SUBCONVEXITY BOUNDS FOR AUTOMORPHIC L–FUNCTIONS

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Abstract. We break the convexity bound in the t–aspect for L–functions attached to cuspforms f for $GL_2(k)$ over arbitrary number fields k. The argument uses asymptotics with error term with a power saving, for second integral moments over spectral families of twists $L(s, f \otimes \chi)$ by grossencharacters χ, from our previous paper [Di-Ga].

§0. Introduction

In many instances, for cuspidal automorphic forms f on reductive adele groups over number fields, the circle of ideas around the Phragmen-Lindelöf principle, together with the functional equation for $L(s, f)$ and asymptotics for $\Gamma(s)$, give an upper bound for $L(s, f)$ on $\Re(s) = \frac{1}{2}$. These are *convexity bounds*, or *trivial bounds*. For example, for the standard L-functions for cuspforms for GL_n, the convexity bound is known. The survey [Iw-Sa] gives a general formulation of the *subconvexity problem.* See also the survey [Mi2].

In particular, the convexity bound for the Riemann zeta function is

$$\zeta(\tfrac{1}{2} + it) \ll_\varepsilon (1 + |t|)^{\frac{1}{4}+\varepsilon} \qquad \text{(for all } \varepsilon > 0)$$

and for χ a primitive Dirichlet character of conductor q

$$L(\tfrac{1}{2}, \chi) \ll_\varepsilon q^{\frac{1}{4}+\varepsilon} \qquad \text{(for all } \varepsilon > 0)$$

A similar estimate holds with $\frac{1}{2}$ replaced by $\frac{1}{2} + it$. The Generalized Lindelöf Hypothesis would replace the exponent $\frac{1}{4} + \varepsilon$ by ε. The *subconvexity problem* for GL_1 over $\mathbb{Q}$ asks for an estimate with exponent strictly below $\frac{1}{4}$. In [We] Weyl proved

$$\zeta(\tfrac{1}{2} + it) \ll_\varepsilon (1 + |t|)^{\frac{1}{6}+\varepsilon} \qquad \text{(for all } \varepsilon > 0)$$

and in [Bu] Burgess showed

$$L(s, \chi) \ll_\varepsilon q^{\frac{3}{16}+\varepsilon} \qquad \text{(fixed } s \text{ with } \Re(s) = \tfrac{1}{2}, \text{ for all } \varepsilon > 0)$$

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It is noteworthy that Burgess' method relies upon Weil's Riemann Hypothesis for curves over finite fields. Hybrid bounds for Dirichlet L–functions were obtained in [HB1] by Heath-Brown, and for quadratic characters, in [Co-Iw] Conrey and Iwaniec showed

$$L(s,\chi) \ll_\varepsilon q^{\frac{1}{6}+\varepsilon} \qquad \text{(fixed } s \text{ with } \Re(s)=\tfrac{1}{2}\text{, for all } \varepsilon>0\text{)}$$

Analogous subconvex estimates for Dedeking zeta-functions and Hecke L–functions of of number fields were established in [HB2], [Ka], and [Sö].

For degree two, t–aspect subconvexity bounds were obtained by Good for holomorphic modular forms in [Go1], and by Meurman for waveforms in [Me]. Duke, Friedlander, and Iwaniec treated subconvexity in the other aspects in several papers, including [DFI1]-[DFI5]. See also Blomer, Harcos, and Michel in [BHM]. For applications of subconvexity to equidistribution and other problems, see [Iw-Sa], [Mi-Ve]., and [Sa3]. For higher-degree L–functions over $\mathbb{Q}$, there are fewer results. Among these are Sarnak [Sa4], Kowalski, Michel, and Vanderkam [KMV], Michel [Mi1] and Michel-Harcos [Ha-Mi], Bernstein-Reznikov [Be-Re], and Lau-Liu-Ye [LLY]. Interesting recent subconvexity results for Gelbart-Jacquet lifts from GL_2 to GL_3 appear in [Li].

For GL_2 over a number field, see [Pe-Sa] and [CPSS]. The latter gives an application to Hilbert's eleventh problem on representability by ternary quadratic forms. A subconvex bound in t–aspect was also needed in [Truelsen] to establish the quantum unique ergodicity conjecture for Eisenstein series over totally real number fields.

In this paper we establish a subconvex bound in the t–aspect for the standard L–function attached to a cuspform f on GL_2 over an arbitrary number field. The approach relies upon results of our previous paper [Di-Ga], with special choice of data entering into the Poincaré series, essentially the choice made by Good in [Go2], and the technique discussed in [Di-Go1] and [Di-Go2]. As usual, we also need a spectral gap, for which we cite [Ki-Sh]. These ideas, together with standard methods from analytic number theory, yield the subconvexity result, our Main Theorem, in Section 4:

Main Theorem. Fix a number field k of degree d over $\mathbb{Q}$, and a cuspform f for $GL_2(k)$. Assume that f is a newform locally everywhere. For a computable constant $\vartheta<1$,

$$L(\tfrac{1}{2}+it,f) \ll_\varepsilon (1+|t|)^{\frac{d-1+\vartheta}{2}+\varepsilon} \qquad \text{(for all } \varepsilon>0\text{)}$$

Recall that [Ca] establishes an adelic form of Atkin-Lehner theory, assuring that we can suppose that f is a newform locally everywhere. That is, the finite-prime part of the Hecke zeta integral of f

$$\text{(gamma factor)}\cdot L(s,f) = \int_{\mathbb{J}/k^\times} |y|^{s-\frac{1}{2}}\, f\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\, d^\times y$$

is equal to the finite-prime L–function $L(s,\pi)$ attached to the automorphic representation π generated by f. Knowing this optimal choice of vector f is fortunate; by contrast, in more complicated scenarios, we do not know an optimal choice.

Note that several prior attempts to prove such a subconvex bound, or an asymptotic formula for the corresponding integral moment, over number fields failed because more naive averages were considered; the family of twists by $|\ |^{it}$ is not *spectrally complete*, in a sense made clear in this paper. See also [Sa1] and for the general situation [DGG2]. One must consider the larger family that includes twists by all unramified grossencharacters (for t-aspect subconvexity).

Remarks:

• A subconvexity result follows even with weaker spectral gap results than [Ki-Sh]: a Selberg-Gelbart-Jacquet-type result (refering to the gap following from Rankin-Selberg considerations) suffices. The result of [Ki-Sh] gives the best meromorphic continuation of the Poincaré series we can assert, namely up to $\frac{1}{2}+\frac{1}{9}=\frac{11}{18}$. However, a subconvexity result follows from *any* meromorphic continuation to $\frac{1}{2}+a$ with effective $a<\frac{1}{2}$.

• This method should yield a subconvex bound for $L(\frac{1}{2}, f\otimes\chi)$ in the conductor of χ; this requires use of the *amplification* method, which in our case amounts to modification of the Poincaré series with *ramified* data at many finite places, both at places dividing and not dividing the conductor of χ. Recently, [Le] has used a similar method to obtain a subconvex bound in conductor-depth aspect.

• The subconvex bound in our main theorem occurs at one archimedean place. Uusing the ideas of the present, one should be able to obtain the meromorphic continuation of the $Z(w)$ (defined by (5.10) in [Di-Ga]) and its polynomial growth in a vertical strip $\frac{11}{18}+\delta<\Im(w)<1+\varepsilon$. This would give a uniform subconvex bound at all archimedean places. That discussion would entail substantial further complications, so we did not pursue it here.

Subconvexity results in the conductor aspect for GL_2 over number fields, and Rankin-Selberg convolutions and triple products for GL_2 over number fields, were also recently obtained in [Ve].

The structure of this paper is as follows. The first section recalls notation and facts from [Di-Ga]. The second and third sections establish the meromorphic continuation and polynomial growth in vertical strips of a generating function attached to the family of twists of a fixed GL_2 cuspform L–function by grossencharacters. We note that this should be viewed as a *spectral* family with respect to $GL_2\times GL_1$ with the GL_2 component fixed. One of the main difficulties in obtaining the subconvexity result is proof of the polynomial growth of the generating function, which is established in section 3. Section 4 is essentially an application of standard techniques to prove subconvexity from the results of sections 2 and 3. Concretely, as usual, we prove a mean value result with a power saving in the error term, from which an estimate on short intervals follows almost immediately. From this the subconvexity bound follows. We also include an appendix consisting of explicit computations and estimates concerning a particular special function appearing throughout the discussion.

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§1. Preliminaries

This section recalls some of the notation, context, and results of [Di-Ga]. Let k be a number field, $G=GL_2$ over k, and define standard subgroups

$$P=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \qquad N=\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \qquad H=\left\{\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}\right\} \qquad M=ZH=\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$$

Let K_ν denote the standard maximal compact in the k_ν–valued points G_ν of G. That is, at finite places $K_\nu=GL_2(\mathfrak{o}_\nu)$, at real places $K_\nu=O(2)$, and at complex places $K_\nu=U(2)$. The **Poincaré series** $\text{Pé}(g)$ discussed in [Di-Ga] is of the form

$$\text{Pé}(g) \quad = \sum_{\gamma\in Z_kH_k\backslash G_k} \varphi(\gamma g) \qquad\qquad (g\in G_{\mathbb{A}}) \tag{1.1}$$

for suitable functions φ on $G_{\mathbb{A}}$ described as follows. For $v \in \mathbb{C}$, let

$$\varphi = \bigotimes_{\nu} \varphi_\nu$$

where for ν *finite*

$$\varphi_\nu(g) = \begin{cases} \left|\frac{a}{d}\right|_\nu^v & \text{for } g = mk \text{ with } m = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu \text{ and } k \in K_\nu \\ 0 & \text{otherwise} \end{cases}$$

and for ν *archimedean* require right K_ν–invariance and left equivariance

$$\varphi_\nu(mg) = \left|\frac{a}{d}\right|_\nu^v \cdot \varphi_\nu(g) \qquad \left(\text{for } g \in G_\nu \text{ and } m = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu\right)$$

Thus, for $\nu|\infty$, the further data determining φ_ν consists of its values on N_ν. The simplest useful choice is

$$\varphi_\nu \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = (1 + |x|^2)^{-d_\nu w_\nu/2} \qquad (\text{with } w_\nu \in \mathbb{C}) \tag{1.2}$$

with $d_\nu = [k_\nu : \mathbb{R}]$. By the product formula, φ is left $Z_{\mathbb{A}} H_k$–invariant. It is critical that we can make another choice of this data, in Section 3.

The specific choice (1.2) of $\varphi_\infty = \otimes_{\nu|\infty} \varphi_\nu$ in the series (1.1) defining Pé(g) produces a Poincaré series converging absolutely and locally uniformly for $\Re(v) > 1$ and $\Re(w_\nu) > 1$ for all $\nu|\infty$. This is essentially a direct computation. For example, see Proposition 2.6 of [Di-Ga].

For $0 < \ell \in \mathbb{Z}$, let Ω_ℓ be the collection of φ_∞ such that the associated function

$$\Phi_\infty(x) = \varphi_\infty \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad (\text{for } x \in k_\infty)$$

is absolutely integrable on the unipotent radical N_∞, and such that the Fourier transform $\widehat{\Phi}_\infty$ along $N_\infty \approx k_\infty$ satisfies the bound

$$\widehat{\Phi}_\infty(x) \ll \prod_{\nu|\infty} (1 + |x|_\nu^2)^{-\ell/2}$$

For example, for φ_∞ to be in Ω_ℓ it *suffices* that Φ_∞ is ℓ times continuously differentiable, with each derivative absolutely integrable. For $\Re(w_\nu) > 1$, $\nu|\infty$, the simple explicit choice (1.2) of φ_∞ lies in Ω_ℓ for *every* $\ell > 0$. For convenience, a monomial vector φ as above will be called *ℓ–admissible*, if $\varphi_\infty \in \Omega_\ell$ when $\Re(v)$ is sufficiently large. When the data is ℓ–admissible for *all* large ℓ, we may say simply that the data is *admissible*. It is not hard to see that, after subtraction of a suitable Eisenstein series, the Poincaré series defined via admissible data is square-integrable on $Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}$. (See Proposition 2.7 of [Di-Ga].) In fact, $(1 + \varepsilon)$–admissibility for $\varepsilon > 0$ suffices.

Let f be a cuspform on $G_{\mathbb{A}}$. Require that f is a special vector locally everywhere in the representation it generates, in the following sense. Let

$$f(g) = \sum_{\xi \in H_k} W_f(\xi g)$$

be the Fourier expansion of f, and let

$$W_f \;=\; \bigotimes_{\nu\le\infty} W_{f,\nu}$$

be the factorization of the Whittaker function W_f into local data. We may require that for all $\nu<\infty$ the Hecke–type local integrals

$$\int\limits_{a\in k_\nu^\times} W_{f,\nu}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} |a|_\nu^{s-\frac{1}{2}}\, da$$

differ by at most an exponential function from the correct local L–factors for the representation generated by f. To be precise, by [Ca] we suppose throughout that f is a *newform* locally everywhere. That is, the finite-prime part of the Hecke zeta integral of f

$$(\text{gamma factor})\cdot L(s,f) \;=\; \int_{\mathbb{J}/k^\times} |y|^{s-\frac{1}{2}}\, f\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} d^\times y$$

is equal to the finite-prime L–function $L(s,\pi)$ attached to the automorphic representation π generated by f. Further, recall the immediate relation

$$L(\tfrac{1}{2}+it,f) \;=\; L(\tfrac{1}{2}+it,\pi) \;=\; L(\tfrac{1}{2},\pi\otimes|\;|^{it}) \;=\; L(\tfrac{1}{2},f\otimes|\;|^{it})$$

which we use throughout.

The integral

$$I(v,w) \;=\; \langle \text{Pé}, |f|^2\rangle \;=\; \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \text{Pé}(g)\,|f(g)|^2\, dg \tag{1.3}$$

can be evaluated in two ways. The first produces a *moment expansion* involving integral second moments of $L(s,f\otimes\chi)$ for grossencharacters χ unramified outside a fixed finite set S of places. The other evaluation uses the *spectral expansion* of Pé. We review this in some detail.

For φ a $(1+\varepsilon)$–admissible monomial vector as above, Theorem 3.12 of [Di-Ga] shows that $I(v,w)$ unwinds to a moment expansion

$$I(v,w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi i}\int\limits_{\Re(s)=\sigma} L(v+1-s,\bar f\otimes\overline{\chi})\cdot L(s,f\otimes\chi)\,\mathcal{K}_\infty(s,\,v,\,\chi)\, ds \tag{1.4}$$

where, for ν infinite and $s\in\mathbb{C}$,

$$\mathcal{K}_\infty(s,v,\,\chi) \;=\; \mathcal{K}_\infty(s,v,\,\chi,\varphi_\infty) \;=\; \prod_{\nu|\infty}\mathcal{K}_\nu(s,\,v,\,\chi_\nu)$$

and

$$\begin{aligned}\mathcal{K}_\nu(s,\,v,\,\chi_\nu) \;=\; &\int_{Z_\nu\backslash M_\nu N_\nu}\int_{Z_\nu\backslash M_\nu} \varphi_\nu(m_\nu n_\nu) W_{f,\nu}(m_\nu n_\nu)\\ &\cdot \overline{W}_{f,\nu}(m'_\nu n_\nu)\,\chi_\nu(m'_\nu)\,|m'_\nu|_\nu^{s-\frac{1}{2}}\,\chi_\nu(m_\nu)^{-1}\,|m_\nu|_\nu^{\frac{1}{2}-s}\, dm'_\nu\, dn_\nu\, dm_\nu\end{aligned}$$

Here $\chi = \bigotimes_\nu \chi_\nu \in \widehat{C}_0$, the unitary dual of the idele class group (of ideles of idele norm 1). For $(1+\varepsilon)$–*admissible* φ, the integral defining K_ν converges absolutely for $\Re(s)$ sufficiently large. When the archimedean data φ_∞ is the simple choice (1.2) or the variant (3.1), depending upon a complex parameter w, we will denote the above $\mathcal{K}_\nu(s,v,\chi_\nu)$ by $\mathcal{K}_\nu(s,v,w,\chi_\nu)$. Let S be a finite set of places including archimedean places, all absolutely ramified primes, and all finite bad places for f. The summands in the moment expansion are non-zero only for χ in a set $\widehat{C}_{0,S}$ of characters unramified outside S, with bounded ramification at finite places in S, depending only upon f.

On the other hand, we also gave the spectral decomposition of the Poincaré series, as follows. Let $\{F\}$ be an orthonormal basis of everywhere locally spherical cuspforms, and let $\mathfrak{d}$ be the idele with ν^{th} component $\mathfrak{d}_\nu$ at finite places ν and component 1 at archimedean places. Then the spectral decomposition is (see (4.13) in [Di-Ga])

(1.5)
$$\text{Pé} \;=\; \left(\int_{N_\infty}\varphi_\infty\right)\cdot E_{v+1} \;+\; \sum_F \left(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}_{F,\infty}\right)\cdot L(v+\tfrac12,\overline{F})\cdot F$$
$$+\sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa}\int\limits_{\Re(s)=\frac12}\left(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot W^E_{1-s,\overline{\chi},\infty}\right)\frac{L(v+1-s,\overline{\chi})\cdot L(v+s,\chi)}{L(2-2s,\overline{\chi}^2)}\,|\mathfrak{d}|^{-(v+s-1/2)}\cdot E_{s,\chi}\,ds$$

When the archimedean data is specialized to (1.2), by Theorem 4.17 in [Di-Ga], the Poincaré series $\text{Pé}(g)$ has meromorphic continuation in the variables v and w, to a region in $\mathbb{C}^2$ containing $v=0$ and $w=1$. And, for $v=0$, as a function of w, it is holomorphic in $\Re(w) > 11/18$ except for a pole at $w=1$ of order r_1+r_2+1.

In the following sections, with the simple choice (1.2) of archimedean data, and the variant (3.1), the ν^{th} local integral of $\varphi_\nu\cdot\overline{W}_\nu$ over $Z_\nu\backslash G_\nu$ in (1.5) will be denoted $\mathcal{G}_\nu(s;v,w)$, and $\mathcal{G}_\infty$ will be the product of the $\mathcal{G}_\nu$ over all archimedean places.

Fix an archimedean place ν_0, and take $1<\alpha\le 2$, to be specified later. If $r_1\ge 1$, we shall assume for convenience that ν_0 is real. This convenient assumption is nevertheless inessential, and we will point out in Section 3 the necessary adaptation when the ground field k is totally complex.

For a character $\chi\in\widehat{C}_0$, and $t\in\mathbb{R}$, $w\in\mathbb{C}$, put

$$\kappa_\chi(t,w) \;=\; \kappa_\chi(t,w,\alpha) \;=\; \big(1+|t+t_{\nu_0}|\big)^{-w}\prod_{\substack{\nu|\infty\\ \nu\ne\nu_0}}\mathcal{K}_\nu(\tfrac12+it,\,0,\,\alpha,\,\chi_\nu)\quad(r_1\ge 1)\tag{1.6}$$

where $\mathcal{K}_\nu(s,v,w,\chi_\nu)$ is as above, corresponding to the choice (1.2) for φ_ν. If $r_1=0$, we define $\kappa_\chi(t,w)$ by replacing $1+|t+t_{\nu_0}|$ in (1.6) with $1+\ell^2_{\nu_0}+4(t+t_{\nu_0})^2$. Here it_ν and ℓ_ν are the parameters of the local component χ_ν of χ. Note that, since χ is trivial on the positive reals, there is a relation among the local parameters, namely,

$$\sum_{\nu|\infty} d_\nu\, t_\nu = 0$$

with $d_\nu = [k_\nu:\mathbb{R}]$ the local degree.

Define

$$Z(w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}}\int\limits_{-\infty}^{\infty}|L(\tfrac12+it,\,f\otimes\chi)|^2\,\kappa_\chi(t,w)\,dt\tag{1.7}$$

By the argument given in [Di-Ga], Section 5, pages 23 and 24, the right-hand side of (1.7) is absolutely convergent for $\Re(w) > 1$. The main objective of this paper is to meromorphically continue this generating function $Z(w)$, with polynomial vertical growth. From this, a subconvexity bound for individual L–functions will follow by essentially traditional techniques.

§2. Meromorphic continuation of $Z(w)$

In this section we establish the following

Theorem 2.1. *The function $Z(w)$, originally defined by (1.7) for $\Re(w) > 1$, has analytic continuation to the half-plane $\Re(w) > 11/18$, except for $w = 1$ where it has a pole of order two.*

Proof: Write $w = \delta + i\eta$. For $C > 0$, possibly depending upon w, let

$$Z_1(w) \;=\; Z_1(w, C) \;=\; \sum_{\substack{\chi\in\widehat{C}_{0,S} \\ |t+t_{\nu_0}|\ll C}} \int |L(\tfrac{1}{2}+it,\, f\otimes\chi)|^2\, \kappa_\chi(t,w)\, dt \tag{2.2}$$

and set $Z_2(w) = Z(w) - Z_1(w)$. We shall be more specific on the choice of C later in the proof. Using (5.11) in [Di-Ga] together with its analog at real places (see also the Appendix), we have

$$\sum_{\substack{\chi\in\widehat{C}_{0,S} \\ |t+t_{\nu_0}|\ll C}} \int |L(\tfrac{1}{2}+it,\, f\otimes\chi)|^2\, |\kappa_\chi(t,w)|\, dt \le \sum_{\substack{\chi\in\widehat{C}_{0,S} \\ |t+t_{\nu_0}|\ll C}} \int |L(\tfrac{1}{2}+it,\, f\otimes\chi)|^2\, \kappa_\chi(t,\delta)\, dt$$

Since

$$\big(1+|t+t_{\nu_0}|\big)^{-\delta} \ll_{\alpha,\,C} \big(1+|t+t_{\nu_0}|\big)^{-\alpha} \qquad (\delta>0)$$

it readily follows that $Z_1(w)$ is holomorphic in the half-plane $\delta > 0$.

To obtain the continuation of the remaining part $Z_2(w)$, consider

$$I(v,w,\alpha) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \frac{1}{2\pi i} \int\limits_{\Re(s)=\sigma} L(\chi^{-1}|\cdot|^{v+1-s},\, \bar{f})\cdot L(\chi|\cdot|^{s},\, f)\, \mathcal{K}_\infty(s,\, v,\, w,\, \alpha,\, \chi)\, ds \tag{2.3}$$

where

$$\mathcal{K}_\infty(s,\, v,\, w,\, \alpha,\, \chi) \;=\; \mathcal{K}_{\nu_0}(s,\, v,\, w,\, \chi_{\nu_0}) \;\cdot\; \prod_{\substack{\nu|\infty \\ \nu\neq\nu_0}} \mathcal{K}_\nu(s,\, v,\, \alpha,\, \chi_\nu)$$

By Proposition 2.6 and Theorem 3.12 in [Di-Ga],

$$I(v,w,\alpha) = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \text{Pé}(g)\, |f(g)|^2\, dg \tag{2.4}$$

provided $\Re(v), \Re(w) > 1$. The choice of initial archimedean data determining the Poincaré series is given by (1.2) with $w_\nu = w$ or α according as $\nu = \nu_0$ or not. On the other hand, we have also established in that paper the analytic continuation of this Poincaré series (see Theorem 4.17 in

[Di-Ga]). These facts were reviewed in Section 1. In the present situation, the only change in the statement of that theorem is that the order of the pole at $w = 1$ of $\mathrm{Pé}(g)$, when $v = 0$, is two instead of $r_1 + r_2 + 1$. It follows that $I(0, w, \alpha)$ is analytic in the half-plane $\Re(w) > 11/18$, except for $w = 1$ where it has a pole of order two. Note that for $\Re(w) = \delta > 1$, we can express

$$
\begin{aligned}
I(0, w, \alpha) \;&=\; \sum_{\chi \in \widehat{C}_{0,S}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \, \mathcal{K}_\infty(\tfrac{1}{2} + it,\, 0,\, w,\, \alpha,\, \chi)\, dt \\
&= I_1(0, w, \alpha) + I_2(0, w, \alpha)
\end{aligned}
\tag{2.5}
$$

where

$$
I_1(0, w, \alpha) \;=\; \frac{1}{2\pi} \sum_{\substack{\chi \in \widehat{C}_{0,S} \\ |t + t_{\nu_0}| \ll C}} \int |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \, \mathcal{K}_\infty(\tfrac{1}{2} + it,\, 0,\, w,\, \alpha,\, \chi)\, dt
$$

We choose $C = C(\eta)$ such that (A.6) or (A.7) (depending on whether ν_0 is complex or real) becomes effective. We remark that $C(\eta)$ is uniform in δ on closed intervals to the right of 11/18, and depends polynomially on $|\eta|$ (see [Di-Go1], the corresponding analysis in [Di-Go2], and the appendix at the end of this paper).

Using (A.6) or (A.7), we can write

$$
I_2(0, w, \alpha) \;=\; (2\pi)^{-1} B(0, w, \mu_{f,\,\nu_0})\, [Z(w) \,-\, Z_1(w) \,+\, Z_3(w)]
$$

with $Z_3(w)$ holomorphic in the half-plane $\delta > 11/18$.

To finish the proof, we show that the series defining $I_1(0, w, \alpha)$ for $\Re(w) > 1$ is in fact absolutely convergent for $\Re(w) > 11/18$. To see this, refer to the Appendix, especially (A.4) and (A.5). We have

$$
\begin{aligned}
I_1(0, w, \alpha) \;&=\; \frac{1}{2\pi} \sum_{\substack{\chi \in \widehat{C}_{0,S} \\ |t + t_{\nu_0}| \ll C}} \int |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \, \mathcal{K}_\infty(\tfrac{1}{2} + it,\, 0,\, w,\, \alpha,\, \chi)\, dt \\
&\ll \sum_{\substack{\chi \in \widehat{C}_{0,S} \\ |t + t_{\nu_0}| \ll C}} \int |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \, \mathcal{K}_\infty(\tfrac{1}{2} + it,\, 0,\, \delta,\, \alpha,\, \chi)\, dt \\
&\ll \sum_{\substack{\chi \in \widehat{C}_{0,S} \\ |t + t_{\nu_0}| \ll C}} \int |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \, \kappa_\chi(t, \alpha)\, dt
\end{aligned}
$$

The first inequality comes from the structure of the integral representation (A.4), and the second comes from the domination (A.8) of the local factors of the kernel by the local factors of the analytic conductor. Notice that the integral representation (A.4) holds for $\Re(w) > 2/9$. The convergence of the last expression is clear. □

§3. Growth of $Z(w)$

Our objective is now to establish that $Z(w)$ defined in the previous section has polynomial growth in a vertical strip $\frac{11}{18}+\varepsilon \le \Re(w) \le 1+\varepsilon$, with ε a small positive number. As we shall see, an important point is that the exponent of $|\Im(w)|$ in our bound is *independent* of the fixed parameter α.

We begin by specializing the Poincaré series as follows. At the archimedean place ν_0, take

$$\varphi_{\nu_0}(n) \;=\; 2^{1-w}\sqrt{\pi}\,\frac{\Gamma(w)(1+x^2)^{-\frac{w}{2}}F(\frac{w}{2},\frac{w}{2};\frac{1}{2}+w;\frac{1}{1+x^2})}{\Gamma(w+\frac{1}{2})} \qquad \left(n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in N_{\nu_0}\right) \tag{3.1}$$

where, as usual,

$$F(\alpha,\beta;\gamma;z) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}\,\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)}\,z^m \qquad (|z|<1)$$

is the Gauss hypergeometric function. This corresponds precisely to the choice made by Good in [Go2] (see also [Di-Go1]), just presented in a form more convenient for us. Recall that we are assuming $\nu_0 \approx \mathbb{R}$. At all the other archimedean places, we shall still keep the choice made in the previous section (i.e., given by (1.2) with $w_\nu=\alpha$).

Notice that with the parameter choices in the hypergeometric function in (3.1), the series representation is absolutely convergent for every real x. An easy estimate proves sufficient *admissibility* of this data in the sense defined above. In fact, the series representation of the hypergeometric function allows the following analysis to be reduced to the case of the simple choice of data (1.2) considered in the previous section. The convenient assumption that there is at least one real place is inessential, since, lacking any real place, for complex ν_0 we can take

$$\varphi_{\nu_0}(n) \;=\; 2^{1-2w}\sqrt{\pi}\,\frac{\Gamma(w)(1+|x|^2)^{-w}\,F(w,w;2w;\frac{1}{1+|x|^2})}{\Gamma(w-\frac{1}{2})} \qquad \left(n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in N_{\nu_0},\, x\neq 0\right)$$

and

$$\varphi_{\nu_0}(n)=0 \qquad \left(\text{if } n=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

This choice corresponds to (3.1) at complex places, but we do not need this unless the field is totally complex.

An overview. Before justifying our statements and giving details of proofs, we find it useful to first present the main ideas in our argument. Accordingly, if we denote the corresponding integral (2.4) by $J(v,w,\alpha)$, it follows again by Proposition 2.6 and Theorem 3.12 in [Di-Ga] that for $\Re(v),\,\Re(w)>1$,

$$J(v,w,\alpha) \;=\; \sum_{\chi\in\widehat{C}_{0,S}}\frac{1}{2\pi i}\int\limits_{\Re(s)=\sigma} L(\chi^{-1}|\cdot|^{v+1-s},\,\bar{f})\cdot L(\chi|\cdot|^{s},\,f)\,\mathcal{K}_\infty(s,\,v,\,w,\,\alpha,\,\chi)\,ds$$

where

$$\mathcal{K}_\infty(s,\,v,\,w,\,\alpha,\,\chi) \;=\; \mathcal{K}_{\nu_0}(s,\,v,\,w,\,\chi_{\nu_0})\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\mathcal{K}_\nu(s,\,v,\,\alpha,\,\chi_\nu)$$

The local kernel $\mathcal{K}_{\nu_0}$ corresponding to archimedean data (3.1) can be continued by expanding the hypergeometric function in a series and reducing the computation to the case (1.2). Furthermore, for $|t + t_{\nu_0}| \to \infty$, it has an asymptotic expansion of type (A.7), with B replaced by a different ratio of gamma functions $B_0(v, w, \mu_{f,\nu_0})$. Taking $v = 0$ and $\sigma = 1/2$ in the above expression for $J(v, w, \alpha)$, one can easily check that the sum over χ together with the vertical integral is absolutely convergent for $\Re(w) = \delta > 1$. Arguing as in the proof of Theorem 2.1 using the asymptotic formula of $\mathcal{K}_{\nu_0}$, we can split

$$J = J_1 + J_2 \qquad J_2(0, w, \alpha) = (2\pi)^{-1} B_0(0, w, \mu_{f,\nu_0}) \left[Z(w) - Z_1(w) + Z_3(w) \right]$$

with both $Z_1(w)$ and $Z_3(w)$ holomorphic in the half-plane $\Re(w) > 11/18$. Since the constant $C = C(\eta)$, where the split of $J(0, w, \alpha)$ was made, depends polynomially on w, it also follows that Z_1 and Z_3 have both polynomial growth in w (obviously with exponents independent of α) for $\Re(w) = \delta > 11/18$. It turns out that the function $B_0(0, w, \mu_{f,\nu_0}) \cos(\pi w/2)$ has polynomial growth, and therefore it suffices to show that $\cos(\pi w/2) J_2(0, w, \alpha)$ has polynomial growth in a vertical strip $\frac{11}{18} + \varepsilon \le \delta \le 1 + \varepsilon$. The precise reason for the choice of $\cos(\pi w/2)$ will become apparent in the next paragraph.

So far, we did not really use the specific choice (3.1) of φ_{ν_0}. In fact, what we have said up to this point equally holds for a variety of choices, including that made in the previous section. As observed by Good in [Go2] over the rationals, the choice (3.1) induces some sort of functional equation relating the values of the corresponding function $\cos(\pi w/2) J(0, w)$ (Since $\mathbb{Q}$ has only one archimedean place, there is no α.) as $w \to 1 - w$. Using a spectral decomposition, we shall establish a similar *relation* for $\cos(\pi w/2) J(0, w, \alpha)$. We call it *relation,* since this function has way too many poles in the vertical strip $0 \le \Re(w) \le 11/18$ (which most probably form dense subsets of the vertical lines $\Re(w) = 1/4$ and $\Re(w) = 1/2$) preventing a *genuine* meromorphic continuation. Nevertheless, following [Di-Go1], we shall consider a similar auxiliary function $J^{\mathrm{aux}}(w, \alpha)$ such that $\cos(\pi w/2) J^{\mathrm{aux}}(w, \alpha)$ has polynomial growth on the vertical lines $\Re(w) = -\varepsilon,\ 1 + \varepsilon$, and such that when subtracted from $J(0, w, \alpha)$ leaves a meromorphic function in the whole vertical strip $-\varepsilon \le \Re(w) \le 1 + \varepsilon$ with finitely many poles. In this strip we can now apply the Phragmen-Lindelöf principle. To obtain the desired growth of $Z(w)$, one just has to observe that $\cos(\pi w/2) J^{\mathrm{aux}}(w, \alpha)$ has, in fact, polynomial growth in every vertical strip of finite width distant from the abundance of poles, and hence in $\frac{11}{18} + \varepsilon \le \Re(w) \le 1 + \varepsilon$.

To be more precise, the Poincaré series built out of the archimedean data specified at the beginning of this section has the spectral decomposition (1.5) (see (4.13) in [Di-Ga]). It follows that $\text{Pé}(g)$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing $v = 0$, $w = 1$. As before, when $v = 0$, it is holomorphic in the half-plane $\Re(w) > 11/18$, except for $w = 1$ where it has a pole of order two. Then with respect to an orthonormal basis $\{F\}$ of $L^2_{\mathrm{cusp}}(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$ of everywhere locally spherical cuspforms, we can write

(3.2)
$$J(w) = J(0, w, \alpha) = M(w, f) + \sum_F \bar{\rho}_F \mathcal{G}_{F_\infty}(w) L(\tfrac{1}{2}, \overline{F}) \langle F, |f|^2 \rangle$$
$$+ \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} \int\limits_{\Re(s) = \frac{1}{2}} \frac{\mathcal{G}_{\chi_\infty}(s, w) L(1 - s, \overline{\chi}) L(s, \chi)}{\Lambda(2 - 2s, \overline{\chi}^2)} \, |\mathfrak{d}|^{-(s - 1/2)} \langle E_{s,\chi}, |f|^2 \rangle \, ds$$

Here

$$\mathcal{G}_{F_\infty}(w) \;=\; \mathcal{G}_{F_\infty}(0,w) \;=\; \mathcal{G}_{\nu_0}(\tfrac{1}{2}+i\bar{\mu}_{F,\nu_0};0,w)\,\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\mathcal{G}_\nu(\tfrac{1}{2}+i\bar{\mu}_{F,\nu};0,\alpha)$$

$$\mathcal{G}_{\chi_\infty}(s,w) \;=\; \mathcal{G}_{\nu_0}(1-s-it_{\nu_0};0,w)\,\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\mathcal{G}_\nu(1-s-it_\nu;0,\alpha)$$

$$M(w,f) \;=\; \lim_{v\to 0}\left(R(w)\langle E_{v+1},|f|^2\rangle \;+\; \frac{\mathcal{G}_1(1-v,v,w)}{\zeta_\infty(2v)}\langle E_{1-v},|f|^2\rangle\right)$$

with

$$R(w)=R(w,\alpha)=\int_{N_\infty}\varphi_\infty \quad\text{and}\quad \mathcal{G}_1(1-v,v,w) \;=\; \mathcal{G}_{\nu_0}(v;v,w)\,\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\mathcal{G}_\nu(v;v,\alpha)$$

We suppressed notation to $J(w)$, as the parameter α plays no role at the moment, and $v=0$ for the remaining part.

Our immediate goal is to establish a *relation* connecting values of the function $\cos(\pi w/2)J(w)$ as $w\to 1-w$. We begin with the following

Lemma 3.3. *With φ_{ν_0} defined by (3.1) and $\Re(w)>1$, we have*

$$\int_{N_{\nu_0}}\varphi_{\nu_0} \;=\; \sqrt{\pi}\,\frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2}+1)}$$

and

$$\mathcal{G}_{\nu_0}(s;0,w) \;=\; 4\,\frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})\Gamma(\frac{w+1}{2})}{(w-s)(w+s-1)\Gamma(\frac{w}{2})}\qquad (\textit{for}\;\; 0<\Re(s)<1)$$

Proof: To compute the first integral, write the hypergeometric function in (3.1) by its series representation. Since this series is absolutely convergent for all $x\in\mathbb{R}$, we can interchange the sum and integral. Using the well-known identity

$$\int_{-\infty}^{\infty}\left(1+x^2\right)^{-\frac{w}{2}-m}dx=\sqrt{\pi}\,\frac{\Gamma(\frac{w-1}{2}+m)}{\Gamma(\frac{w}{2}+m)}\qquad (\text{for}\;\; m=0,1,2,3,\dots)$$

we recognize the hypergeometric function $F(w/2,(w-1)/2;1/2+w;z)$ evaluated at $z=1$. Applying Gauss' formula

$$F(\alpha,\beta;\gamma;1) \;=\; \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\qquad (\text{when}\;\; \Re(\gamma)>\Re(\alpha+\beta))$$

the identity follows after canceling some of the gamma functions involved.

The second identity can be verified in the same way, using formula (A.10) in the Appendix after interchanging the sum and integral. $\square$

Remark: More generally, for $\Re(w) > 1$ and $-\Re(v) < \Re(s) < 1 + \Re(v)$, the same idea can be used to compute $\mathcal{G}_{\nu_0}(s; v, w)$ as

$$\pi^{-v}\,\frac{\Gamma(\frac{w+1}{2})\Gamma(\frac{v+1-s}{2})\Gamma(\frac{v+s}{2})\Gamma(\frac{v+w-s}{2})\Gamma(\frac{v+w+s-1}{2})\,{}_3F_2(\frac{w}{2},\frac{v+w-s}{2},\frac{v+w+s-1}{2}; w+\frac{1}{2}, v+\frac{w}{2}; 1)}{\Gamma(\frac{1}{2}+w)\Gamma(v+\frac{w}{2})}$$

where ${}_3F_2$ is a generalized hypergeometric function, see [Gr-Ry], page 1071. One can easily verify that the series defining ${}_3F_2$ above is absolutely convergent away from its poles.

From the lemma, we deduce that $\cos(\pi w/2)R(w)$, $\cos(\pi w/2)\mathcal{G}_{F_\infty}(w)$ and $\cos(\pi w/2)\mathcal{G}_{\chi_\infty}(s, w)$ are all invariant under $w \to 1 - w$. To establish that $M(w, f)$ appearing in the spectral identity of $J(w)$ satisfies the same invariance, first recall that

$$M(w, f) \;=\; \lim_{v\to 0}\Big(R(w)\langle E_{v+1}, |f|^2\rangle \;+\; \frac{\mathcal{G}_1(1-v, v, w)}{\zeta_\infty(2v)}\langle E_{1-v}, |f|^2\rangle\Big)$$

with

$$R(w) = \int_{N_\infty}\varphi_\infty = 2^{r_2}\,\pi^{\frac{d}{2}}\,\frac{\Gamma(\frac{w-1}{2})\Gamma(\frac{\alpha-1}{2})^{r_1-1}}{\Gamma(\frac{w}{2}+1)\Gamma(\frac{\alpha}{2})^{r_1-1}}\cdot(\alpha-1)^{-r_2}$$

in view of the lemma. We can compute the limit by writing the Laurent expansions around $v = 0$ of the functions involved:

$$E_{v+1} = \frac{c}{v} + c_0 + \cdots$$

and

$$\begin{aligned}\frac{\mathcal{G}_1(1-v, v, w)}{\zeta_\infty(2v)}\,E_{1-v} &= \frac{\mathcal{G}_{\nu_0}(v; v, w)}{\pi^{-v}\Gamma(v)}\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\frac{\mathcal{G}_\nu(v; v, \alpha)}{\zeta_\nu(2v)}E_{1-v}\\ &= (b_0(w)+b_1(w)v+\cdots)(d_0(\alpha)+d_1(\alpha)v+\cdots)\Big(-\frac{c}{v}+c_0+\cdots\Big)\end{aligned}$$

One can easily find that

$$b_0(w) = \sqrt{\pi}\,\frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2}+1)} \;\text{ and }\; d_0(\alpha) = \Big(\sqrt{\pi}\frac{\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{\alpha}{2})}\Big)^{r_1-1}\cdot\;[2\pi/(\alpha-1)]^{r_2}$$

Also, by considering the function (A.10) with $s = v$ and $w \to w + 2m$, expanding it in a power series around $v = 0$, and then proceeding as in the lemma, we find that

$$b_1(w) = \pi\,2^{-w+2}\cdot\frac{\Gamma(w-1)}{(1-w)\Gamma(1+\frac{w}{2})^2} = \frac{2b_0(w)}{w(1-w)}$$

Since $R(w) = b_0(w)d_0(\alpha)$, we obtain

$$\lim_{v\to 0}\Big(R(w)E_{v+1} \;+\; \frac{\mathcal{G}_1(1-v, v, w)}{\zeta_\infty(2v)}\,E_{1-v}\Big) \;=\; 2c_0R(w) \;-\; c\cdot(b_0(w)d_1(\alpha)+b_1(w)d_0(\alpha))$$

It follows now easily that $\cos(\pi w/2)M(w, f)$ is invariant as $w \to 1 - w$.

Following our plan, consider the auxiliary function $J^{\rm aux}(w)$ defined by

(3.4)
$$J^{\rm aux}(w) = \sum_F \bar\rho_F\, \mathcal{G}^{\rm aux}_{F_\infty}(w) L(\tfrac12, \overline{F})\langle F, |f|^2\rangle + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa} \int\limits_{\Re(s)=\frac12} \frac{\mathcal{G}^{\rm aux}_{\chi_\infty}(s,w)L(1-s,\overline{\chi})L(s,\chi)}{\Lambda(2-2s,\overline{\chi}^2)} |\mathfrak{d}|^{-(s-1/2)}\langle E_{s,\chi}, |f|^2\rangle\, ds$$

where $\mathcal{G}^{\rm aux}_{F_\infty}$ and $\mathcal{G}^{\rm aux}_{\chi_\infty}$ are obtained by just replacing $\mathcal{G}_{\nu_0}$ in the products defining $\mathcal{G}_{F_\infty}$ and $\mathcal{G}_{\chi_\infty}$ above by the function

$$\mathcal{G}^{\rm aux}_{\nu_0}(s;w) := \pi\, 8^{-w+1}\cdot \frac{\Gamma(2w-1)}{\Gamma(\frac{w}{2})\Gamma(w+\frac12)}\cdot\left[\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{w-s}{2}\Big) + \Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{w+s-1}{2}\Big)\right] \tag{3.5}$$

For instance,
$$\mathcal{G}^{\rm aux}_{F_\infty}(w) = \mathcal{G}^{\rm aux}_{\nu_0}(\tfrac12+i\bar\mu_{F,\nu_0};w)\cdot \prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}} \mathcal{G}_\nu(\tfrac12+i\bar\mu_{F,\nu};0,\alpha)$$

With this notation, we establish the following propositions:

Proposition 3.6. *For $\varepsilon>0$ sufficiently small, the difference*

$$H(w) := J(w) - J^{\rm aux}(w) = M(w,f) + \sum_F \bar\rho_F(\mathcal{G}_{F_\infty}(w)-\mathcal{G}^{\rm aux}_{F_\infty}(w))L(\tfrac12,\overline{F})\langle F,|f|^2\rangle + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa}\int\limits_{\Re(s)=\frac12} \frac{(\mathcal{G}_{\chi_\infty}(s,w)-\mathcal{G}^{\rm aux}_{\chi_\infty}(s,w))L(1-s,\overline{\chi})L(s,\chi)}{\Lambda(2-2s,\overline{\chi}^2)}|\mathfrak{d}|^{-(s-1/2)}\langle E_{s,\chi},|f|^2\rangle\, ds$$

restricted to $11/18<\Re(w)\le 1+\varepsilon$, extends holomorphically to the whole vertical strip $-\varepsilon\le\Re(w)\le 1+\varepsilon$, except for at most $w=0,1/2,1$, where it may have poles. For $w\neq 0,1/2,1$ in this strip, the expression in the right is absolutely convergent.

Proof: By the above computation, $M(w,f)$ is clearly holomorphic in the strip $-\varepsilon\le\Re(w)\le 1+\varepsilon$, except for $w=0$, 1 where it has poles. Note that $w=1$ is a double pole.

To analyze the remaining part, remark that, by design, the function $\mathcal{G}^{\rm aux}_{\nu_0}$ cancels the relevant poles of $\mathcal{G}_{\nu_0}$ in the region we are interested in. Specifically, using the expression of $\mathcal{G}_{\nu_0}(s;0,w)$ in Lemma 3.3, it follows that
$$\mathcal{G}_{\nu_0}(s;0,w) - \mathcal{G}^{\rm aux}_{\nu_0}(s;w)$$
is holomorphic as a function of w around $w=s$, $w=1-s$, for $s\neq 1/2$, and as a function of s around $s=w$, $s=1-w$, for $w\neq 1/2$. Applying this with $s=\frac12+i\bar\mu_{F,\nu_0}$, it follows that

$$\mathcal{G}_{F_\infty}(w) - \mathcal{G}^{\rm aux}_{F_\infty}(w) = [\mathcal{G}_{\nu_0}(\tfrac12+i\bar\mu_{F,\nu_0};0,w) - \mathcal{G}^{\rm aux}_{\nu_0}(\tfrac12+i\bar\mu_{F,\nu_0};w)]\cdot\prod_{\substack{\nu|\infty\\ \nu\neq\nu_0}}\mathcal{G}_\nu(\tfrac12+i\bar\mu_{F,\nu};0,\alpha)$$

is a holomorphic function for $-\varepsilon \le \Re(w) \le 1+\varepsilon$, except for $w = 1/2$ where it has a pole. Recall that we can assume $|\Re(i\mu_{F,\nu_0})| < 1/9$ (see [Ki], [Ki-Sh]).

The absolute convergence of the discrete part follows by combining the exponential decay of

$$\bar{\rho}_F(\mathcal{G}_{F_\infty}(w) - \mathcal{G}^{\mathrm{aux}}_{F_\infty}(w))L(\tfrac{1}{2}, \overline{F})\langle F, |f|^2\rangle$$

in the archimedean parameters of F with the fact that the number of cuspforms with archimedean data within a given bound grows polynomially, from Weyl's Law [La-Ve], or from the upper-bound of [Do]. In fact, shortly we shall need that $\bar{\rho}_F\langle F, |f|^2\rangle$ grows at worst polynomially in the archimedean data of F (and by the convexity bound, so does $\bar{\rho}_F L(\frac{1}{2}, \overline{F})\langle F, |f|^2\rangle$). This important observation first made by Selberg in [Se], was proved by Sarnak in [Sa2].

On the continuous spectrum-part, the variable w can cross the vertical line of integration $\Re(s) = 1/2$, for each χ, since by the above observation, the function

$$\mathcal{G}_{\chi_\infty}(s,w) - \mathcal{G}^{\mathrm{aux}}_{\chi_\infty}(s,w) = [\mathcal{G}_{\nu_0}(1-s-it_{\nu_0}; 0, w) - \mathcal{G}^{\mathrm{aux}}_{\nu_0}(1-s-it_{\nu_0}; w)] \cdot \prod_{\substack{\nu|\infty \\ \nu\neq\nu_0}} \mathcal{G}_\nu(1-s-it_\nu; 0, \alpha)$$

is holomorphic around $s = 1-w-it_{\nu_0}$, $w - it_{\nu_0}$. Also, this function has no poles other than $w = 1/2$ in the vertical strip $-\varepsilon \le \Re(w) \le 1+\varepsilon$ when $\Re(s) = 1/2$.

To justify the absolute convergence of the sum and integral, we note that by standard estimates

$$\frac{L(1-s,\overline{\chi})L(s,\chi)}{\Lambda(2-2s,\overline{\chi}^2)} \cdot \langle E_{s,\chi}, |f|^2\rangle$$

has at worst polynomial growth in $|\Im(s)|$ and the archimedean parameters of χ; the exponential decay of the twisted Rankin-Selberg convolution $\langle E_{s,\chi}, |f|^2\rangle$ matches the exponential decay of the gamma factors in $\Lambda(2-2s,\overline{\chi}^2)$. Since $\mathcal{G}_{\chi_\infty}(s,w) - \mathcal{G}^{\mathrm{aux}}_{\chi_\infty}(s,w)$ has exponential decay in $|\Im(s)|$ and the archimedean parameters of χ, the sum and integral are absolutely convergent for $-\varepsilon \le \Re(w) \le 1+\varepsilon$, $w \neq 1/2$. $\square$

Proposition 3.7. *Fix a small positive ε, and write $w = \delta + i\eta$. For $11/18 + \varepsilon \le \delta \le 1+\varepsilon$, or $\delta = -\varepsilon$, and large $|\eta|$, we have the estimate*

$$\cos\frac{\pi w}{2} \cdot \left(\sum_F \bar{\rho}_F \mathcal{G}^{\mathrm{aux}}_{F_\infty}(w) L(\tfrac{1}{2}, \overline{F})\langle F, |f|^2\rangle\right)$$
$$+ \cos\frac{\pi w}{2} \cdot \left(\sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa} \int\limits_{\Re(s)=\frac{1}{2}} \frac{\mathcal{G}^{\mathrm{aux}}_{\chi_\infty}(s,w)L(1-s,\overline{\chi})L(s,\chi)}{\Lambda(2-2s,\overline{\chi}^2)} |\mathfrak{d}|^{-(s-1/2)}\langle E_{s,\chi}, |f|^2\rangle\, ds\right) \ll_{\varepsilon,\alpha} |\eta|^{N_\delta}$$

with a computable $N_\delta > 0$ independent of α.

Proof: We begin by applying Sarnak's estimate [Sa2] on integrals of triple products of automorphic forms. That result, together with a convexity estimate, implies a polynomial bound for $\bar{\rho}_F L(\frac{1}{2}, \overline{F})\langle F, |f|^2\rangle$ in the archimedean data of F. The exponent in the polynomial bound as well as the implied constant is independent of F. An analogous fact occurs in the continuous part, where

$$\frac{L(1-s,\overline{\chi})L(s,\chi)}{\Lambda(2-2s,\overline{\chi}^2)}\langle E_{s,\chi}, |f|^2\rangle \qquad (s = \tfrac{1}{2} + it)$$

is bounded polynomially in $|t+t_\nu|$ with $\nu|\infty$. We remark that the continuous part estimate can also be obtained by traditional methods. Applying Stirling's asymptotic formula to the gamma functions present in $\mathcal{G}^{\rm aux}_{F_\infty}$ and $\mathcal{G}^{\rm aux}_{\chi_\infty}$, we observe that the relevant terms of the expression in the proposition are those corresponding to $|\mu_{F,\nu_0}|$, $|t+t_{\nu_0}| \le 2|\eta|$, say. The remaining part of the expression is negligible.

Recalling that

$$\mathcal{G}^{\rm aux}_{\nu_0}(s;w) \,=\, \pi\, 8^{-w+1}\cdot\frac{\Gamma(2w-1)}{\Gamma(\frac{w}{2})\Gamma(w+\frac12)}\cdot\left[\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{w-s}{2}\Big)\,+\,\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{w+s-1}{2}\Big)\right]$$

it follows from Stirling's asymptotic formula that

$$\mathcal{G}^{\rm aux}_{\nu_0}(\tfrac12+i\bar\mu_{F,\nu_0};w) \ll_\varepsilon |\eta|^{N'_\delta}e^{-\frac{\pi}{2}|\eta|} \qquad (\text{for } |\mu_{F,\nu_0}|\le 2|\eta|)$$

A similar estimate (with the same exponential decay) holds on the continuous side.

The proposition follows immediately from Weyl's Law [La-Ve] (the upper-bound of [Do] suffices). □

We can relate the functions J and Z by splitting, as in the previous section,

$$J(w) \,=\, J_1(w)\,+\,J_2(w) \qquad J_2(w) \,=\, (2\pi)^{-1}B_0(0,w,\mu_{f,\nu_0})\,[Z(w)\,-\,Z_1(w)\,+\,Z_3(w)] \tag{3.8}$$

with both $Z_1(w)$ and $Z_3(w)$ holomorphic in the half-plane $\Re(w)>11/18$. In fact, the local kernel $\mathcal{K}_{\nu_0}$ determined by (3.1) has an asymptotic expansion which can be easily deduced from (A.7) in the Appendix by representing the hypergeometric function in (3.1) by its series, and reducing to the previous choice of φ_{ν_0}. It follows that

$$B_0(0,w,\mu_{f,\nu_0}) \,=\, 2^{-w+1}\sqrt{\pi}\,\frac{\Gamma(w)}{\Gamma(\frac12+w)}\cdot B(0,w,\mu_{f,\nu_0}) \tag{3.9}$$

where

$$B(v,w,\mu) \,=\, 2^{w-2}\,\pi^{-v}\,\frac{\Gamma(\frac{w+v+i\mu+i\bar\mu}{2})\Gamma(\frac{w+v-i\mu+i\bar\mu}{2})\Gamma(\frac{w+v+i\mu-i\bar\mu}{2})\Gamma(\frac{w+v-i\mu-i\bar\mu}{2})}{\Gamma(w+v)}$$

See also the Appendix. For holomorphic discrete series, see (3.1), (4.1) and Proposition 4.2 in [Di-Go1]. Using Stirling's asymptotic formula, it can be observed that $B_0(0,w,\mu_{f,\nu_0})\cos(\pi w/2)$ is of polynomial growth. Furthermore, the constant $C=C(\eta)$, where the split of $J(w)$ occurs, depends polynomially on w (as explained in the Appendix). This fact was established for holomorphic discrete series in [Di-Go1], Proposition 4.2, and in general it follows by keeping track of all the constants depending on w in the process of obtaining the asymptotic formula for $\mathcal{K}_{\nu_0}$. Then by their definitions (see the previous section), Z_1 and Z_3 are both of polynomial growth in w, for $\Re(w)=\delta>11/18$, with exponents independent of α. Therefore, to achieve our goal, that is, the function $Z(w)$ is of polynomial growth, it suffices to show the same for $\cos(\pi w/2)J_2(w)$.

To prove that $\cos(\pi w/2)J_2(w)$ is of polynomial growth in a vertical strip $\frac{11}{18}+\varepsilon\le\delta\le 1+\varepsilon$ with small positive ε, consider the function $\cos(\pi w/2)(H(w)-J_1(w))$, where $H(w)$ is defined in Proposition 3.6. We shall see that this function has polynomial growth in $-\varepsilon\le\delta\le 1+\varepsilon$, away from its (finitely many) poles. For $11/18<\delta\le 1+\varepsilon$, we have

$$\cos(\pi w/2)\cdot(H(w)\,-\,J_1(w)) \,=\, \cos(\pi w/2)\cdot(J_2(w)\,-\,J^{\rm aux}(w)) \tag{3.10}$$

First, suppose that $w = 1 + \varepsilon + i\eta$. As $B_0(0, w, \mu_{f,\nu_0}) \cos(\pi w/2)$ is of polynomial growth and $Z(w) = O(1)$ (w being in the region of absolute convergence of its defining sum and integral), it follows from Proposition 3.7 that the right hand side of the above identity is polynomial in $|\eta|$.

Recall that $J_1(w)$ is defined by

$$J_1(w) \;=\; \frac{1}{2\pi} \sum_{\substack{\chi \in \widehat{C}_{0,S} \\ |t+t_{\nu_0}| \ll C}} \int |L(\tfrac{1}{2}+it, f \otimes \chi)|^2 \, \mathcal{K}_\infty(\tfrac{1}{2}+it,\, 0,\, w,\, \alpha,\, \chi)\, dt \qquad (\text{for } \Re(w) > 1)$$

The integral representing the local kernel $\mathcal{K}_{\nu_0}(\frac{1}{2}+it,\, 0,\, w,\, \chi_{\nu_0})$ continues to $-\varepsilon \le \delta \le 1+\varepsilon$ with finitely many poles in this region (its poles coincide with those of $B_0(0, w, \mu_{f,\nu_0})$). Being a function of $t + t_{\nu_0}$ and w, we can estimate $\mathcal{K}_{\nu_0}$, for $|t + t_{\nu_0}| \ll C$ and w not a pole, as

$$\mathcal{K}_{\nu_0}(\tfrac{1}{2}+it,\, 0,\, w,\, \chi_{\nu_0}) \ll_{w,\,\alpha} \big(1 + |t + t_{\nu_0}|\big)^{-\alpha} \qquad (\text{for } -\varepsilon \le \Re(w) \le 1+\varepsilon)$$

See also the discussion in the Appendix. By a simple comparison with $Z(\alpha)$, we deduce that the defining expression of $J_1(w)$ converges absolutely, away from the poles of $\mathcal{K}_{\nu_0}$, throughout the strip.

Now assume $w = -\varepsilon + i\eta$. Using Proposition 3.6 and the invariance of $\cos(\pi w/2) M(w, f)$, $\cos(\pi w/2)\mathcal{G}_{F_\infty}(w)$, $\cos(\pi w/2)\mathcal{G}_{\chi_\infty}(s, w)$ as $w \to 1 - w$, we have

$$\begin{aligned}
\cos(\pi w/2) \cdot (H(w) \,-\, J_1(w)) \;&=\; \cos\frac{\pi(1-w)}{2}\, J(1-w) \,-\, \cos(\pi w/2)\, J_1(w) \\
&\quad -\, \cos(\pi w/2) \cdot \Big(\sum_F \bar{\rho}_F\, \mathcal{G}^{\text{aux}}_{F_\infty}(w) L(\tfrac{1}{2}, \overline{F}) \langle F, |f|^2 \rangle \Big) \\
-\, \cos(\pi w/2) \cdot \Bigg(\sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} &\int_{\Re(s)=\frac{1}{2}} \frac{\mathcal{G}^{\text{aux}}_{\chi_\infty}(s, w) L(1-s, \overline{\chi}) L(s, \chi)}{\Lambda(2-2s, \overline{\chi}^2)}\, |\mathfrak{d}|^{-(s-1/2)} \langle E_{s,\chi}, |f|^2 \rangle\, ds \Bigg)
\end{aligned}$$

In the right hand side, write $J = J_1 + J_2$. As already observed, $\cos[\pi(1-w)/2] J_2(1-w)$ is bounded polynomially in $|\eta|$.

To see that

$$\cos\frac{\pi(1-w)}{2}\, J_1(1-w) \,-\, \cos\frac{\pi w}{2}\, J_1(w)$$

is also bounded polynomially, we use the fact that $J_1(w)$ can be expressed by its original representation throughout the vertical strip $-\varepsilon \le \Re(w) \le 1+\varepsilon$, except for finitely many poles of the local kernel $\mathcal{K}_{\nu_0}(\frac{1}{2}+it,\, 0,\, w,\, \chi_{\nu_0})$. Since away from the poles and $|t + t_{\nu_0}| \ll C$,

$$\cos\frac{\pi(1-w)}{2} \cdot \mathcal{K}_{\nu_0}(\tfrac{1}{2}+it,\, 0,\, 1-w,\, \chi_{\nu_0}) \,-\, \cos\frac{\pi w}{2} \cdot \mathcal{K}_{\nu_0}(\tfrac{1}{2}+it,\, 0,\, w,\, \chi_{\nu_0}) \;\ll\; (1 + |t + t_{\nu_0}|)^{A} |\eta|^{B}$$

for some constants A, B (see [Di-Go1], Proposition 4.6, for holomorphic discrete series, and in general by a similar idea), we have the estimate

$$\cos\frac{\pi(1-w)}{2}\, J_1(1-w) \,-\, \cos\frac{\pi w}{2}\, J_1(w) \;\ll\; (1+C)^{A+2} |\eta|^{B}\, Z(\alpha) \qquad (\text{for } \alpha \le 2)$$

Recall that $C = C(\eta)$ depends polynomially on w.

From Proposition 3.7, it follows now that $\cos(\pi w/2)(H(w) - J_1(w))$ is a function of polynomial growth on the vertical line $\Re(w) = -\varepsilon$.

As $H(w)$ and $J_1(w)$ have both finitely many poles in the strip $-\varepsilon \leq \Re(w) \leq 1+\varepsilon$, we can apply the Phragmen-Lindelöf principle. Accordingly, the function $\cos(\pi w/2)(H(w) - J_1(w))$ has polynomial growth throughout the strip (away from its poles). In particular, for $\frac{11}{18} + \varepsilon \leq \delta \leq 1+\varepsilon$, the identity (3.10) implies that $\cos(\pi w/2)(J_2(w) - J^{\mathrm{aux}}(w))$ has polynomial growth in this region, and hence, so has $\cos(\pi w/2) J_2(w)$ by Proposition 3.7.

From the above discussion, we obtain the main result of this section. This is contained in the following

Theorem 3.11. *For fixed small positive ε, the function $Z(w)$ has polynomial growth in the half-plane $\Re(w) \geq \frac{11}{18} + \varepsilon$; on the vertical line $\Re(w) = \frac{11}{18} + \varepsilon$,*

$$Z(w) \ll_{\varepsilon,\,\alpha} |\eta|^{M_\varepsilon} \qquad\qquad (w = \tfrac{11}{18} + \varepsilon + i\eta)$$

with a computable $M_\varepsilon > 0$ independent of α.

Remark. To optimize the exponent M_ε in Theorem 3.11, it is necessary to invoke [Ho-Lo] and [Ho-Ra], or an extension of [Be-Re1]. In this direction, see [Kr-St].

§4. Subconvexity

Fix a cuspform f on $GL_2(k)$ for a number field k of degree d over $\mathbb{Q}$. As above, we assume that f is a newform locally everywhere. Recall that the t–aspect *convexity bound* on the standard L–function $L(s,f)$ is

$$L(\tfrac{1}{2} + it, f) \ \ll_\varepsilon\ (1+|t|)^{2d\cdot\frac{1}{4}+\varepsilon} \ =\ (1+|t|)^{\frac{d}{2}+\varepsilon} \qquad\qquad (\text{for all } \varepsilon > 0)$$

We *break convexity* in the t–aspect by decreasing the exponent, proving

Main Theorem. Fix a number field k of degree d over $\mathbb{Q}$, and a cuspform f for $GL_2(k)$, with f generating a local newform everywhere locally. For a computable constant $\vartheta < 1$,

$$L(\tfrac{1}{2} + it, f) \ \ll_\varepsilon\ (1+|t|)^{\frac{d-1+\vartheta}{2}+\varepsilon} \qquad\qquad (\text{for all } \varepsilon > 0)$$

Note that the hypothesis that f is a newform locally everywhere exactly assures the equality $L(s,f) = L(s,\pi)$ of the finite-prime part $L(s,f)$ of the Hecke zeta integral of f with the Langlands L–function $L(s,\pi)$ attached to the automorphic representation π attached to f.

Some further preparation is required before we can begin the proof of the Main Theorem.

For convenience, assume that k has at least one real place ν_0, although this assumption is inessential. Given a grossencharacter χ of k, let

$$q_\nu(\chi_\nu, t) \ =\ \begin{cases} 1 + |t + t_\nu| & (\text{for } \nu \text{ real}) \\ \\ 1 + \ell_\nu^2 + 4(t+t_\nu)^2 & (\text{for } \nu \text{ complex}) \end{cases}$$

with data t_ν and ℓ_ν attached to the local component χ_ν of χ by

$$\begin{cases} \chi_\nu(y) & = & |y|^{it_\nu} & (\nu \text{ real}, \, y > 0) \\ \chi_\nu(z) & = & \left(\frac{z}{|z|}\right)^{\ell_\nu} |z|^{it_\nu} & (\nu \text{ complex}) \end{cases}$$

The archimedean part of the analytic conductor of $L(s,\chi)$ is $Q(\chi,t) = \prod_{\nu|\infty} q_\nu(\chi_\nu,t)$. As f is fixed, ignore the dependence on the (archimedean) local parameters of f, and approximate the archimedean part of the analytic conductor of $L(s, f\otimes\chi)$ by $Q(\chi,t)^2$. Fix real $1 < \alpha \le 2$, as before. Instead of the $(w/2)^{\text{th}}$ power of the archimedean part of analytic conductor of $L(s, f\otimes\chi)$, we use

$$q_{\nu_0}(\chi_{\nu_0},t)^{-w} \cdot \prod_{\nu\neq\nu_0} \mathcal{K}_\nu(\tfrac{1}{2}+it, 0, \alpha, \chi_\nu)$$

Abbreviate

$$q_0(\chi,t) \;=\; q_{\nu_0}(\chi_{\nu_0},t) \qquad\qquad \kappa_0(\chi,t) \;=\; \Pi_{\nu\neq\nu_0} \mathcal{K}_\nu(\tfrac{1}{2}+it, 0, \alpha, \chi_\nu)$$

Recall that

$$Z(w) \;=\; \sum_{\chi\in\widehat{C}_{0,S}} \int_{-\infty}^{+\infty} |L(\tfrac{1}{2}+it, f\otimes\chi)|^2 \cdot \frac{\kappa_0(\chi,t)}{q_0(\chi,t)^w}\, dt$$

In the sequel, $\widehat{C}_{0,S}$ will be implicit, and we will refer to the combination of sum over χ and integral over t simply as *the integral*.

From Theorem 2.1, the integral for $Z(w)$ converges absolutely for $\Re(w) > 1$, with analytic continuation to $\Re(w) > \frac{11}{18}$, except for a pole of order 2 at $w = 1$. By Theorem 3.11, away from the pole at $w = 1$, the function $Z(w)$ has *polynomial growth* vertically on every vertical strip inside $\frac{11}{18} + \varepsilon \le \Re(w) \le \alpha$. The function $Z(w)$ is bounded for $1+\varepsilon \le \Re(w) \le \alpha$. Via Phragmen-Lindelöf, choose $\frac{11}{18} < \delta_0 < 1$ such that $Z(w)$ has polynomial growth of exponent $< \frac{1}{2}$ in the vertical strip $\delta_0 \le \Re(w) \le 1+\varepsilon$. Notice that this assures that $\eta \to Z(\delta_0+i\eta)/(\delta_0+i\eta)$ is square-integrable on $\mathbb{R}$. For instance, for fixed sufficiently small $\varepsilon > 0$ ($\varepsilon = 10^{-3}$, say) one can take any

$$1 - \left(\frac{7-18\varepsilon}{36\, M_\varepsilon}\right) \;<\; \delta_0 \;<\; 1$$

where M_ε is the exponent in Theorem 3.11. Note that this is independent of the parameter α, since (as in Theorem 3.11), the constant M_ε is independent of α. We will make repeated use of the fact that, for real $w > 1$, the kernel $\mathcal{K}_\nu(\frac{1}{2}+it, 0, w, \chi_\nu)$ is positive, with $\nu \neq \nu_0$, and is dominated by $q_\nu(\chi_\nu,t)^{-w}$. See the Appendix.

Let R be the rectangle with vertices $\alpha \pm iT$ and $\delta_0 \pm iT$, traced counter-clockwise, with T eventually going to $+\infty$. On one hand, the integral over R can be evaluated by Cauchy's theorem, as

$$\frac{1}{2\pi i}\int_R \frac{Z(w)\, x^w}{w}\, dw \;=\; x\, P(\log x)$$

with a linear polynomial P. By choice of δ_0, the integrals of $Z(w)\, x^w/w$ along the top and bottom of the rectangle R go to 0 as $T \to +\infty$.

On the other hand, from [Da], recall Perron's formula

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{x^w}{w}\,dw \;=\; \begin{cases}1 & (\text{for } x>1)\\ 0 & (\text{for } x<1)\end{cases} \;+\; x^\alpha\cdot O_\alpha\Big(\min\Big\{1,\frac{1}{T|\log x|}\Big\}\Big)$$

Apply the identity, with x replaced by $x/q_{\scriptscriptstyle 0}(\chi,t)$, to rewrite the integral as

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{Z(w)\,x^w}{w}\,dw$$

$$= \sum_\chi\int_{-\infty}^{+\infty}|L(\tfrac12+it,f\otimes\chi)|^2\cdot\kappa_{\scriptscriptstyle 0}(\chi,t)\cdot\Bigg(\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\big(x/q_{\scriptscriptstyle 0}(\chi,t)\big)^w}{w}\,dw\Bigg)\,dt$$

$$= \sum_\chi\int_{q_0(\chi,t)\le x}|L(\tfrac12+it,f\otimes\chi)|^2\cdot\kappa_{\scriptscriptstyle 0}(\chi,t)\,dt \;+\; E(x,T)$$

where the *error term* $E(x,T)$ is estimated by the corresponding sum of integrals

$$E(x,T)\;\ll_\alpha\;\sum_\chi\int_{-\infty}^{+\infty}|L(\tfrac12+it,f\otimes\chi)|^2\cdot\kappa_{\scriptscriptstyle 0}(\chi,t)\cdot\Big(\frac{x}{q_{\scriptscriptstyle 0}(\chi,t)}\Big)^\alpha\,\min\Big\{1,\;\frac{1}{T\cdot|\log x/q_{\scriptscriptstyle 0}(\chi,t)|}\Big\}\,dt$$

Lemma 4.1. For fixed $x>0$,

$$\lim_{T\to+\infty}E(x,T)\;=\;0$$

Proof: This uses the positivity of the kernels $\mathcal{K}_\nu(\frac12+it,0,\alpha,\chi_v)$ (for $\nu\neq\nu_{\scriptscriptstyle 0}$) and their bounds by $q_\nu(\chi,t)^{-\alpha}$. In the estimate on E, for each χ break the integral over t into two pieces

$$\{t\;:\;\frac{1}{T\cdot|\log x/q_{\scriptscriptstyle 0}(\chi,t)|}\;\le\;\frac{1}{\sqrt{T}}\}\qquad\text{and}\qquad\{t\;:\;\frac{1}{T\cdot|\log x/q_{\scriptscriptstyle 0}(\chi,t)|}\;\ge\;\frac{1}{\sqrt{T}}\}$$

The sum of the integrals in the first case is dominated by

$$\frac{x^\alpha}{\sqrt{T}}\cdot\sum_\chi\int_{\mathbb{R}}|L(\tfrac12+it,f\otimes\chi)|^2\cdot\frac{\kappa_{\scriptscriptstyle 0}(\chi,t)}{q_{\scriptscriptstyle 0}(\chi,t)^\alpha}\,dt\;=\;\frac{1}{\sqrt{T}}\cdot x^\alpha\cdot Z(\alpha)$$

Since the integral for $Z(\alpha)$ converges absolutely, the factor $1/\sqrt{T}$ assures that the integral in the first subset goes to 0 as T becomes large, for fixed x.

The second case breaks into two parts. Since the integral for $Z(w)$ converges absolutely for $\Re(w)>1$, the definition of absolute convergence implies that the *tail* of the integral goes to 0. That is, the integral over the set in which $Q(\chi,t)\ge\log T$ goes to 0 as T becomes large, so the same is certainly true for the integral over the subset meeting the second case.

For the part of the second case where $Q(\chi,t)\le\log T$, observe that, for each $u>0$, one trivially has

$$\sum_\chi\int_{Q(\chi,t)\le u}1\;\ll\;\text{polynomial bound in } u$$

In fact, there is a bound of $\ll u^{1+\varepsilon}$, although for the present argument *any* polynomial bound in u will suffice. Apply this bound with $u = \log T$, and use the convexity bound of $L(\frac{1}{2} + it, f \otimes \chi)$, to obtain a bound on the integrand *logarithmic* in T. The defining condition for the second case implies

$$x \cdot e^{-1/\sqrt{T}} \ < \ 1 + |t + t_{\nu_0}| \ < \ x \cdot e^{1/\sqrt{T}}$$

which restricts t to a set of measure $\ll 1/\sqrt{T}$. Thus, this part of the second case also goes to 0 for large T. This proves the Lemma 4.1. $\square$

Thus, taking the limit as T becomes large produces an equality of the integral along $\Re(w) = \alpha$ with a sum of integrals over regions $q_0(\chi, t) \leq x$. Thus, so far, we have

$$\sum_{\chi} \int_{q_0(\chi,t)\leq x} |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \cdot \kappa_0(\chi, t)\, dt \ = \ x\, P(\log x) \ + \ \frac{1}{2\pi i} \int_{\delta_0 - i\infty}^{\delta_0 + i\infty} \frac{Z(w)\, x^w}{w}\, dw \tag{4.2}$$

Theorem 4.3. We have the estimate

$$\frac{1}{2\pi i} \int_{\delta_0 - i\infty}^{\delta_0 + i\infty} \frac{Z(w)\, x^w}{w}\, dw \ \ll_\alpha \ x^{\frac{2\delta_0 + 1}{3}} \cdot \log x$$

and thus,

$$\sum_{\chi} \int_{q_0(\chi,t)\leq x} |L(\tfrac{1}{2} + it, f \otimes \chi)|^2 \cdot \kappa_0(\chi, t)\, dt \ = \ x\, P(\log x) + O\left(x^{\frac{2\delta_0+1}{3}} \log x\right)$$

Proof: Let $E(x)$ denote the first integral in the statement of the theorem, $w = \delta_0 + i\eta$, and $x = e^{-2\pi u}$. Then

$$E(e^{-2\pi u}) \ = \ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-2\pi i u \eta} \cdot f(\eta) \cdot e^{-2\pi u \delta_0}\, d\eta$$

is essentially a Fourier transform of $f(\eta) = Z(\delta_0 + i\eta)/(\delta_0 + i\eta)$ in η, namely,

$$2\pi \cdot (e^{-2\pi u})^{-\delta_0} \cdot E(e^{-2\pi u}) \ = \ \widehat{f}(u)$$

By our choice of δ_0, we have convergence of

$$\int_{-\infty}^{+\infty} \left| \frac{Z(\delta_0 + i\eta)}{\delta_0 + i\eta} \right|^2 \, d\eta$$

Then Plancherel gives

$$1 \gg \int_{-\infty}^{+\infty} |f(\eta)|^2\, d\eta \ = \ \int_{-\infty}^{+\infty} |\widehat{f}(u)|^2\, du \ = \ (2\pi)^2 \int_{-\infty}^{\infty} |(e^{-2\pi u})^{-\delta_0} \cdot E(e^{-2\pi u})|^2\, du$$

$$= \ 2\pi \int_0^\infty y^{-2\delta_0} \cdot |E(y)|^2 \, \frac{dy}{y} \ = \ 2\pi \int_0^\infty y^{-(2\delta_0+1)} \cdot |E(y)|^2\, dy$$

since $du = dy/2\pi y$. The elementary estimate

$$1 \gg \int_0^x |E(y)|^2 \, y^{-(2\delta_0+1)} \, dy \; \geq \; \int_0^x |E(y)|^2 \, x^{-(2\delta_0+1)} \, dy \; = \; x^{-(2\delta_0+1)} \int_0^x |E(y)|^2 \, dy$$

gives

$$\int_0^x |E(y)|^2 \, dy \; \ll \; x^{2\delta_0+1} \tag{4.4}$$

The latter yields a pointwise estimate with the exponent reduced by a factor of 3, by an argument reminiscent of a part of [Iw-Mi], as follows. That is, we claim that

$$E(x) \; \ll \; x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

Again, use the positivity of $\mathcal{K}_\nu(\frac{1}{2}+it, 0, \alpha, \chi_\nu)$ with real $\alpha > 1$, for $\nu \neq \nu_0$. Let

$$\mathfrak{I}(\chi, x) \; = \; \{t \; : \; q_0(\chi, t) \leq x\}$$

For any χ and $0 < x \leq y$, certainly $\mathfrak{I}(\chi, x) \subset \mathfrak{I}(\chi, y)$. As

$$E(y) \; = \; \sum_\chi \int_{\mathfrak{I}(\chi,\, y)} |L(\tfrac{1}{2}+it, f \otimes \chi)|^2 \cdot \kappa_0(\chi, t) \, dt \; - \; y \, P(\log y)$$

and because the integrand is *positive*,

$$E(y) - E(x) \; \geq \; -\big(y \, P(\log y) - x \, P(\log x)\big) \qquad \text{(for } 0 < x \leq y\text{)} \tag{4.5}$$

Fix $x \geq 3$. Replacing y by $x + u$ in this inequality, with $0 \leq u \leq x$, we have

$$E(x) \; \leq \; E(x+u) + C \cdot u \cdot \log x \qquad \text{(for some positive } C\text{)}$$

since P is a linear polynomial. Similarly, replacing x by $x - u$ with $0 \leq u < x$ and y by x gives

$$E(x) \; \geq \; E(x-u) - C \cdot u \cdot \log x$$

with the same constant C.

Integrating these inequalities in u, over $0 \leq u \leq H$, with $0 < H \leq x$, gives

$$\frac{1}{H} \int_{x-H}^x E(t) \, dt - CH \cdot \log x \; \leq \; E(x) \; \leq \; \frac{1}{H} \int_x^{x+H} E(t) \, dt + CH \cdot \log x \tag{4.6}$$

with a new constant C (half of the previous constant C). The second of these inequalities is applied for $E(x) \geq 0$ and the first for $E(x) \leq 0$.

Indeed, for $E(x) \geq 0$, the estimate

$$\int_0^x |E(y)|^2 \, dy \; \ll \; x^{2\delta_0+1}$$

in (4.4) together with

$$E(x) \;\le\; \frac{1}{H}\int_x^{x+H} E(t)\,dt + CH\cdot\log x$$

in (4.6) gives

$$E(x)^2 \;\le\; \frac{2}{H^2}\Big(\int_x^{x+H} E(t)\,dt\Big)^2 + 2C^2H^2\cdot(\log x)^2 \;\le\; \frac{2}{H}\int_x^{x+H} |E(t)|^2\,dt + 2C^2H^2\cdot(\log x)^2$$

$$\ll\; \frac{1}{H}\cdot x^{2\delta_0+1} + H^2\cdot(\log x)^2$$

using $(a+b)^2 \le 2(a^2+b^2)$, and Cauchy-Schwarz. Setting $H = x^{\frac{2\delta_0+1}{3}}$, and recalling that $E(x)\ge 0$, we obtain

$$E(x) \;\ll\; x^{\frac{2\delta_0+1}{3}}\cdot\log x$$

A similar argument applies when $E(x)\le 0$, proving the theorem. $\square$

Proof of the Main Theorem: Recall that $\kappa_{_0}(\chi,t)$ is positive, for all χ,t. Let

$$S(x) \;=\; \sum_{\chi}\int_{q_0(\chi,t)\le x} |L(\tfrac{1}{2}+it, f\otimes\chi)|^2\,\kappa_{_0}(\chi,t)\;dt \tag{4.7}$$

For $H>0$ and fixed χ, certainly $\mathfrak{I}(\chi,x)\subset\mathfrak{I}(\chi,x+H)$. Also, the integrands are *positive*. For $\chi=1$, with $\nu_{_0}$ *real*, $q_{_0}(1,t)=1+|t|$. Ignoring all terms but for trivial χ, and taking

$$x^{\frac{2\delta_0+1}{3}} \;\ll\; H \;\ll\; x^{\frac{2\delta_0+1}{3}} \tag{4.8}$$

we have

$$\int_x^{x+H} |L(\tfrac{1}{2}+it,f)|^2\,\kappa_{_0}(1,t)\,dt \;\le\; S(x+H+1)-S(x) \tag{4.9}$$

$$=\; (x+H+1)\,P(\log(x+H+1)) - x\,P(\log x) + E(x+H+1) - E(x)$$

$$\ll\; x^{\frac{2\delta_0+1}{3}}\cdot\log x$$

where the last inequality comes from Theorem 4.3.

Recall that δ_0 is independent of α, and that $\kappa_{_0}(1,t)\gg(1+|t|)^{-(d-1)\alpha}$ (see the lower-bound (A.9) in the Appendix). Choosing $\alpha = 1+\frac{\varepsilon}{2d-2}$, we obtain

$$\int_x^{x+H} |L(\tfrac{1}{2}+it,f)|^2\,dt \;\ll\; (x+H)^{(d-1)\alpha}\cdot x^{\frac{2\delta_0+1}{3}}\cdot\log x \tag{4.10}$$

$$\ll\; x^{d-1+\frac{2\delta_0+1}{3}+\frac{\varepsilon}{2}}\cdot\log x \;\ll_\varepsilon\; x^{d-1+\frac{2\delta_0+1}{3}+\varepsilon}$$

The short-interval estimate (4.10) yields a pointwise estimate, via Cauchy's theorem and use of the functional equation of $L(s,f)$, by an argument broadly analogous to that at the end of [Go1] for $GL_2(\mathbb{Q})$, yielding the conclusion of the Main Theorem. $\square$

§**Appendix**

Here we record some explicit formulas and related estimates used throughout. These facts were discussed in detail in [Di-Go1] and [Di-Go2]. In the holomorphic discrete series case everything is worked out completely in [Di-Go1]. For the waveform case, the corresponding results can be obtained analogously using the computations made in [Di-Go2]. Accordingly, we organize these facts for the convenience of the reader. Throughout this appendix, we refer very precisely to certain points in [Di-Go2].

First, the kernel $\mathcal{K}_\infty(s,\, v,\, \chi)$ appearing on the moment side (1.4) decomposes over the infinite primes

$$\mathcal{K}_\infty(s,\, v,\, \chi) \;=\; \prod_{\nu|\infty} \mathcal{K}_\nu(s,\, v,\, \chi_\nu)$$

We shall discuss some properties of $\mathcal{K}_\nu(s, v, w, \chi_\nu)$, with the choice of archimedean data corresponding to (1.2). When archimedean data is given by (3.1), the study of the corresponding kernel $\mathcal{K}_\nu$ can be reduced to that for the choice (1.2). Thus, our conclusions concerning $\mathcal{K}_\nu$ for the choice (1.2) hold equally for $\mathcal{K}_\nu$ attached to (3.1).

For ν complex, we have

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; 2^{-4v-1}\pi^{1-2v}\mathcal{K}_{\ell_\nu}(2s+2it_\nu, 2v, 2w) \tag{A.1}$$

where $\mathcal{K}_\ell(s, v, w)$ is defined in [Di-Go2], page 71, (4.15). In that formula for $\mathcal{K}_\ell(s, v, w)$, one should also replace μ and ν by $2\mu_{f,\,\nu}$ and $2\bar{\mu}_{f,\,\nu}$, respectively.

For ν real, we have

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) \;=\; 2^{1-v}\pi^{-v}\mathcal{K}_{-\frac{1}{2}}(s+\tfrac{1}{2}+it_\nu, v, w) \tag{A.2}$$

This follows easily by comparing (1.9), (1.17), and (1.18) transformed by (2.2) in [Zh2] with the formula for $\mathcal{K}_\ell(s, v, w)$ (following (4.15)) in [Di-Go2]. Here again $\mathcal{K}_{-\frac{1}{2}}(s+\frac{1}{2}+it_\nu, v, w)$ in the right must have the parameters μ, ν replaced by $\mu_{f,\,\nu}$, $\bar{\mu}_{f,\,\nu}$, respectively.

In what follows we shall refer to specific facts about $\mathcal{K}_\ell(s, v, w)$ discussed in [Di-Go2]; the corresponding properties of the local kernel $\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu)$ can then be easily translated using (A.1) and (A.2).

From the representation (4.17), page 72, it follows that $\mathcal{K}_\ell(s, v, w)$ is holomorphic in the region $\mathcal{D}$ consisting of $(s, v, w) \in \mathbb{C}^3$ such that there exist δ_1 and δ_2 for which

$$-\frac{\sigma \pm \Re(i\mu)}{2} < \delta_1 < \frac{\ell}{2},\;\; -\frac{2+\Re(v)-\sigma \pm \Re(i\nu)}{2} < \delta_2 < \frac{\ell}{2},\;\; -1 < \delta_1+\delta_2 < \frac{\Re(w)-2}{2} \tag{A.3}$$

Here ℓ is either $-1/2$ or a non-negative integer. In particular, if $\Re(s) = 1/2$ and $v = 0$, the local kernel $\mathcal{K}_\nu(s,\, 0,\, w,\, \chi_\nu)$ is holomorphic for $\Re(w) > 2/9$. Note that we used Kim-Shahidi bound $|\Re(i\mu_{f,\nu})| < 1/9$ (see [Ki], [Ki-Sh]) for the local archimedean parameters of f. Using (4.17), one can also meromorphically continue $\mathcal{K}_\ell(s, v, w)$ by shifting the contour in δ_1, say, to the right. To see this, assume $\ell = -1/2$, $s = 1 + iT$, $v = 0$, $|\Re(i\mu)| \le 1/9$ and $\nu = \bar{\mu}$ in (6.1) on page 76. By (A.3), we can take $\delta_1 = \delta_2 = -\frac{1}{2} + \frac{1}{18} + \varepsilon$, and $\Re(w) = \frac{2}{9} + 6\varepsilon$ with small positive ε. Aiming to continue $\mathcal{K}_{-1/2}$ in w, shift the contour $\delta_1 = -\frac{1}{2} + \frac{1}{18} + \varepsilon$ slightly to the right such that the only pole crossed is $\xi_1 = -1 - \xi_2 + w/2$. The shifted integral is holomorphic for $-c \le \Re(w) \le \frac{2}{9} + 6\varepsilon$ with a small positive constant c. The residue at $\xi_1 = -1 - \xi_2 + w/2$ is given in (6.25) on page 82.

Shifting again to the right in (6.25) such that the only poles crossed are at $\xi_2 = (w-1 \pm i\mu)/2$, one obtains the meromorphic continuation of $\mathcal{K}_{-1/2}$ for $\Re(w) \geq -c$. It also follows that the poles of $\mathcal{K}_{-1/2}$ in this region coincide with those of $B(0, w, \mu)$ in (A.7). This last fact also holds for the local kernel $\mathcal{K}_{\nu_0}(\frac{1}{2}+it, 0, w, \chi_{\nu_0})$ attached to (3.1), by expanding the hypergeometric function in its series, thereby reducing to the immediately previous discussion.

Combining (4.13), (4.14) with $\nu = \bar{\mu}$, and (4.15) on page 71, one can see that

$$\mathcal{K}_\ell(1+it, 0, w) \;=\; \int_0^{\frac{\pi}{2}} (\cos\phi)^{w-1} \sin\phi \cdot |V_{\mu,\ell}(t,\phi)|^2 \, d\phi \tag{A.4}$$

where

$$V_{\mu,\ell}(t,\phi) \;=\; 2^{-1+it} \cos^{-\ell-1-it}(\phi)\, \sin^{\ell}(\phi) \cdot \frac{\Gamma(\frac{\ell+1+it-i\mu}{2})\Gamma(\frac{\ell+1+it+i\mu}{2})}{\Gamma(\ell+1)} \cdot F\Big(\frac{\ell+1+it-i\mu}{2}, \frac{\ell+1+it+i\mu}{2}; \ell+1; -\tan^2(\phi)\Big) \tag{A.5}$$

Initially, the integral representation (A.4) holds for $\Re(w)$ sufficiently large. Then, by Landau's Lemma (adapting the proof of Theorem 6, page 115 in [Ch], as in section §5 of [Di-Ga]), the holomorphy up to $\Re(w) = 2/9$ above implies that the integral representation holds for $\Re(w) > 2/9$. Clearly, for real $w > 2/9$, the kernel in (A.4) is positive. This positivity was used repeatedly throughout.

Recall from Section 4 that $q_\nu(\chi_\nu, t)$ is defined to be

$$q_\nu(\chi_\nu, t) \;=\; \begin{cases} 1 + |t + t_\nu| & \text{(for } \nu \text{ real)} \\ 1 + \ell_\nu^2 + 4(t+t_\nu)^2 & \text{(for } \nu \text{ complex)} \end{cases}$$

By Theorem 6.2, one has the following asymptotic expansions when $|\Im(w)|^{2+\varepsilon} \ll q_\nu(\chi_\nu, t)$ with small $\varepsilon > 0$: for ν complex,

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = A(v, w, \mu_{f,\nu}) \cdot q_\nu(\chi_\nu, t)^{-w} \cdot \Big[1 \;+\; \mathcal{O}_{\sigma, v, w, \mu_{f,\nu}}\Big(q_\nu(\chi_\nu, t)^{-1/2}\Big)\Big] \tag{A.6}$$

where $A(v, w, \mu_{f,\nu})$ is the ratio of gamma functions

$$2^{2w-2v-4}\, \pi^{1-2v}\, \frac{\Gamma(w+v+i\mu+i\bar{\mu})\Gamma(w+v-i\mu+i\bar{\mu})\Gamma(w+v+i\mu-i\bar{\mu})\Gamma(w+v-i\mu-i\bar{\mu})}{\Gamma(2w+2v)}$$

and, for ν real,

$$\mathcal{K}_\nu(s,\, v,\, w,\, \chi_\nu) = B(v, w, \mu_{f,\nu}) \cdot q_\nu(\chi_\nu, t)^{-w} \cdot \Big[1 \;+\; \mathcal{O}_{\sigma, v, w, \mu_{f,\nu}}\big(q_\nu(\chi_\nu, t)^{-1}\big)\Big] \tag{A.7}$$

where

$$B(v, w, \mu) \;=\; 2^{w-2}\, \pi^{-v}\, \frac{\Gamma(\frac{w+v+i\mu+i\bar{\mu}}{2})\Gamma(\frac{w+v-i\mu+i\bar{\mu}}{2})\Gamma(\frac{w+v+i\mu-i\bar{\mu}}{2})\Gamma(\frac{w+v-i\mu-i\bar{\mu}}{2})}{\Gamma(w+v)}$$

The dependence upon $\Im(w)$ in these asymptotics is *polynomial* (in fact, essentially quadratic), and the dependence upon $\Re(w)$ is *continuous.* This is explicit in Lemma 6.5 on page 77 and in the proof of Lemma 6.6 on page 82. The underlying point in the above asymptotics is the asymptotic

$$\frac{\Gamma(s+a)}{\Gamma(s)} \;=\; s^a\Big(1+\mathcal{O}\left(\frac{|a|^2}{|s|}\right)\Big)$$

for $|a| = o(\sqrt{|s|})$. This is used in the proof of Lemma 6.7.

We have an estimate

$$\mathcal{K}_\nu(\tfrac{1}{2}+it, 0, w, \chi_\nu) \;\ll_w\; q_\nu(\chi_\nu, t)^{-\delta} \tag{A.8}$$

When $|\Im(w)|^{2+\varepsilon} \ll q_\nu(\chi_\nu, t)$, with small $\varepsilon > 0$, and $\Re(w) > 2/9$, inequality (A.8) follows from (A.6) and (A.7). When $|\Im(w)|^{2+\varepsilon} \gg q_\nu(\chi_\nu, t)$, the inequality (A.8) holds wherever $\mathcal{K}_\nu$ is defined (e.g., away from its poles).

For $C < q_\nu(\chi_\nu, t)$ with sufficiently large positive C, and real $w > 2/9$, it also follows from (A.6) and (A.7) that

$$q_\nu(\chi_\nu, t)^{-w} \;\ll_w\; \mathcal{K}_\nu(\tfrac{1}{2}+it, 0, w, \chi_\nu) \tag{A.9}$$

Using the integral representation (A.4), and the fact that the hypergeometric function in (A.5) is not identically 0, it follows that the inequality (A.9) also holds for $q_\nu(\chi_\nu, t) \le C$ and real $w > 2/9$.

In Section 3, we needed specific choices of archimedean data, producing corresponding functions $\mathcal{G}_\nu$ on the spectral side (3.2) of the identity. For φ_ν given by (1.2), the corresponding $\mathcal{G}_\nu$ on the spectral side is, for ν real,

$$\mathcal{G}_\nu(s; v, w) = \pi^{-v}\,\frac{\Gamma\big(\frac{v+1-s}{2}\big)\Gamma\big(\frac{v+w-s}{2}\big)\Gamma\big(\frac{v+s}{2}\big)\Gamma\big(\frac{v+w+s-1}{2}\big)}{\Gamma\big(\frac{w}{2}\big)\Gamma\big(v+\frac{w}{2}\big)} \tag{A.10}$$

and, at complex places ν,

$$\mathcal{G}_\nu(s; v, w) = (2\pi)^{-2v}\,\frac{\Gamma(v+1-s)\Gamma(v+w-s)\Gamma(v+s)\Gamma(v+w+s-1)}{\Gamma(w)\Gamma(2v+w)} \tag{A.11}$$

When the field k is totally complex, we take

(A.12)

$$\varphi_{\nu_0}(n) \;=\; 2^{1-2w}\,\sqrt{\pi}\,\frac{\Gamma(w)(1+|x|^2)^{-w}\,F(w,w;2w;\frac{1}{1+|x|^2})}{\Gamma(w-\frac{1}{2})} \qquad \left(n = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \in N_{\nu_0},\, x\neq 0\right)$$

and

$$\varphi_{\nu_0}(n) = 0 \qquad \left(\text{if } n = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

With this choice, the archimedean integrals in (1.5) at ν_0 (i.e., the analog of Lemma 3.3) are computed as

$$\int_{N_{\nu_0}} \varphi_{\nu_0} = \pi\cdot\frac{2w-1}{w(w-1)} \tag{A.13}$$

and

$$\mathcal{G}_{\nu_0}(s; 0, w) \;=\; (w-\tfrac{1}{2})\cdot\frac{\Gamma(1-s)\Gamma(s)}{(w-s)(w+s-1)} \tag{A.14}$$

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