(*February 26, 2015*)

Standard archimedean integrals for $GL(2)$

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1. *Basic identities involving* $\Gamma(s)$, $B(\alpha,\beta)$, *etc.*

We take the Gamma function $\Gamma(s)$ to be defined by Euler's integral

$$\Gamma(s) \;=\; \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

Integration by parts proves the functional equation

$$\Gamma(s+1) \;=\; s\cdot\Gamma(s)$$

For $0 < s \in \mathbb{Z}$, this relation and induction show the connection to *factorials*,

$$\Gamma(n) \;=\; (n-1)! \qquad\qquad (\text{for } n = 1, 2, \ldots)$$

From the functional equation, we get a meromorphic continuation of $\Gamma(s)$ to the entire complex plane, except for poles at non-positive integers $-n$. The poles are *simple*, with residue $(-1)^n/n!$ at $-n$.

The identity

$$\int_0^\infty t^s\, e^{-ty}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{y^s} \qquad\qquad (\text{for } y > 0 \text{ and } \mathrm{Re}(s) > 0)$$

for $y > 0$ first follows for $\mathrm{Re}(s) > 0$ by replacing t by t/y in the integral. Then

$$\int_0^\infty t^s\, e^{-tz}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{z^s} \qquad\qquad (\text{for } \mathrm{Re}(z) > 0 \text{ and } \mathrm{Re}(s) > 0)$$

by complex analysis, since both sides are holomorphic in s and agree on the positive reals.

The latter identity allows non-obvious evaluation of a Fourier transform. Namely, let

$$f(x) \;=\; \begin{cases} x^\alpha\cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

For $\mathrm{Re}(\alpha) > -1$ this function is locally integrable at 0, and in any case is of rapid decay at infinity. We can compute its Fourier transform:

$$\int_{\mathbb{R}} e^{-2\pi i\xi x}\, f(x)\, dx \;=\; \int_0^\infty e^{-2\pi i\xi x}\, x^{\alpha+1}\, e^{-x}\, \frac{dx}{x} \;=\; \int_0^\infty x^{\alpha+1}\, e^{-x(1+2\pi i\xi)}\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha+1)}{(1+2\pi i\xi)^{\alpha+1}}$$

Further, Fourier inversion gives the non-obvious

$$\int_{\mathbb{R}} e^{2\pi i\xi x}\,\frac{1}{(1+2\pi i\xi)^{\alpha+1}}\,d\xi \;=\; \frac{1}{\Gamma(\alpha+1)}\cdot\begin{cases} x^\alpha\cdot e^{-x} & (\text{for } x>0)\\ 0 & (\text{for } x<0)\end{cases}$$

For $\alpha\in\mathbb{Z}$, the same conclusion can be reached by evaluation by residues.

Next, we recall the argument that expresses Euler's beta integral in terms of gamma functions, as

$$B(a,b) \;=\; \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Indeed, replacing x by $\frac{t}{t+1}=1-\frac{1}{t+1}$ in the integral gives

$$\int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \int_0^\infty \Big(\frac{t}{t+1}\Big)^{a-1}\Big(1-\frac{t}{t+1}\Big)^{b-1}\,\frac{dt}{(t+1)^2} \;=\; \int_0^\infty t^a\,\Big(\frac{1}{t+1}\Big)^{a+b}\,\frac{dt}{t}$$

Use the gamma identity in the form

$$\Big(\frac{1}{t+1}\Big)^s \;=\; \frac{1}{\Gamma(s)}\int_0^\infty e^{-u(t+1)}\,u^s\,\frac{du}{u}$$

to rewrite the beta integral further as

$$\frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^{a+b}\,t^a\,e^{-u(t+1)}\,\frac{du}{u}\frac{dt}{t} \;=\; \frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^b\,t^a\,e^{-u}\,e^{-t}\,\frac{dt}{t}\frac{du}{u} \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

as claimed. ///

A similar sort of integral, with one more factor, is Euler's integral representation for **hypergeometric functions**, namely,

$$F(\alpha,\beta,\gamma;z) \;=\; \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 x^{\beta-1}\,(1-x)^{\gamma-\beta-1}\,(1-xz)^{-\alpha}\,dx$$

where B is the beta function

$$B(a,b) \;=\; \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

This F is the ${}_2F_1$ hypergeometric function, whose *series* definition is

$$F(\alpha,\beta,\gamma;z) \;=\; 1+\frac{a\,b}{c}\,\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\,\frac{z^2}{2!}+\ldots \;=\; \sum_{n=0}^\infty \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

The notation $(a)_n$ is the *Pockhammer* symbol.

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2. *Holomorphic discrete series*

Whittaker functions $W(g)$ for *holomorphic discrete series* for $GL_2(\mathbb{R})$ are completely elementary. For the holomorphic discrete series of weight $0<\kappa\in\mathbb{Z}$, the Whittaker function corresponding to the lowest K-type is essentially an exponential function, namely,

$$W\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} \;=\; y^{\frac{\kappa}{2}}\,e^{-2\pi y} \qquad\qquad (\text{for } y>0)$$

The extension of this to the whole group is

$$W(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) \;=\; \omega(z) e^{2\pi i x} \cdot y^{\frac{\kappa}{2}}\, e^{-2\pi y} \cdot e^{i\kappa\theta} \qquad (\text{for } y > 0,\ z \in \mathbb{R}^\times)$$

where the central character ω must be compatible with κ, in the sense that

$$\omega \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \;=\; e^{\pi i \kappa}$$

That is, the central character must have the same *parity*, as a character, as the weight κ has, as an integer.

Mellin transforms of holomorphic discrete series Whittaker functions are easy to evaluate, with the exponent $s - \frac{1}{2}$ rather than s, as usual, to have the functional equations $s \to 1 - s$ rather than $s \to -s$. Namely, regardless of the parity of κ,

$$\int_{\mathbb{R}^\times} \operatorname{sgn}(y)^\kappa \cdot |y|^{s-\frac{1}{2}}\, W \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{|y|} \;=\; 2 \cdot \int_0^\infty |y|^{s-\frac{1}{2}}\, W \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y}$$

$$=\; 2 \cdot \int_0^\infty |y|^{s-\frac{1}{2}+\frac{\kappa}{2}}\, e^{-2\pi y}\, \frac{dy}{y} \;=\; 2 \cdot (2\pi)^{-(s-\frac{1}{2}+\frac{\kappa}{2})}\, \Gamma(s - \tfrac{1}{2} + \frac{\kappa}{2})$$

The archimedean integral arising in the *Rankin-Selberg convolution* L-function for two holomorphic cuspforms *of the same weight* κ is similarly easy to evaluate explicitly:

$$\int_0^\infty |y|^{s-1}\, W \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \overline{W} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \int_0^\infty |y|^{s-1+\kappa}\, e^{-4\pi y}\, \frac{dy}{y} \;=\; (4\pi)^{-(s-1+\kappa)} \cdot \Gamma(s - 1 + \kappa)$$

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3. *Spherical Whittaker functions for* $GL_2(\mathbb{R})$

First, we want *integral representations* of Whittaker functions for *spherical principal series* with trivial central character for $G = GL_2(\mathbb{R})$. As usual, define subgroups

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \qquad N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$$

For $\mu \in \mathbb{C}$, let I_μ be the μ^{th} (naively normalized) spherical principal series of G with trivial central character, by definition consisting of the collection of all smooth functions f on G such that

$$f(p \cdot g) \;=\; \chi(p) \cdot f(g) \qquad (\text{for } \chi(p) = |a/d|^\mu \text{ with } p = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in P, \text{ and } g \in G)$$

Computation of the Fourier expansion of Eisenstein series suggests presenting the Whittaker function as an image under an intertwining operator from I_μ to the *Whittaker space*

$$\text{Wh} \;=\; \{f \text{ on } GL_2(\mathbb{R}) : f(ng) = \psi(n) \cdot f(g) \text{ for all } n \in N,\ g \in G\} \qquad (\text{where } \psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{2\pi i x})$$

Such an intertwining amounts to taking a Fourier transform, as follows. Let the spherical function be

$$\varphi(\begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) \;=\; |y|^\mu \qquad (\text{for } y > 0)$$

with *trivial* central character. This spherical function gets mapped to the Whittaker space by an intertwining

$$W_\mu^{\rm nf}(g) \;=\; \int_{\mathbb{R}} e^{-2\pi i x}\, \varphi(w_o\, n_x\, g)\, dx$$

For diagonal elements $m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, we can explicitly determine the Iwasawa decomposition of $w_o n_x m_y$, namely (see the appendix)

$$w_o n_x m_y \;=\; \begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & x/r \\ -x/r & y/r \end{pmatrix} w_o \qquad\qquad (\text{where } r = \sqrt{x^2+y^2})$$

Therefore,

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \int_{\mathbb{R}} e^{-2\pi i x}\, \varphi(\begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix} \begin{pmatrix} y/r & x/r \\ -x/r & y/r \end{pmatrix} w_o)\, dx$$

The spherical-ness is simply right K-invariance, and we are assuming trivial central character, so this simplifies to

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \int_{\mathbb{R}} e^{-2\pi i x} \left(\frac{|y|}{y^2+x^2}\right)^\mu dx \qquad\qquad (\text{for } {\rm Re}\,(\mu) > \tfrac{1}{2})$$

Replace x by xy to obtain the useful integral representation

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; |y|^{1-\mu} \int_{\mathbb{R}} e^{-2\pi i x y}\, \frac{1}{(1+x^2)^\mu}\, dx \qquad\qquad (\text{for } {\rm Re}\,(\mu) > \tfrac{1}{2})$$

It is awkward that ${\rm Re}\,(\mu) > \frac{1}{2}$ is necessary for absolute convergence, since the *unitary* principal series have ${\rm Re}\,(\mu) = \frac{1}{2}$, in the naive normalization. However, we can rearrange the integral to a form that shows the meromorphic continuation, and explains the normalization constant at the same time. Namely, using the identity involving $\Gamma(s)$,

$$|y|^{1-\mu} \int_{\mathbb{R}} e^{-2\pi i x y}\, \frac{1}{(1+x^2)^\mu}\, dx \;=\; \frac{|y|^{1-\mu}}{\Gamma(\mu)} \int_{\mathbb{R}} \int_0^\infty e^{-2\pi i x y}\, t^\mu\, e^{-t(1+x^2)}\, dx\, \frac{dt}{t}$$

Replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ to obtain

$$\frac{\sqrt{\pi}\,|y|^{1-\mu}}{\Gamma(\mu)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x y \cdot \frac{\sqrt{\pi}}{\sqrt{t}}}\, t^{\mu-\frac{1}{2}}\, e^{-t}\, e^{-\pi x^2}\, dx\, \frac{dt}{t}$$

Taking the Fourier transform of the Gaussian in x, this is

$$\frac{\sqrt{\pi}\,|y|^{1-\mu}}{\Gamma(\mu)} \int_0^\infty e^{-\pi y^2 \cdot \frac{\pi}{t}}\, t^{\mu-\frac{1}{2}}\, e^{-t}\, \frac{dt}{t}$$

Replacing t by $\pi|y| \cdot t$ gives

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \frac{\pi^\mu\, |y|^{1/2}}{\Gamma(\mu)} \int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)}\, t^{\mu-\frac{1}{2}}\, \frac{dt}{t}$$

This last integral converges nicely for any complex μ.

A less naive normalization of the parametrization of the principal series $I_\mu^{\rm nf}$ replaces μ by $\mu+\frac{1}{2}$, taking

$$I_\mu \;=\; I_{\mu+\frac{1}{2}}^{\rm nf}$$

And a less naive normalization of the Whittaker function not only shifts μ, but also gets rid of the $\Gamma(\mu)$ and π^μ, by taking

$$W_\mu \;=\; \frac{\Gamma(\mu+\frac{1}{2})}{\pi^\mu}\cdot W^{\rm nf}_{\mu+\frac{1}{2}}$$

That is,

$$W_\mu\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} \;=\; \frac{\Gamma(\mu+\frac{1}{2})}{\pi^\mu}\int_{\mathbb{R}} e^{-2\pi i x}\left(\frac{|y|}{x^2+y^2}\right)^{\mu+\frac{1}{2}} dx \;=\; |y|^{1/2}\int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)}\, t^\mu\,\frac{dt}{t}$$

Note that, in this normalization,

$$W_{-\mu} \;=\; W_\mu$$

One might further make a comparison to classical special functions.

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4. *Spherical Mellin transforms for* $GL_2(\mathbb{R})$

The Hecke-Maaß integral representation of L-functions attached to *waveforms* already requires that we understand archimedean local integrals, namely, Mellin transforms of Whittaker functions, given by

$$\int_0^\infty y^s\, W^{\rm nf}_\mu\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\frac{dy}{y}$$

where $W^{\rm nf}_\mu$ is the Whittaker function of the μ^{th} (naively normalized) spherical principal series $I^{\rm nf}_\mu$. As with holomorphic discrete series, we will later renormalize the exponent of y to $s-\frac{1}{2}$, as a global normalization to give L-functions functional equation $s\to 1-s$ rather than $s\to -s$. The integral representation obtained above by consideration of an intertwining operator from $I^{\rm nf}_\mu$ to the Whittaker space gives us what we need:

$$\int_0^\infty y^s\, W^{\rm nf}_\mu\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\frac{dy}{y} \;=\; \int_0^\infty y^s\int_{\mathbb{R}} e^{-2\pi i x}\left(\frac{y}{x^2+y^2}\right)^\mu dx\,\frac{dy}{y}$$

The obvious thing to do is to interchange the order of integration, and apply the gamma identity, obtaining

$$\int_{\mathbb{R}}\int_0^\infty e^{-2\pi i x}\, y^{s+\mu}\left(\frac{1}{x^2+y^2}\right)^\mu \frac{dy}{y}\,dx \;=\; \frac{1}{\Gamma(\mu)}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty e^{-2\pi i x}\, y^{s+\mu}\, t^\mu\, e^{-t(x^2+y^2)}\,\frac{dy}{y}\,dx\,\frac{dt}{t}$$

Replace y by $y/\sqrt{t}$

$$\frac{1}{\Gamma(\mu)}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty e^{-2\pi i x}\, y^{s+\mu}\, t^{\frac{\mu-s}{2}}\, e^{-tx^2}\, e^{-y^2}\,\frac{dy}{y}\,dx\,\frac{dt}{t}$$

Replace y by $\sqrt{y}$

$$\frac{1}{2\,\Gamma(\mu)}\int_0^\infty\int_{\mathbb{R}}\int_0^\infty e^{-2\pi i x}\, y^{\frac{s+\mu}{2}}\, t^{\frac{\mu-s}{2}}\, e^{-tx^2}\, e^{-y}\,\frac{dy}{y}\,dx\,\frac{dt}{t} \;=\; \frac{\Gamma(\frac{s+\mu}{2})}{2\,\Gamma(\mu)}\int_0^\infty\int_{\mathbb{R}} e^{-2\pi i x}\, t^{\frac{\mu-s}{2}}\, e^{-tx^2}\,dx\,\frac{dt}{t}$$

Then replace x by $x\cdot\frac{\sqrt{\pi}}{\sqrt{t}}$ and take the Fourier transform of the Gaussian

$$\frac{\sqrt{\pi}\,\Gamma(\frac{s+\mu}{2})}{2\,\Gamma(\mu)}\int_0^\infty\int_{\mathbb{R}} e^{-2\pi i x\cdot\frac{\sqrt{\pi}}{\sqrt{t}}}\, t^{\frac{\mu-s-1}{2}}\, e^{-\pi x^2}\,dx\,\frac{dt}{t} \;=\; \frac{\sqrt{\pi}\,\Gamma(\frac{s+\mu}{2})}{2\,\Gamma(\mu)}\int_0^\infty e^{-\pi\cdot\frac{\pi}{t}}\, t^{\frac{\mu-s-1}{2}}\,\frac{dt}{t}$$

Replace t by $1/t$ and then by $t\cdot\pi^2$

$$\frac{\sqrt{\pi}\,\Gamma(\frac{s+\mu}{2})\,\pi^{-(s+1-\mu)}}{2\,\Gamma(\mu)}\int_0^\infty e^{-t}\, t^{\frac{s+1-\mu}{2}}\,\frac{dt}{t} \;=\; \frac{\pi^{\mu-s-\frac{1}{2}}\,\Gamma(\frac{s+\mu}{2})}{2\,\Gamma(\mu)}\cdot\Gamma(\tfrac{s+1-\mu}{2})$$

That is, in a naive normalization,

$$\int_0^\infty |y|^s\, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\frac{dy}{y} \;=\; \frac{\pi^{\mu-s-\frac{1}{2}}}{2\,\Gamma(\mu)}\cdot \Gamma(\tfrac{s+\mu}{2})\,\Gamma(\tfrac{s+1-\mu}{2})$$

The less naive normalization replaces μ by $\mu+\frac{1}{2}$, s by $s-\frac{1}{2}$, and uses

$$W_\mu \;=\; \frac{\Gamma(\mu+\frac{1}{2})}{\pi^{\mu+\frac{1}{2}}}\cdot W^{\rm nf}_{\mu+\frac{1}{2}}$$

This gives

$$\int_0^\infty |y|^{s-\frac{1}{2}}\, W_\mu\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\frac{dy}{y} \;=\; \tfrac{1}{2}\cdot\pi^{-s}\,\Gamma(\tfrac{s+\mu}{2})\cdot\Gamma(\tfrac{s-\mu}{2})$$

Note the nature of the product of gammas, specifically, the symmetry in $\pm\mu$.

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5. *Spherical Rankin-Selberg integrals for* $GL_2(\mathbb{R})$

Let $W_\mu^{\rm nf}$ and $W_\nu^{\rm nf}$ be spherical Whittaker functions for the μ^{th} and ν^{th} spherical principal series for $GL_2(\mathbb{R})$ in the naive normalization. The local Rankin-Selberg integral is

$$\int_0^\infty y^{s-1}\, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} W_\nu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\frac{dy}{y}$$

The exponent $s-1$ arises naturally from unwinding an Eisenstein series with a functional equation $s\to 1-s$. Also, one might complex conjugate the second Whittaker function, but this would only complex-conjugate the parameter ν (and replace y by $-y$, but the function is even). Replace the Whittaker functions by their integral representations, letting $\psi(x)=e^{2\pi ix}$, and apply the gamma identity:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty y^{s-1}\,y^{1-\mu}\,\frac{\overline{\psi}(xy)}{(1+x^2)^\mu}y^{1-\nu}\,\frac{\overline{\psi}(uy)}{(1+u^2)^\nu}\,\frac{dy}{y}\,dx\,du$$

$$=\;\frac{1}{\Gamma(\mu)\,\Gamma(\nu)}\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty y^{s+1-\mu-\nu}\,\overline{\psi}(xy)\,t^\mu\,e^{-t(1+x^2)}\overline{\psi}(uy)\,\tau^\nu\,e^{-\tau(1+u^2)}\,\frac{dy}{y}\,dx\,du\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

Replace x by $x\cdot\frac{\sqrt{\pi}}{\sqrt{t}}$ and u by $u\cdot\frac{\sqrt{\pi}}{\sqrt{\tau}}$ and take the Fourier transforms of the Gaussians, giving

$$\frac{\pi}{\Gamma(\mu)\,\Gamma(\nu)}\int_0^\infty\int_0^\infty\int_0^\infty y^{s+1-\mu-\nu}\,t^{\mu-\frac{1}{2}}\,e^{-t}\,e^{-\pi y^2(\frac{\pi}{t})}\tau^{\nu-\frac{1}{2}}\,e^{-\tau}e^{-\pi y^2(\frac{\pi}{\tau})}\,\frac{dy}{y}\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

Replace y by $\sqrt{y}$ and then replace y by y/π^2, to obtain

$$\frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\int_0^\infty\int_0^\infty\int_0^\infty y^{\frac{s+1-\mu-\nu}{2}}\,t^{\mu-\frac{1}{2}}\,e^{-t}\,e^{-y(\frac{1}{\tau}+\frac{1}{t})}\tau^{\nu-\frac{1}{2}}\,e^{-\tau}\,\frac{dy}{y}\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

Replace y by $y/(\tau^{-1}+t^{-1})$:

$$\frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2})\int_0^\infty\int_0^\infty t^{\mu-\frac{1}{2}}\,e^{-t}\,\frac{1}{(\frac{1}{\tau}+\frac{1}{t})^{\frac{s+1-\mu-\nu}{2}}}\tau^{\nu-\frac{1}{2}}\,e^{-\tau}\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

$$=\;\frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2})\int_0^\infty\int_0^\infty t^{\mu-\frac{1}{2}}\,e^{-t}\,\frac{(t\tau)^{\frac{s+1-\mu-\nu}{2}}}{(t+\tau)^{\frac{s+1-\mu-\nu}{2}}}\tau^{\nu-\frac{1}{2}}\,e^{-\tau}\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

$$=\;\frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2})\int_0^\infty\int_0^\infty t^{\frac{s+\mu-\nu}{2}}\,e^{-t}\,\frac{1}{(t+\tau)^{\frac{s+1-\mu-\nu}{2}}}\tau^{\frac{s+\nu-\mu}{2}}\,e^{-\tau}\,\frac{dt}{t}\,\frac{d\tau}{\tau}$$

Replace t by $t\tau$

$$= \frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2}) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, e^{-t\tau}\, \frac{1}{(t\tau+\tau)^{\frac{s+1-\mu-\nu}{2}}} \tau^s\, e^{-\tau}\, \frac{dt}{t}\, \frac{d\tau}{\tau}$$

$$= \frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2}) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, e^{-(t+1)\tau}\, \frac{1}{(t+1)^{\frac{s+1-\mu-\nu}{2}}} \tau^{\frac{s-1+\mu+\nu}{2}}\, \frac{dt}{t}\, \frac{d\tau}{\tau}$$

Replace τ by $\tau/(t+1)$:

$$= \frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2}) \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, e^{-\tau}\, \frac{1}{(t+1)^s} \tau^{\frac{s-1+\mu+\nu}{2}}\, \frac{dt}{t}\, \frac{d\tau}{\tau}$$

$$= \frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)}\Gamma(\tfrac{s+1-\mu-\nu}{2})\Gamma(\tfrac{s-1+\mu+\nu}{2}) \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, \frac{1}{(t+1)^s}\, \frac{dt}{t}$$

As usual,

$$\int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, \frac{1}{(t+1)^s}\, \frac{dt}{t} \;=\; \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, x^s\, e^{-x(1+t)}\, \frac{dt}{t}\, \frac{dx}{x}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty t^{\frac{s+\mu-\nu}{2}}\, x^{\frac{s-\mu+\nu}{2}}\, e^{-x}\, e^{-t}\, \frac{dt}{t}\, \frac{dx}{x} \;=\; \frac{\Gamma(\frac{s+\mu-\nu}{2})\,\Gamma(\frac{s-\mu+\nu}{2})}{\Gamma(s)}$$

Thus, the whole is

$$\frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)} \cdot \frac{\Gamma(\frac{s+\mu-\nu}{2})\,\Gamma(\frac{s-\mu+\nu}{2})\,\Gamma(\frac{s+1-\mu-\nu}{2})\,\Gamma(\frac{s-1+\mu+\nu}{2})}{\Gamma(s)}$$

That is, with the naive normalization,

$$\int_0^\infty y^{s-1}\, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} W_\nu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \frac{\pi^{\mu+\nu-s}}{2\,\Gamma(\mu)\,\Gamma(\nu)} \cdot \frac{\Gamma(\frac{s+\mu-\nu}{2})\,\Gamma(\frac{s-\mu+\nu}{2})\,\Gamma(\frac{s+1-\mu-\nu}{2})\,\Gamma(\frac{s-1+\mu+\nu}{2})}{\Gamma(s)}$$

For the less naive normalization, replace μ by $\mu+\frac{1}{2}$ and ν by $\nu+\frac{1}{2}$, and recall the renormalization

$$W_\mu \;=\; \frac{\Gamma(\mu+\frac{1}{2})}{\pi^{\mu+\frac{1}{2}}} \cdot W^{\rm nf}_{\mu+\frac{1}{2}}$$

Then the local Rankin-Selberg integral is

$$\int_0^\infty y^{s-1}\, W_\mu\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} W_\nu\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \frac{\pi^{1-s}}{2} \cdot \frac{\Gamma(\frac{s+\mu-\nu}{2})\,\Gamma(\frac{s-\mu+\nu}{2})\,\Gamma(\frac{s-\mu-\nu}{2})\,\Gamma(\frac{s+\mu+\nu}{2})}{\Gamma(s)}$$

Note the pleasing symmetry in the parameters $\pm\mu$ and $\pm\nu$. The extra 2 in the denominator can be rationalized away in several different ways.

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6. *Moment integrals for discrete series of $GL_2(\mathbb{R})$*

In the study of *moments* of L-functions, we encounter *local moment integrals* such as

$$\int_0^\infty \int_0^\infty y^{it}\, y'^{-it}\, W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \overline{W}\begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} \int_{\mathbb{R}} \overline{\psi}((y-y')\cdot x)\, \frac{dx}{(1+x^2)^{w/2}}\, \frac{dy}{y}\, \frac{dy'}{y'}$$

where the W's are Whittaker functions for irreducibles of $GL_2(\mathbb{R})$ or $GL_2(\mathbb{C})$, with $\psi(x) = e^{2\pi i x}$ as usual, and where we have made the simplest useful choice $1/(1+x^2)^{w/2}$ of archimedean data. By Diaconu's computations, these integrals are not generally elementary, except for holomorphic discrete series of $GL_2(\mathbb{R})$, where we can evaluate them as follows. For holomorphic discrete series with lowest weights both equal to κ, the local moment integral is

$$\int_0^\infty \int_0^\infty y^{it}\, y'^{-it}\, y^{\kappa/2} e^{-2\pi y}\, y'^{\kappa/2} e^{-2\pi y'} \int_{\mathbb{R}} \overline{\psi}((y-y')\cdot x)\, \frac{dx}{(1+x^2)^{w/2}}\, \frac{dy}{y}\, \frac{dy'}{y'}$$

$$= \int_0^\infty \int_0^\infty \int_{\mathbb{R}} y^{it+\frac{\kappa}{2}}\, y'^{-it+\frac{\kappa}{2}} e^{-(2\pi+2\pi ix)y} e^{-(2\pi-2\pi ix)y'}\, \frac{dx}{(1+x^2)^{w/2}}\, \frac{dy}{y}\, \frac{dy'}{y'}$$

Using the gamma identity in both y and y', this gives

$$(2\pi)^{-(1+\kappa)}\, \Gamma(it+\frac{\kappa}{2})\, \Gamma(-it+\frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+ix)^{it+\frac{\kappa}{2}}} \cdot \frac{1}{(1-ix)^{-it+\frac{\kappa}{2}}} \cdot \frac{dx}{(1+x^2)^{w/2}}$$

Because of the simple choice of data, this is

$$(2\pi)^{-(1+\kappa)}\, \Gamma(it+\frac{\kappa}{2})\, \Gamma(-it+\frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+ix)^{it+\frac{\kappa}{2}+\frac{w}{2}}} \cdot \frac{1}{(1-ix)^{-it+\frac{\kappa}{2}+\frac{w}{2}}}\, dx$$

This is the inner product on $\mathbb{R}$ of two functions whose Fourier transforms we essentially know from the gamma identity. That is, we know

$$\int_{\mathbb{R}} e^{2\pi i\xi x}\, \frac{1}{(1+2\pi i\xi)^{\alpha+1}}\, d\xi \;=\; \frac{1}{\Gamma(\alpha+1)} \cdot \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x>0) \\ 0 & (\text{for } x<0) \end{cases}$$

Thus, replace x by $2\pi x$ in the given integral, to have

$$(2\pi)^{-\kappa}\, \Gamma(it+\frac{\kappa}{2})\, \Gamma(-it+\frac{\kappa}{2}) \int_{\mathbb{R}} \frac{1}{(1+2\pi ix)^{it+\frac{\kappa}{2}+\frac{w}{2}}} \cdot \frac{1}{(1-2\pi ix)^{-it+\frac{\kappa}{2}+\frac{w}{2}}}\, dx$$

By the Plancherel identity,

$$\int_{\mathbb{R}} \frac{1}{(1+2\pi ix)^{\alpha+1}}\, \frac{1}{(1-2\pi ix)^{\beta+1}}\, dx \;=\; \frac{1}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_{\mathbb{R}} x^\alpha\, e^{-x} \cdot x^\beta\, e^{-x}\, dx \;=\; \frac{2^{-(\alpha+\beta+1)}\,\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}$$

In the case at hand,

$$\alpha+1 \;=\; it+\frac{\kappa}{2}+\frac{w}{2} \quad \text{and} \quad \beta+1 \;=\; -it+\frac{\kappa}{2}+\frac{w}{2}$$

Thus, the whole integral is

$$(2\pi)^{-\kappa}\, \Gamma(it+\frac{\kappa}{2})\, \Gamma(-it+\frac{\kappa}{2})\, \frac{2^{-(\kappa+w)}\; \Gamma(\kappa+w)}{\Gamma(it+\frac{\kappa}{2}+\frac{w}{2})\,\Gamma(-it+\frac{\kappa}{2}+\frac{w}{2})}$$

Thus, for weight κ holomorphic discrete series on $GL_2(\mathbb{R})$, and with the simplest choice of data,

$$(\text{moment integral}) \;=\; (2\pi)^{-\kappa} \cdot 2^{-(\kappa+w)} \cdot \frac{\Gamma(it+\frac{\kappa}{2})\,\Gamma(-it+\frac{\kappa}{2})\,\Gamma(\kappa+w)}{\Gamma(it+\frac{\kappa}{2}+\frac{w}{2})\,\Gamma(-it+\frac{\kappa}{2}+\frac{w}{2})}$$

From the asymptotic result

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; s^{-a} \cdot (1+O(\frac{1}{|s|})) \qquad\qquad (\text{for } a \text{ fixed})$$

for fixed w and for $s=\frac{1}{2}+it$ this moment integral has simple asymptotics, namely

$$(\text{moment integral}) \;\sim\; (\text{constant}) \cdot t^{-w}$$

To clarify the dependence upon w, use the *duplication* formula

$$\Gamma(s)\cdot\Gamma(s+\tfrac{1}{2}) \;=\; 2^{1-2s}\cdot\sqrt{\pi}\cdot\Gamma(2s)$$

to rewrite

$$\Gamma(\kappa+w) \;=\; \Gamma(2\cdot\frac{\kappa+w}{2}) \;=\; 2^{\kappa+w-1}\cdot\sqrt{\pi}\cdot\Gamma(\frac{\kappa+w}{2})\cdot\Gamma(\frac{\kappa+w+1}{2})\cdot$$

Thus, the exponential depending on w disappears, and

$$(\text{moment integral}) \;=\; (2\pi)^{-\kappa}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(it+\frac{\kappa}{2})\,\Gamma(-it+\frac{\kappa}{2})\,\Gamma(\frac{\kappa}{2}+\frac{w}{2})\cdot\Gamma(\frac{\kappa}{2}+\frac{w}{2}+\frac{1}{2})\cdot}{\Gamma(it+\frac{\kappa}{2}+\frac{w}{2})\,\Gamma(-it+\frac{\kappa}{2}+\frac{w}{2})}$$

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7. *Spherical Whittaker functions for* $GL_2(\mathbb{C})$

First, we want *integral representations* of Whittaker functions for *spherical principal series* with trivial central character for $G = GL_2(\mathbb{C})$. As usual, let

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \qquad N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$$

For compatibility with global normalizations, take additive character and norm on $\mathbb{C}$ to be

$$\psi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;=\; e^{2\pi i(x+\overline{x})} \qquad\qquad |\alpha|_{\mathbb{C}} \;=\; |\alpha\cdot\overline{\alpha}|$$

where the norm on the right-hand side is the usual norm on $\mathbb{R}$. For $\mu\in\mathbb{C}$, let I_μ be the μ^{th} (naively normalized) spherical principal series of G with trivial central character, by definition consisting of the collection of all smooth functions f on G such that

$$f(p\cdot g) \;=\; \chi(p)\cdot f(g) \qquad\qquad (\text{for } \chi(p) = |a/d|_{\mathbb{C}}^{\mu} \text{ with } p = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}\in P, \text{ and } g\in G)$$

As over $\mathbb{R}$, computation of the Fourier expansion of Eisenstein series suggests presenting the Whittaker function as an image under an intertwining operator from I_μ to the *Whittaker space*

$$\mathrm{Wh} \;=\; \{f \text{ on } GL_2(\mathbb{C}) : f(ng) = \psi(n)\cdot f(g) \text{ for all } n\in N,\, g\in G\}$$

Such an intertwining amounts to taking a Fourier transform, as follows. Let the spherical function be

$$\varphi(\begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix}\cdot k) \;=\; |y|_{\mathbb{C}}^{\mu} \;=\; y^{2\mu} \qquad\qquad (\text{for } y>0,\; k\in U(2))$$

with *trivial* central character. This spherical function gets mapped to the Whittaker space by an intertwining

$$W_\mu^{\mathrm{nf}}(g) \;=\; \int_{\mathbb{C}} \overline{\psi}(x)\,\varphi(w_o\,n_x\,g)\,dx$$

where the measure is *twice* the usual measure on $\mathbb{C}$, for compatibility with the normalization of the character. For diagonal elements $m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ with $y>0$, we can explicitly determine the Iwasawa decomposition of $w_o n_x m_y$, namely (see the appendix)

$$w_o n_x m_y \;=\; \begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix}\begin{pmatrix} y/r & \overline{x}/r \\ -x/r & y/r \end{pmatrix} w_o \qquad\qquad (\text{where } r = \sqrt{x\overline{x}+y^2})$$

Therefore,

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \int_{\mathbb{C}} \overline{\psi}(x)\,\varphi(\begin{pmatrix} y/r & * \\ 0 & r \end{pmatrix}\begin{pmatrix} y/r & \overline{x}/r \\ -x/r & y/r \end{pmatrix}\, w_o)\,dx$$

The spherical-ness is right $U(2)$-invariance, and we are assuming trivial central character, so this simplifies to

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \int_{\mathbb{C}} e^{-2\pi i(x+\overline{x})}\left|\frac{y}{y^2+x\overline{x}}\right|^{2\mu} dx \;=\; \int_{\mathbb{C}} e^{-2\pi i(x+\overline{x})}\left|\frac{y}{y^2+x\overline{x}}\right|^{2\mu} dx \qquad (\text{for } \mathrm{Re}(\mu) > \tfrac{1}{2})$$

where we reverted to the usual absolute value on $\mathbb{R}$, causing the exponent μ to effectively double. Replace x by xy to obtain a useful integral representation, valid for $y \in \mathbb{C}^\times$,

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; |y|_{\mathbb{C}}^{1-\mu}\int_{\mathbb{C}} \overline{\psi}(x\cdot y)\,\frac{1}{(1+x\overline{x})^{2\mu}}\,dx \qquad (\text{for } \mathrm{Re}(\mu) > \tfrac{1}{2})$$

As in the case of $GL_2(\mathbb{R})$, the integral can be rearranged to show the meromorphic continuation, and explain subsequent renormalization. Namely, using the identity involving $\Gamma(s)$, for $y > 0$,

$$y^{2-2\mu}\int_{\mathbb{C}} e^{-2\pi i(x+\overline{x})y}\,\frac{1}{(1+x\overline{x})^{2\mu}}\,dx \;=\; \frac{y^{2-2\mu}}{\Gamma(2\mu)}\int_{\mathbb{C}}\int_0^\infty e^{-2\pi i(x+\overline{x})y}\,t^{2\mu}\,e^{-t(1+x\overline{x})}\,dx\,\frac{dt}{t}$$

Replace x by $x\cdot\frac{\sqrt{\pi}}{\sqrt{t}}$ to obtain

$$\frac{\pi\,|y|^{1-\mu}}{\Gamma(\mu)}\int_0^\infty\int_{\mathbb{C}} e^{-2\pi i(x+\overline{x})y\cdot\frac{\sqrt{\pi}}{\sqrt{t}}}\,t^{2\mu-1}\,e^{-t}\,e^{-\pi x\overline{x}}\,dx\,\frac{dt}{t}$$

Taking the Fourier transform of the Gaussian in $x \in \mathbb{C}$, this is

$$\frac{\pi\,|y|^{2-2\mu}}{\Gamma(2\mu)}\int_0^\infty e^{-\pi y^2\cdot\frac{\pi}{t}}\,t^{2\mu-1}\,e^{-t}\,\frac{dt}{t}$$

Replacing t by $\pi|y|\cdot t$ gives

$$W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \frac{\pi^{2\mu}\,|y|}{\Gamma(2\mu)}\int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)}\,t^{2\mu-1}\,\frac{dt}{t}$$

This last integral converges nicely for any complex μ.

A less naive normalization of the parametrization of the principal series $I_\mu^{\rm nf}$ replaces μ by $\mu+\frac{1}{2}$, taking

$$I_\mu \;=\; I^{\rm nf}_{\mu+\frac{1}{2}}$$

and gets rid of the $\Gamma(2\mu)$ and $\pi^{2\mu}$, by taking

$$W_\mu \;=\; \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}}\cdot W^{\rm nf}_{\mu+\frac{1}{2}}$$

That is,

$$W_\mu\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}}\int_{\mathbb{C}} e^{-2\pi i(x+\overline{x})}\left(\frac{|y|}{x\overline{x}+y^2}\right)^{2\mu+1} dx \;=\; |y|\int_0^\infty e^{-\pi|y|(\frac{1}{t}+t)}\,t^{2\mu}\,\frac{dt}{t}$$

Thus, in this normalization, visibly

$$W_{-\mu} \;=\; W_\mu$$

8. *Spherical Mellin transforms for* $GL_2(\mathbb{C})$

For $GL_2(\mathbb{C})$, the Mellin transform of the spherical Whittaker function gives an integral of the form

$$\int_0^\infty |y|_{\mathbb{C}}^s \, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \int_0^\infty y^{2s}\, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y}$$

where $W_\mu^{\rm nf}$ is the Whittaker function of the μ^{th} (naively normalized) spherical principal series $I_\mu^{\rm nf}$. As for $GL_2(\mathbb{R})$, later we will rewrite this with $|y|_{\mathbb{C}}^s$ replaced by $|y|_{\mathbb{C}}^{s-\frac{1}{2}}$, in order to give L-functions functional equation $s \to 1-s$ rather than $s \to -s$. To compute this integral in terms of gamma functions, use the integral representation obtained above from an intertwining operator from $I_\mu^{\rm nf}$ to the Whittaker space:

$$\int_0^\infty y^{2s}\, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \int_0^\infty y^{2s} \int_{\mathbb{C}} \overline{\psi}(x) \left(\frac{y}{x\overline{x}+y^2}\right)^{2\mu} dx\, \frac{dy}{y}$$

As over $\mathbb{R}$, the obvious thing to do is to interchange the order of integration, and apply the gamma identity, obtaining

$$\int_{\mathbb{C}}\int_0^\infty \overline{\psi}(x)\, y^{2s+2\mu} \left(\frac{1}{x\overline{x}+y^2}\right)^{2\mu} \frac{dy}{y}\, dx \;=\; \frac{1}{\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}}\int_0^\infty \overline{\psi}(x)\, y^{2s+2\mu}\, t^{2\mu}\, e^{-t(x\overline{x}+y^2)}\, \frac{dy}{y}\, dx\, \frac{dt}{t}$$

Replace y by $y/\sqrt{t}$

$$\frac{1}{\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}}\int_0^\infty \overline{\psi}(x)\, y^{2s+2\mu}\, t^{\mu-s}\, e^{-tx\overline{x}}\, e^{-y^2}\, \frac{dy}{y}\, dx\, \frac{dt}{t}$$

Replace y by $\sqrt{y}$

$$\frac{1}{2\,\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}}\int_0^\infty \overline{\psi}(x)\, y^{s+\mu}\, t^{\mu-s}\, e^{-tx\overline{x}}\, e^{-y}\, \frac{dy}{y}\, dx\, \frac{dt}{t} \;=\; \frac{\Gamma(s+\mu)}{2\,\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \overline{\psi}(x)\, t^{\mu-s}\, e^{-tx\overline{x}}\, dx\, \frac{dt}{t}$$

Then replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$ and take the Fourier transform of the Gaussian

$$\frac{\pi\,\Gamma(s+\mu)}{2\,\Gamma(2\mu)} \int_0^\infty \int_{\mathbb{C}} \overline{\psi}\left(x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}\right) t^{\mu-s-1}\, e^{-\pi x\overline{x}}\, dx\, \frac{dt}{t} \;=\; \frac{\pi\,\Gamma(s+\mu)}{2\,\Gamma(2\mu)} \int_0^\infty e^{-\pi\cdot\frac{\pi}{t}}\, t^{\mu-s-1}\, \frac{dt}{t}$$

Replace t by $1/t$ and then by t/π^2

$$\frac{\pi\,\Gamma(s+\mu)\,\pi^{-2(s+1-\mu)}}{2\,\Gamma(\mu)} \int_0^\infty e^{-t}\, t^{s+1-\mu}\, \frac{dt}{t} \;=\; \frac{\pi^{2\mu-2s-1}\,\Gamma(s+\mu)}{2\,\Gamma(2\mu)} \cdot \Gamma(s+1-\mu)$$

That is, in a naive normalization,

$$\int_0^\infty |y|_{\mathbb{C}}^s \, W_\mu^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \frac{\pi^{2\mu-2s-1}}{2\,\Gamma(2\mu)} \cdot \Gamma(s+\mu)\,\Gamma(s+1-\mu)$$

The less naive normalization replaces μ by $\mu+\frac{1}{2}$, s by $s-\frac{1}{2}$, and uses

$$W_\mu \;=\; \frac{\Gamma(2\mu+1)}{\pi^{2\mu+1}} \cdot W_{\mu+\frac{1}{2}}^{\rm nf}$$

In this normalization, the Mellin transform of the spherical $GL_2(\mathbb{C})$ Whittaker function is

$$\int_0^\infty |y|_{\mathbb{C}}^{s-\frac{1}{2}} \, W_\mu\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \frac{dy}{y} \;=\; \tfrac{1}{2} \cdot \pi^{-2s}\, \Gamma(s+\mu) \cdot \Gamma(s-\mu)$$

As usual, note the nature of the product of gammas, especially, the symmetry in $\pm\mu$.

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9. *Appendix: explicit Iwasawa decompositions*

To compute the images of various vectors under natural intertwinings from principal series to Whittaker spaces, as well as for other purposes, we need explicit Iwasawa decompositions of group elements. Let

$$w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

Given x in $\mathbb{R}$ or $\mathbb{C}$ and $y > 0$, let $r = \sqrt{y^2 + x\overline{x}}$. We claim that we have an Iwasawa decomposition

$$w_o\, n_x\, m_y \;=\; \begin{pmatrix} y/r & -\overline{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \overline{x}/r \\ -x/r & y/r \end{pmatrix} \cdot w_o$$

Of course, once asserted, this can be *verified* by direct computation. However, instead, we should see how the decomposition could be found in the first place.

Over either $\mathbb{R}$ or $\mathbb{C}$, the maximal compact K can be defined as

$$K \;=\; \{g \;:\; g^*\, g = 1_2\} \;=\; \{g \;:\; g \cdot g^* = 1_2\} \qquad \text{(where } g^* \text{ is } g\text{-conjugate-transpose)}$$

Of course, over $\mathbb{R}$ the conjugation does nothing. Assuming that $g = pk$ with p upper triangular and k in K,

$$g\, g^* \;=\; (pk)\,(pk)^* \;=\; p(kk^*)p = pp^*$$

Letting $p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a > 0$ and $d > 0$, we can solve for a, b, d. Here,

$$g \;=\; w_o\, n_x\, m_y\, w_o^{-1} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix}$$

Then

$$g\, g^* \;=\; \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix}\begin{pmatrix} 1 & -\overline{x} \\ 0 & y \end{pmatrix} \;=\; \begin{pmatrix} 1 & -\overline{x} \\ x & y^2 + x\overline{x} \end{pmatrix}$$

From an equality

$$\begin{pmatrix} a^2 + b\bar{b} & bd \\ \bar{b}d & d^2 \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^* \;=\; \begin{pmatrix} 1 & -\overline{x} \\ x & y^2 + x\overline{x} \end{pmatrix}$$

immediately $d = r$, and $b = -\overline{x}/d = -\overline{x}/r$, and then

$$a^2 \;=\; 1 - b\bar{b} \;=\; 1 - x\overline{x}/r^2 \;=\; \frac{y^2 + x\overline{x} - x\overline{x}}{r^2} \;=\; \frac{y^2}{r^2}$$

from which $a = y/r$. Thus,

$$g \;=\; w_o\, n_x\, m_y\, w_o^{-1} \;=\; \begin{pmatrix} y/r & -\overline{x}/r \\ 0 & r \end{pmatrix} \cdot k$$

for some $k \in K$. Then

$$k \;=\; p^{-1}\, g \;=\; \begin{pmatrix} r/y & \overline{x}/ry \\ 0 & 1/r \end{pmatrix} \;=\; \begin{pmatrix} r/y - x\overline{x}/ry & \overline{x}/ry \\ -x/r & y/r \end{pmatrix} \;=\; \begin{pmatrix} y/r & \overline{x}/ry \\ -x/r & y/r \end{pmatrix}$$

Thus, we have the Iwasawa decomposition

$$w_o\, n_x\, m_y\, w_o^{-1} \;=\; \begin{pmatrix} y/r & -\overline{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \overline{x}/ry \\ -x/r & y/r \end{pmatrix}$$

This can be slightly rewritten as

$$w_o\, n_x\, m_y \;=\; \begin{pmatrix} y/r & -\overline{x}/r \\ 0 & r \end{pmatrix} \cdot \begin{pmatrix} y/r & \overline{x}/ry \\ -x/r & y/r \end{pmatrix} \cdot w_o$$

---

10. *Appendix:* $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin\pi s$

Take $0 < \mathrm{Re}(s) < 1$ for convergence of both integrals, and compute

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty \int_0^\infty u^s\, e^{-u}\cdot v^{1-s}\, e^{-v}\, \frac{du}{u}\,\frac{dv}{v} \;=\; \int_0^\infty \int_0^\infty u\, e^{-u(1+v)}\, v^{1-s}\, \frac{du}{u}\,\frac{dv}{v}$$

by replacing v by uv. Replacing u by $u/(1+v)$ (another instance of the basic *gamma identity*) and noting that $\Gamma(1) = 1$ gives

$$\int_0^\infty \frac{v^{-s}}{1+v}\, dv$$

Replace the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of v^{-s} given by $(e^{2\pi i}v)^{-s} = e^{-2\pi i s}v^{-s}$, integrate from $+\infty$ to $\varepsilon > 0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of v^{-s}. For $\mathrm{Re}(-s) > -1$ the integral around the little circle goes to 0 as $\varepsilon \to 0$. Thus,

$$\int_0^\infty \frac{v^{-s}}{1+v}\, dv \;=\; \lim_{\varepsilon\to 0} \frac{1}{1-e^{-2\pi i s}} \int_{H_\varepsilon} \frac{v^{-s}}{1+v}\, dv$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\mathrm{Re}(-s) < 0$. Thus, this integral is equal to the limit as $R \to +\infty$ and $\varepsilon \to 0$ of the integral

from R to ε
from ε clockwise back to ε
from ε to R
from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty \frac{v^{-s}}{1+v}\, dv \;=\; \frac{2\pi i}{1-e^{-2\pi i s}}\cdot (e^{\pi i})^{-s} \;=\; \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}} \;=\; \frac{\pi}{\sin \pi s}$$

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11. *Appendix: Duplication:* $\Gamma(s)\cdot\Gamma(s+\frac{1}{2}) \;=\; 2^{1-2s}\cdot\sqrt{\pi}\cdot\Gamma(2s)$

From the Eulerian integral definition,

$$\Gamma(s)\cdot\Gamma(s+\tfrac{1}{2}) \;=\; \int_0^\infty e^{-t}\, t^s\, \frac{dt}{t} \cdot \int_0^\infty e^{-u}\, u^{s+\frac{1}{2}}\, \frac{du}{u}$$

Replacing t by t/u

$$\int_0^\infty \int_0^\infty e^{-(\frac{t}{u}+u)}\, t^s\, u^{\frac{1}{2}}\, \frac{du}{u}\,\frac{dt}{t}$$

In the Fourier transform identity

$$e^{-\pi\xi^2} \;=\; \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}\, e^{-\pi x^2}\, dx$$

let $\xi = \sqrt{t}/\sqrt{u}$ and replace x by $x/\sqrt{\pi}$:

$$e^{-\pi \frac{t}{u}} \;=\; \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2\pi i x \cdot \frac{\sqrt{t}}{\sqrt{u}\sqrt{\pi}}} e^{-x^2} \; dx$$

and replace t by t/π to obtain

$$e^{-\frac{t}{u}} \;=\; \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2ix \cdot \frac{\sqrt{t}}{\sqrt{u}}} \, e^{-x^2} \; dx$$

Substituting the Fourier transform expression in place of $e^{-\frac{t}{u}}$ gives

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix \cdot \frac{\sqrt{t}}{\sqrt{u}}} \, e^{-x^2} e^{-u} \, t^s \, u^{\frac{1}{2}} \; dx \, \frac{du}{u} \, \frac{dt}{t}$$

Replace x by $x\sqrt{u}$, and then u by $u/(x^2+1)$:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix \cdot \sqrt{t}} \, e^{-u(x^2+1)} \, t^s \, u \, dx \, \frac{du}{u} \, \frac{dt}{t} \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix \cdot \sqrt{t}} \, \frac{1}{x^2+1} e^{-u} \, t^s \, u \, dx \, \frac{du}{u} \, \frac{dt}{t}$$

$$=\; \frac{1}{\sqrt{\pi}} \Gamma(1) \int_0^\infty \int_{\mathbb{R}} e^{-2ix \cdot \sqrt{t}} \, \frac{1}{x^2+1} \, t^s \, dx \, \frac{dt}{t} \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} e^{-2ix \cdot \sqrt{t}} \, \frac{1}{x^2+1} \, t^s \, dx \, \frac{dt}{t}$$

The inner integral over x can be evaluated by residues: it captures the negative of the residue of $x \to e^{-2ix\sqrt{t}}/(x^2+1)$ in the *lower* half-plane, giving

$$\int_{\mathbb{R}} e^{-2ix \cdot \sqrt{t}} \, \frac{1}{x^2+1} \, dx \;=\; -2\pi i \cdot e^{-2i(-i)\sqrt{t}} \cdot \frac{1}{(-i)-i} \;=\; \pi \, e^{-2\sqrt{t}}$$

Summarizing, and then replacing t by t^2 and t by $t/2$:

$$\Gamma(s) \cdot \Gamma(s+\tfrac{1}{2}) \;=\; \sqrt{\pi} \int_0^\infty e^{-2\sqrt{t}} \, t^s \, \frac{dt}{t} \;=\; 2\sqrt{\pi} \int_0^\infty e^{-2t} \, t^{2s} \, \frac{dt}{t} \;=\; 2^{1-2s} \sqrt{\pi} \int_0^\infty e^{-t} \, t^{2s} \, \frac{dt}{t} \;=\; \sqrt{\pi} \, 2^{1-2s} \Gamma(2s)$$
