P f$ of f (along P) is defined to be

$$c_P f(g) = \int_{N\cap\Gamma\backslash N} f(ng)\, dn$$

As usual, a (locally integrable) function f on $\Gamma\backslash G$ is a **cuspform** if for almost all $g \in G$

$$c_P f(g) = 0$$

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1. *Pseudo-Eisenstein series*

While cuspforms are mysterious, a completely not-mysterious type of automorphic form is constructed directly from functions $\varphi \in C_c^\infty(N\backslash G)$ by forming the **incomplete theta series** or **pseudo-Eisenstein series**[1]

$$\Psi_\varphi(g) = \sum_{P\cap\Gamma\backslash\Gamma} \varphi(\gamma g)$$

[1.0.1] Remark: Note that $P \cap \Gamma$ differs from $N \cap \Gamma$ just by $\{\pm 1_2\}$, which are both in the center of Γ and are in K. Since our interest for the moment is only in right K-invariant functions, everything here will be invariant under $\{\pm 1_2\}$.

[1.0.2] Lemma: The series for an incomplete theta series is absolutely and uniformly convergent for g in compacts, and yields a function in $C_c^\infty(\Gamma\backslash G)$.

Proof: Given $\varphi \in C_c^\infty(N\backslash G/K$, let C be a compact set in G so that $N \cdot C$ contains the support of φ. Fix a compact subset C_o of G in which $g \in G$ is constrained to lie. Then a summand $\varphi(\gamma g)$ is non-zero only if $\gamma g \in N \cdot C$, which is to say

$$\gamma \in \Gamma \cap N \cdot C \cdot g^{-1}$$

which requires that

$$\gamma \in \Gamma \cap N \cdot C \cdot C_o^{-1}$$

[1] In 1966 Godement called these *incomplete theta series*, but more recently Moeglin-Waldspurger strengthened the precedent of calling them *pseudo-Eisenstein series*

If this held, then

$$(N \cap \Gamma) \cdot \gamma \subset \Gamma \cap N \cdot C \cdot C_o^{-1}$$

and then

$$(N \cap \Gamma) \cdot \gamma \in (N \cap \Gamma)\backslash\Gamma \cap (N \cap \Gamma)\backslash(N \cdot C \cdot C_o^{-1})$$

The second term on the right-hand side is compact, since $(N \cap \Gamma)\backslash N$ is compact. The first term on the right-hand side is discrete, since $N \cap \Gamma$ is a closed subgroup. Thus, the right hand side is compact and discrete, so is finite. Thus, the series is in fact locally finite, and defines a smooth function on $\Gamma\backslash G/K$.

To show that it has compact support in $\Gamma\backslash G$, proceed similarly. That is, for a summand $\varphi(\gamma g)$ to be non-zero, it must be that $g \in \Gamma \cdot C$, which implies $\Gamma \cdot g \subset \Gamma \cdot C$, and $\Gamma \cdot g \in \Gamma\backslash(\Gamma \cdot C)$. The right-hand side is compact, being the continuous image of a compact set under the continuous map $G \to \Gamma\backslash G$, proving the compact support. ///

[1.0.3] Remark: We will make incessant use of the lemma that for a countably-based locally compact Hausdorff topological group G, and for a Hausdorff space X on which G acts transitively, X is homeomorphic to the quotient of G by the isotropy group of a chosen point in X. And we will use standard integration theory on quotients such as $\Gamma\backslash G$ and $N\backslash G$, etc.

Let $\langle , \rangle$ be the complex bilinear form

$$\langle f_1, f_2 \rangle = \int_{\Gamma\backslash G} f_1(g)\, f_2(g)\, dg$$

with respect to a fixed right G-invariant measure on $\Gamma\backslash G$.

[1.0.4] Proposition: A locally integrable automorphic form f is a cuspform if and only if $\langle f, \Psi_\varphi \rangle = 0$ for all pseudo-Eisenstein series Ψ_φ, with $\varphi \in C_c^\infty(N\backslash G)$.

Proof: Since Ψ_φ has compact support on $\Gamma\backslash G$, as long as f is locally integrable the integral makes sense.

On one hand, by unwinding the sum defining the theta series,

$$\langle f, \Psi_\varphi \rangle = \int_{\Gamma\backslash G} \sum_{\gamma \in N \cap \Gamma\backslash\Gamma} f(g)\, \varphi(\gamma g)\, dg = \int_{N \cap \Gamma\backslash G} f(g)\, \varphi(g)\, dg$$

$$= \int_{N\backslash G} \int_{N\cap\Gamma\backslash N} f(ng)\, \varphi(ng)\, dn\, dg = \int_{N\backslash G} \varphi(g) \int_{N\cap\Gamma\backslash N} f(ng)\, dn\, dg = \int_{N\backslash G} cf(g)\, \varphi(g)\, dg$$

where the next-to-last inequality is obtained from the fact that φ is left N-invariant. Certainly if $cf = 0$ then this integral vanishes for all φ. On the other hand, cf is locally integrable on $N\backslash G$, and if it were non-zero on a set of positive measure, then (by density of C_c^∞ in C_c^0) there would be some φ which would make this integral non-zero. ///

[1.0.5] Corollary: The square-integrable cuspforms are the orthogonal complement of the (closed) space spanned by the pseudo-Eisenstein series in $L^2(\Gamma\backslash G)$. ///

[1.0.6] Remark: It is immediate that the pseudo-Eisenstein series are not cuspforms: if Ψ_φ is not identically zero, then

$$0 < \langle \Psi_\varphi, \Psi_\varphi \rangle$$

since the inner product is the integral of a not-identically-zero non-negative smooth function. That is, if Ψ_φ is not identically zero then it is certainly not a cuspform.

2. *Fourier-Laplace-Mellin transforms*

Recall that Fourier inversion for Schwartz functions on the real line asserts that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i\xi x}\, d\xi$$

where the Fourier transform $\hat{f}$ of f is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx$$

Replacing ξ by $\xi/(2\pi)$ gives the variant identity

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, e^{-it\xi}\, dt\right) e^{i\xi x}\, d\xi$$

Now suppose that $F \in C_c^\infty(0,+\infty)$, and take $f(x) = F(e^x)$. Then let $y = e^x$ (and $r = e^t$ in the innermost integral) and rewrite the identity as

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_0^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}\right) y^{i\xi}\, d\xi$$

We define a related transform [2] $\mathcal{M}F$ of F by

$$\mathcal{M}F(i\xi) = \int_{-\infty}^{\infty} F(r)\, r^{-i\xi}\, \frac{dr}{r}$$

or (for complex s)

$$\mathcal{M}F(s) = \int_{-\infty}^{\infty} F(r)\, r^{-s}\, \frac{dr}{r}$$

Then the previous identity gives the inversion formula

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}F(i\xi) y^{i\xi}\, d\xi$$

If we view ξ as being the imaginary part of a complex variable s, and rewrite the latter integral as a complex path integral, then it becomes (since $d\xi = -i\, ds$)

$$F(y) = \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \mathcal{M}F(s) y^s\, ds$$

(where the integral is along the obvious vertical line).

[2.0.1] **Remark:** It is very important to know that for $f \in C_c^\infty(\mathbb{R})$ the Fourier transform $\hat{f}(\xi)$ extends to an *entire* function in ξ which is of rapid decay on horizontal lines. [3] Then certainly the same is true for the transform $\mathcal{M}F$ of $F \in C_c^\infty(0,+\infty)$. In this case, for *any* real σ, the inversion formula yields

$$F(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}F(s) y^s\, ds$$

[2] In these coordinates, this is called a *Mellin transform*, but it is a Fourier transform.

[3] More can be said about decay of Fourier/Mellin transforms: see the Paley-Wiener theorem.

[2.0.2] **Remark:** The fact that the integral defining the Fourier transform $\hat{f}$ does not converge for all f in $L^2(\mathbb{R})$ is a portent of what happens generally: integral formulas are valid only on a small but dense subspace of a (concrete) Hilbert space of functions, and the maps are extended by Hilbert-space isometry to the whole space. Of course, this presumes that we have shown that the integral expression does give an isometry.

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3. *Recollection of facts about Eisenstein series*

We review some basic features of the (spherical) Eisenstein series for $SL_2(\mathbb{Z})$, and the causal mechanisms. Rather than use explicit formulas, easily possible here, we use methods which will scale upward as well as possible. Let

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

be the usual G-invariant Laplacian on $\mathfrak{H} = G/K$ in upper half-plane coordinates. Because

$$\Delta(y^s) = s(s-1) \cdot y^s$$

we have

$$\Delta E_s = s(s-1) \cdot E_s$$

And since Δ commutes with the map $f \to c_P f$, we see that $c_P E_s$ is a function $u(y)$ of y satisfying the Eulerian equation

$$y^2 \frac{\partial^2}{\partial\, y^2}\, u(y) = s(s-1)\, u(y)$$

For $s \neq 1/2$ this has the two linearly independent elementary solutions y^s and y^{1-s}. So, for some meromorphic functions a_s and c_s,

$$c_P E_s = a_s\, y^s + c_s\, y^{1-s}$$

In fact, a direct computation shows that the first of the two summands is entirely elementary, and in particular $a_s = 1$. That is,

$$c_P\, E_s = y^s + c_s\, y^{1-s}$$

Following the Selberg-Bernstein method of analytic continuation, for example, we find

[3.0.1] **Theorem:** The equations

$$\Delta w = s(s-1) \cdot w \qquad\qquad \left(y \frac{\partial}{\partial y} - (1-s) \right) cw = (2s-1) \cdot y^s$$

uniquely determine $w = E_s$, and imply that it has a meromorphic continuation and functional equation

$$E_{1-s} = c_{1-s}\, E_s$$

(This is non-trivial!)

Granting the unique characterization of the Eisenstein series, the functional equation is readily obtained, as follows. One can check directly that $c_{1-s}^{-1} E_{1-s}$ satisfies those equations as well, so by *uniqueness*

$$c_{1-s}^{-1} E_{1-s} = E_s$$

which is the functional equation. Continuing in this vein, applying the functional equation twice gives

$$c_s c_{1-s} = 1$$

Since $E_{\bar{s}} = \overline{E_s}$, we have $\overline{c_s} = c_{\bar{s}}$ and $|c_{\frac{1}{2}+it}|^2 = 1$. For real t. In particular, c_s does not vanish on that line.

From the spectral formula below expressing pseudo-Eisenstein series in terms of Eisenstein series, poles of E_s sufficiently far to the right can be made to play a role in the decomposition of $L^2(\Gamma\backslash G)$.

[3.0.2] **Remark:** If we imagine that (for example) we can arrange to have $\mathcal{M}\varphi(s_o) = 1$ while φ is adjusted so that $\mathcal{M}\varphi(\frac{1}{2} + it) \to 0$ (and all other residue terms go to 0), then we would conclude from the previous paragraph that all negative-order terms of all poles of Eisenstein series (if not cancelled in some manner) are in the closure of the space spanned by the pseudo-Eisenstein series. That is, (if we can arrange things as required) these residues of Eisenstein series are in $L^2(\Gamma\backslash G)$ and are orthogonal to cuspforms.

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4. *Decomposition of pseudo-Eisenstein series*

We restrict out attention to right K-invariant functions on $\Gamma\backslash G$. Invoking the Iwasawa decomposition, we have

$$G = N \cdot A^o \cdot K$$

with A^o consisting of elements

$$m_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{1/y} \end{pmatrix}$$

Define a function

$$a : G \to A^o$$

by sending $g \in G$ to the element $a(g) \in A^o$ so that

$$g \in N \cdot a(g) \cdot K$$

Then a left N-invariant right K-invariant function on G can be identified with a function on

$$N\backslash G/K \approx A^o \approx (0, +\infty)$$

by the map

$$g \to a(g) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{1/y} \end{pmatrix} \to y$$

We will use the Laplace transform and inversion formula to decompose the pseudo-Eisenstein series.

Let $\varphi \in C_c^\infty(N\backslash G/K)$. By the inversion formula

$$\varphi(a_y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) y^s \, ds$$

or

$$\varphi(g) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) a(g)^s \, ds$$

Then the pseudo-Eisenstein series is expressible as

$$\Psi_\varphi(g) = \frac{1}{2\pi i} \sum_{\gamma \in (\Gamma \cap N)\backslash\Gamma} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) a(\gamma g)^s \, ds$$

The case that $\sigma = 0$ might seem to be the natural spectral decomposition. However, with $\sigma = 0$ the double integral (sum and integral) is *not* absolutely convergent, and the two integrals cannot be interchanged. (And we'll subsequently decide that the 'correct' line is $\sigma = 1/2$). But for $\sigma > 0$ sufficiently large, say

$\sigma > 1$, relatively elementary estimates show that the double integral *is* absolutely convergent, so (by Fubini's theorem) the two integrals can be interchanged:

$$\Psi_\varphi(g) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) \sum_{\gamma\in(\Gamma\cap N)\backslash\Gamma} a(\gamma g)^s\, ds$$

The inner sum is (two times) the usual (spherical) Eisenstein series

$$E_s(g) = \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} a(\gamma g)^s = \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \mathrm{Im}\,(\gamma z)^s$$

where we let Γ act as usual on the upper half-plane, and we know the identity

$$\mathrm{Im}\,(\begin{pmatrix} a & b \\ c & d\end{pmatrix} z) = \frac{y}{|cz+d|^2}$$

where $z = x + iy$. That is, for $\sigma > 1$,

$$\Psi_\varphi(g) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\, E_s(g) \quad (\sigma = \mathrm{Re}\,(s) > 1)$$

If we *grant* the meromorphic continuation of the Eisenstein series, then we can move the vertical line of integration to the left, say to the line $\sigma = 1/2$. (Preference for this particular vertical line will be clear shortly.) Of course, this uses the fact that integrals over small horizontal line segments

$$[\frac{1}{2} + it, \sigma + it]$$

go to zero as $t \to \pm\infty$. And this is slightly complicated by the fact that we are actually looking at meromorphic function-valued functions. Granting these two sorts of things for the moment, we obtain

$$\Psi_\varphi(g) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\, E_s(g) + \sum_{s_o} \mathrm{res}_{s=s_o}(E_s \cdot \mathcal{M}\varphi(s))$$

By luck the $1/2\pi i$ from the Laplace inversion formula exactly cancels the $2\pi i$ in the residue formula.

[4.0.1] Remark: We also need to know that E_s has no pole on the line $\mathrm{Re}\,(s) = 1/2$.

And we'd prefer to have $\mathcal{M}c_P(\Psi_\varphi)$ enter, not just $\mathcal{M}\varphi$, because we would want whatever integral formulas we have to be expressed in terms of the automorphic forms themselves, not in terms of the auxiliary functions from which they're made. To this end, we need a standard unwinding trick:

[4.0.2] Proposition: For $f \in C_c^\infty(\Gamma\backslash G/K)$

$$\int_{\Gamma\backslash G} E_s(g)\, f(g)\, dg = \mathcal{M}c_P(f)(1-s)$$

Proof: We use the fairly standard unwinding trick (which is certainly legitimate for $\mathrm{Re}\,(s) > 1$ and for $f \in C_c^\infty(\Gamma\backslash G)$,

$$\int_{\Gamma\backslash G} E_s(g)\, f(g)\, dg = \int_{\Gamma\backslash G} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} a(\gamma g)^s\, f(g)\, dg = \int_{P\cap\Gamma\backslash G} a(g)^s\, f(g)\, dg$$

$$= \int_{N(P\cap\Gamma)\backslash G} \int_{N\cap\Gamma\backslash N} a(ng)^s\, f(ng)\, dn\, dg = \int_{N(P\cap\Gamma)\backslash G} a(g)^s\, c_P f(g)\, dg$$

Note that $c_P f$ is not generally in $C_c^\infty(N\backslash G)$. Nevertheless, the absolute convergence of the initial integral (and of the sum defining the Eisenstein series) assures that the last integral is absolutely convergent for $\mathrm{Re}(s) > 1$. Thus, in the case that f is right K-invariant, using the fact that

$$d(nm_y k) = y^{-1}\, dn\, \frac{dy}{y}\, dk$$

with Haar measures dn and dk on N and K, respectively, this becomes

$$\mathcal{M}c_P f(1-s)$$

and the integral defining the latter converges absolutely at first for $\mathrm{Re}(s) > 1$. Then this identity holds also for the analytically continued Eisenstein series since it is continuous and f is in $C_c^\infty(\Gamma\backslash G)$. ///

On the other hand, we also have

[4.0.3] Proposition: For an pseudo-Eisenstein series Ψ_φ,

$$\int_{\Gamma\backslash G} E_s(g)\, \Psi_\varphi(g)\, dg = \mathcal{M}\varphi(1-s) + c_s\, \mathcal{M}\varphi(s)$$

Proof: Use the known form of the constant term of the Eisenstein series

$$c_P E_s = a(g)^s + c_s\, a(g)^{1-s}$$

and the properties of the pseudo-Eisenstein series observed earlier:

$$\int_{\Gamma\backslash G} E_s(g)\, \Psi_\varphi(g)\, dg = \int_{N\backslash G} c_P E_s(g)\, \varphi(g)\, dg$$

This is

$$\int_{N\backslash G} \left(a(g)^s + c_s a(g)^{1-s}\right) \varphi(g)\, dg$$

which immediately yields the proposition. ///

This allows us to understand the constant term of pseudo-Eisenstein series without direct computation:

[4.0.4] Corollary:

$$\mathcal{M}c_P \Psi_\varphi(s) = \mathcal{M}\varphi(1-s) + c_{1-s}\, \mathcal{M}\varphi(s)$$

Proof: Compare the two computations of the last two propositions. ///

Then the integral part of the expression of Ψ_φ in terms of Eisenstein series can be rearranged to

$$\Psi_\varphi - (\text{residual part}) = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\, E_s + \mathcal{M}\varphi(1-s)\, E_{1-s}\, ds = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}\varphi(s)\, E_s + \mathcal{M}\varphi(1-s)\, c_{1-s} E_s\, ds$$

$$= \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (\mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s))\, E_s\, ds = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\, E_s\, ds = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_{1-s}\rangle\, E_s\, ds$$

That is, an pseudo-Eisenstein series is expressible as an integral of Eisenstein series E_s on the line $\mathrm{Re}(s) = 1/2$, plus a sum of residues:

$$\Psi_\varphi - (\text{residual part}) = \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\, E_s\, ds = \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\, E_s\, ds$$

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5. *Interlude: the constant function*

In the case of $SL_2(\mathbb{Z})$, we know by various means that there is a single pole of E_s in the half-plane $\mathrm{Re}(s) \geq 1/2$ at $s = 1$, and it is simple, so

$$\Psi_\varphi(g) = \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathcal{M}c_P\Psi_\varphi(s)\, E_s(g) + \mathcal{M}\varphi(1)\cdot \mathrm{res}_{s=1}E_s(g)$$

The coefficient $\mathcal{M}\varphi(1)$ is

$$\mathcal{M}\varphi(1) = \int_o^{+\infty} \varphi(m_y)\, y^{-1}\,\frac{dy}{y} = \int_{N\backslash G} \varphi(g)\; dg$$

since a right Haar measure on G is given by

$$d(nm_yk) = y^{-1}\; dn\cdot\frac{dy}{y}\cdot dk$$

for Haar measure dn on N and Haar measure dk on K. Rearranging, we have

$$\mathcal{M}\varphi(1) = \int_{N\backslash G} \varphi(g)\; dg = \int_{N\backslash G}\int_{N\cap\Gamma\backslash N} \varphi(ng)\; dn\, dg$$

$$= \int_{N\backslash G} \varphi(ng)\int_{N\cap\Gamma\backslash N} 1\; dn\, dg = \int_{N\cap\Gamma\backslash G} \varphi(g)\; dg$$

since the natural volume of $(N\cap\Gamma)\backslash N$ is 1 and φ is left N-invariant. Then, winding up, we have

$$\mathcal{M}\varphi(1) = \int_{\Gamma\backslash G}\sum_{\gamma\in(N\cap\Gamma\backslash\Gamma} \varphi(g)\; dg = \int_{\Gamma\backslash G} \Psi_\varphi(g)\; dg = \langle\Psi_\varphi, 1\rangle$$

That is, $\mathcal{M}\varphi(1)$ is the inner product of Ψ_φ with the constant function 1 (which *is* square-integrable).

[5.0.1] Remark: This would cause a person to speculate that the residue of the Eisenstein series E_s at $s = 1$ should be a constant function, depending upon the normalization of measures. Of course this presumes that the rest of the expression decomposing Ψ_φ in terms of Eisenstein series is for some reason orthogonal to the constant functions. This is certainly not clear *a priori*.

6. *Plancherel for the continuous spectrum*

Ignoring constants for a moment, for $f \in C_c^\infty(\Gamma\backslash G)$, using the expression for Ψ_φ in terms of Eisenstein series,

$$\langle \Psi_\varphi, f\rangle \;=\; \Big\langle \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_{1-s}\rangle \cdot E_s\, ds,\;\; f\Big\rangle \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_{1-s}\rangle \cdot \langle E_s, f\rangle\, ds$$

This proves that $f \to (s \to \langle f, E_s\rangle)$ is an inner-product-preserving map from the Hilbert-space span of the pseudo-Eisenstein series to $L^2(\frac{1}{2} + i\mathbb{R})$.

Functions $u(t) = \langle \Psi_\varphi, E_{\frac{1}{2}-it}\rangle$ satisfy

$$u(-t) \;=\; \langle \Psi_\varphi, E_s\rangle \;=\; \langle \Psi_\varphi, c_s E_{1-s}\rangle \;=\; c_s \langle \Psi_\varphi, E_{1-s}\rangle \;=\; c_s \cdot u(t)$$

We claim that *any* $u \in L^2(\frac{1}{2} + i\mathbb{R})$ satisfying $u(-t) = c_s\, u(t)$ is in the image. First, claim that, for compactly-supported u satisfying $u(-t) = c_s\, u(t)$

$$\Phi_u \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t) \cdot E_{\frac{1}{2}+it}\, dt \;\neq\; 0$$

It suffices to show $c_P \Phi_u$ is not 0. With $s = \frac{1}{2} + it$, the relation implies $u(-t)E_{1-s} = u(t)c_s \cdot E_{1-s}/c_s = u(t)E_s$. Then

$$\Phi_u \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t) \cdot E_s\, dt \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+0}^{\frac{1}{2}+i\infty} u(t) \cdot E_s\, dt$$

The constant term of Φ_u is

$$c_P\Phi_u = \frac{1}{2\pi i}\int_{\frac{1}{2}+0i}^{\frac{1}{2}+i\infty} u(t)\cdot(y^s + c_s y^{1-s})\, dt \;=\; \frac{1}{2\pi i}\int_{\frac{1}{2}+0i}^{\frac{1}{2}+i\infty} u(t) y^{\frac{1}{2}+it} + u(-t) y^{\frac{1}{2}-it}\, dt \;=\; \frac{\sqrt{y}}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t)\, e^{it\log y}\, dt$$

This Fourier transform does not vanish for non-vanishing u.

Since the E_s integrate to 0 against cuspforms, an integral Φ_u of them does, also. Thus, Φ_u is in the topological closure of pseudo-Eisenstein series Ψ_φ with test-function data φ. Thus, given u, there is φ such that $\langle \Psi_\varphi, \Phi_u\rangle \neq 0$. Then

$$0 \;\neq\; \langle \Psi_\varphi, \Phi_u\rangle \;=\; \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} u(t) \cdot \langle \Psi_\varphi, E_{1-s}\rangle\, dt$$

Thus, the functions $s \to \langle \Psi_\varphi, E_s\rangle$ are dense in the space of $L^2(\frac{1}{2} + i\mathbb{R})$ functions u satisfying $u(-t) = c_s\, u(t)$. Thus, there is an *isometry*

$$\{\text{cuspforms}\}^\perp \cap L^2(\Gamma\backslash G)^K \;\;\approx\;\; \{u \in L^2(\Gamma\backslash G/K) : u(-t) = c_s \cdot u(t)\}$$