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Averages of symmetric square L-functions, and applications

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We exhibit a spectral identity involving $L(s, \mathrm{Sym}^2 f)$ for f on SL_2. Perhaps contrary to expectations, we do not treat $L(s, \mathrm{Sym}^2 f)$ directly as a GL_3 object. Rather, we take advantage of the coincidence that the *standard* L-function for SL_2 is the symmetric square for a cuspform on GL_2 restricted to SL_2. [1] As $SL_2 = Sp_2$, the integral identities obtained from $Sp_{2n} \times Sp_{2n} \subset Sp_{4n}$ produce standard L-functions for Sp_{2n}, giving the symmetric square for GL_2 as a special case. This computation is done in an appendix.

The same general argument applies to classical groups and their standard L-functions. Indeed, it is useful to note that the twist $Sp^*(\Phi)$ of Sp_{2n} consisting of isometries of a rationally anisotropic skew-quaternion form Φ has *compact* arithmetic quotients, avoiding certain problems of regularization if desired.

The *initial* form of the spectral identity relates a sum of second integral moments of all automorphic forms on Sp_{2n} to a sum over automorphic forms on Sp_{4n} of *global* integrals. Due to vanishing of $Sp_{2n} \times Sp_{2n}$ periods, the expansion on Sp_{4n} involves only automorphic forms generating *degenerate* principal series at finite primes. [2]

We give two archimedean *deformations*[3] of the initial spectral relation.

1. The spectral identity
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The *general recipe* includes the case of interest as follows. Let G be a reductive group defined over a number field k, and H a k-subgroup of G, assumed without loss of generality to contain the center Z of G. Consider two chains of subgroups inside $G \times G$,

$$\begin{array}{ccccc} H^\Delta & \subset & H \times H & \subset & G \times G \\ H^\Delta & \subset & G^\Delta & \subset & G \times G \end{array}$$

where the superscript Δ denotes diagonal copies. Pictorially, this is

$$\begin{array}{ccccc} & & G \times G & & \\ & \nearrow & & \nwarrow & \\ H \times H & & & & G^\Delta \\ & \nwarrow & & \nearrow & \\ & & H^\Delta & & \end{array}$$

[1] One can be more precise about this, and discuss how various notions of *packet* behave under restriction.

[2] This is literally true of *cuspforms*, namely, that the only ones appearing in the spectral expansion of the initial distribution or certain of its deformations are degenerate principal series attached to the Siegel parabolic. Typically, degenerate principal series are irreducible, and this is true of *unitary* ones generally. Proof of this follows from an argument similar to Casselman's treatment of the Borel-Matsumoto theorem in his 1980 Compositio paper.

[3] More precisely, the initial relation is a limiting case of the deformations, as the Dirac delta on the real line is a limiting case of suitably renormalized integration against $1/(1+x^2)^s$ as $s \to +\infty$.

Consider the *initial distribution* u on suitable [4] automorphic forms on $G \times G$ defined by

$$u(f_1 \otimes f_2) \;=\; \int_{Z_{\mathbb{A}}^{\Delta} H_k^{\Delta} \backslash H_{\mathbb{A}}^{\Delta}} f_1 \otimes f_2 \;=\; \int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} f_1 \cdot f_2$$

where Z is the center of G. The spectral expansion of this diagonal initial distribution along $H \times H$ is

$$u \circ \mathrm{Res}^{G \times G}_{H \times H} \;=\; \mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } H} F \otimes \overline{F}$$

where F runs over what *would* be an orthonormal basis if the decomposition were discrete, but in general must include continuous-spectrum contributions. On the other hand, the spectral expansion of this diagonal distribution along G^{Δ} is

$$u \circ \mathrm{Res}^{G \times G}_{G^{\Delta}} \;=\; \mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } G^{\Delta}} F_H \cdot F$$

where $u(F) = F_H$ is the *period* of F along H.

Let f be an automorphic form on G, with contragredient $f^{\vee}$. The general recipe gives [5]

$$\mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } H} |\langle f, F \rangle_H|^2 \;=\; u(f \otimes f^{\vee}) \;=\; \mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } G} F_H \;\cdot\; \langle F,\, |f|^2 \rangle_G$$

Diagrammatically, this is

$$\begin{array}{ccccc}
\text{(moment side)} & & & & \text{(spectral side)} \\[1ex]
\displaystyle\mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } H} |\langle f, F \rangle_H|^2 & \longleftarrow & \begin{array}{c} f \otimes f^{\vee} \\ G \times G \end{array} & \longrightarrow & \displaystyle\mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } G} F_H \;\cdot\; \langle F,\, |f|^2 \rangle_G \\[1ex]
\uparrow & \nearrow & & \nwarrow & \uparrow \\[1ex]
\displaystyle\mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } H} F \otimes \overline{F} & H \times H & & G^{\Delta} & \displaystyle\mathop{\sum\!\!\!\!\!\!\int}_{F \text{ on } G^{\Delta}} F_H \cdot F \\[1ex]
\uparrow & \nwarrow & & \nearrow & \uparrow \\[1ex]
 & & \begin{array}{c} H^{\Delta} \\ u \sim 1 \end{array} & &
\end{array}$$

The *positivity* of the left-hand side is a virtue of this relation. The weakness of this *initial* identity is that the archimedean contributions in the left side will make the summands converge *too well*, being of exponential decrease. We *deform* u in order to extract more information.

We call a *Poincaré series* any deformation of the initial distribution u to (integration against) a classical function on $Z_{\mathbb{A}}^{\Delta} G_k^{\Delta} \backslash G_{\mathbb{A}}^{\Delta}$. One natural *non-elementary* deformation is as follows. Let v_0 be archimedean,

[4] The indicated integral literally converges at least for *cuspforms*, and for wave packets of Eisenstein series with cuspidal data. If the spectral coefficients of a packet of Eisenstein series are extremely smooth, then the packet will be of rapid decay. The L^2 spectral decomposition of automorphic forms behaves well with respect to restriction to various notions of *Schwartz spaces* of automorphic forms. Thus, via duality, suitably *tempered* automorphic distributions admit spectral decompositions.

[5] In general, since u will not have compact support, this evaluation has an immediate sense only for f in a suitable Schwartz space. That is, only *packets* of Eisenstein series allow literal evaluation of the functional. Nevertheless, suitable *regularization* can extend the domain of the functional. Further, *deformation* of the initial distribution to a classical function already effectively extends the functional to a degree.

and let Ω be Casimir on G_{v_0}. Let $\lambda \in \mathbb{C}$, and consider the (distributional) partial differential equation on $H_{v_0}\backslash G_{v_0}$

$$(\Omega - \lambda)\,\beta^\lambda \;=\; u$$

where β^λ is left H_{v_0}-invariant and right K_{v_0}-invariant. Assume there is a locally integrable solution [6] β^λ with sufficient decay at infinity. For simplicity, suppose that there is a unique archimedean place v_0 of k, and that β^λ solves the previous equation on $H_{v_0}\backslash G_{v_0}$. For $v \neq v_0$, let

$$\varphi_v(g) \;=\; \begin{cases} 1 & (\text{for } g \in H_v \cdot K_v) \\ \\ 0 & (\text{off } H_v \cdot K_v) \end{cases}$$

where K_v is a maximal compact subgroup of G_v. Let

$$\varphi^\lambda(g) \;=\; \beta^\lambda(g_{v_0}) \cdot \prod_{v \neq v_0} \varphi_v(g_v)$$

Form [7] the *Poincaré series*

$$\text{Pé}^\lambda(g) \;=\; \sum_{\gamma \in H_k\backslash G_k} \varphi^\lambda(\gamma \cdot g)$$

Compute the *spectral* components [8] of Pé^λ as follows. Take a spherical automorphic form F on G with eigenvalue λ_F for Ω on G_{v_0}. Unwinding as usual, and integrating by parts at v_0, the F^{th} spectral component of Pé^λ is

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} F \cdot \text{Pé}^\lambda \;=\; \int_{Z_{\mathbb{A}}H_k\backslash G_{\mathbb{A}}} F \cdot \varphi^\lambda \;=\; \int_{Z_{\mathbb{A}}H_k\backslash G_{\mathbb{A}}} \frac{\Omega - \lambda}{\lambda_F - \lambda}\, F \cdot \varphi^\lambda \;=\; \int_{Z_{\mathbb{A}}H_k\backslash G_{\mathbb{A}}} F \cdot \frac{\Omega - \lambda}{\lambda_F - \lambda}\,\varphi^\lambda$$

$$= \int_{Z_{\mathbb{A}}H_k\backslash G_{\mathbb{A}}} F \cdot \frac{1}{\lambda_F - \lambda}\,\big(u \otimes \bigotimes_{v \neq v_0} \varphi_v\big) \;=\; \frac{1}{\lambda_F - \lambda} \cdot \int_{Z_{\mathbb{A}}H_k\backslash H_{\mathbb{A}}} F \;=\; \frac{u(F)}{\lambda_F - \lambda} \;=\; \frac{F_H}{\lambda_F - \lambda}$$

Visibly, this F^{th} spectral coefficient has a pole [9] at $\lambda = \lambda_F$.

On the other hand, compute the *moment expansion* as follows. For f on G with contragredient $f^\vee$, do an initial unwinding

$$\langle \text{Pé}, |f|^2 \rangle \;=\; \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \text{Pé}^\lambda \cdot |f|^2 \;=\; \int_{Z_{\mathbb{A}}H_k\backslash G_{\mathbb{A}}} \varphi^\lambda \cdot |f|^2 \;=\; \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi^\lambda(g) \int_{Z_{\mathbb{A}}H_k\backslash H_{\mathbb{A}}} f(hg)\, f^\vee(hg)\; dh\; dg$$

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[6] Smoothness of a solution of $(\Omega - \lambda)\,\beta = u$ away from the singular support H_{v_0} of u follows from hypoellipticity of Ω on G_{v_0}/K_{v_0}.

[7] Often this sum will not converge classically, requiring regularization via analytic continuation in a further auxiliary parameter, but this further deformation is relatively elementary, and we suppress it here.

[8] To compute spectral components, the obvious heuristic is to pretend that everything is L^2 and compute following Selberg, Langlands, Arthur, Jacquet Moeglin-Waldspurger, *et alia.* However, most interesting deformations are not L^2. Indeed, with $G = GL_n$ and $H = GL_{n-1}$, for $n \geq 3$, deformations as here have *no* genuine L^2 components remaining after *singular* components are removed. This can be remedied by a further deformation, identifying the Poincaré series as an iterated residue of an object *with* cuspidal spectral components. Luckily, in the examples considered here, this additional device is unnecessary for understanding the spectral decomposition, though additional regularization may be convenient.

[9] For cuspforms F this does promise a genuine pole in the spectral expansion.

since φ^λ is left $H_{\mathbb{A}}$-invariant. Expand $f(hg)$ along H, as

$$f(hg) \;=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} F(h)\,\Big(\int_{Z_{\mathbb{A}}H_k\backslash H_{\mathbb{A}}} f(\eta g)\,\overline{F}(\eta)\,d\eta\Big)\,d\eta \;=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} F(h)\cdot \langle g\cdot f,\, F\rangle_H$$

where the action of $g \in G_{\mathbb{A}}$ on functions f is by right translation:

$$\big(g\cdot f\big)(h) \;=\; f(hg)$$

Thus,

$$\begin{aligned}\langle \text{Pé}, |f|^2\rangle \;&=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi^\lambda(g) \int_{Z_{\mathbb{A}}H_k\backslash H_{\mathbb{A}}} F(h)\cdot\langle g\cdot f,\, F\rangle_H \cdot f^\vee(hg)\,dh\,dg \\ &=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi^\lambda(g)\cdot |\langle g\cdot f,\, F\rangle_H|^2\,dg\end{aligned}$$

Because φ^λ is significantly deformed only at the single archimedean place v_0, in the integral over $H_{\mathbb{A}}\backslash G_{\mathbb{A}}$ the adele group element $g = \{g_v\}$ can be taken in H_v except at the single place v_0. Thus,

$$\langle \text{Pé}, |f|^2\rangle \;=\; \sum\!\!\!\!\!\!\!\!\int_{F \text{ on } H} \int_{H_{v_0}\backslash G_{v_0}} \beta^\lambda(g_{v_0})\cdot |\langle g_{v_0}\cdot f,\, F\rangle_H|^2\,dg_{v_0}$$

The specific structure of the case $H = Sp_{2n}\times Sp_{2n}$ and $G = Sp_{4n}$ with the direct-sum imbedding allows an an *elementary* further unwinding [10] when f is an Eisenstein series

$$E(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \varepsilon(\gamma\cdot g)$$

induced from a one-dimensional character of the Siegel parabolic [11] P, with $\varepsilon = \bigotimes_v \varepsilon_v$ spherical in the appropriate induced representation. Among the finitely-many double cosets $P_k\backslash G_k/H_k$, there is a unique [12] *cuspidal*[13] double coset, $P\xi H$, with

$$\xi \;=\; \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1\end{pmatrix}$$

with n-by-n blocks. With this particular choice, the isotropy subgroup is

$$\Theta \;=\; \xi^{-1}P\xi\cap H \;=\; \{(g, g^\sigma)\;:\; g\in Sp_n\} \;\approx\; Sp_n$$

[10] In the anomalously simple case that $G = GL_n$ and $H = GL_{n-1}$, global Whittaker-Fourier expansions for *cuspforms* (at least) allow the integrals $\langle g\cdot f, F\rangle_H$ to be unwound to products of local integrals as in the Rankin-Selberg convolution integrals for $GL_n\times GL_{n-1}$. Then the deformation at v_0 has an impact visibly confined to the local integral at v_0. At the other extreme, in many interesting situations, an unwinding produces not only an Euler product of local integrals, but also a global integral, a *period*. The present scenario, so-called *doubling*, has no period on the moment side of the relation.

[11] The parabolic stabilizing the standard maximal totally isotropic subspace of the symplectic space is the *Siegel* or *popular* parabolic.

[12] This double-coset computation is non-trivial, but by now standard.

[13] With arbitrary parabolic P and subgroup H, a double coset PxH is *cuspidal*, or *non-negligible*, when $P\cap xHx^{-1}$ contains *no* unipotent radical of any parabolic of G as a normal subgroup.

where, in n-by-n blocks,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

since all the other integrals vanish due to the Gelfand cuspform condition [14] on F. Thus, with cuspform F, the usual unwinding gives

$$\langle g_{v_0} \cdot E,\, F\rangle_H = \int_{H_k\backslash H_{\mathbb{A}}} \sum_{\gamma\in P_k\backslash G_k} \varepsilon(\gamma \cdot hg_{v_0})\, \overline{F}(h)\, dh$$

$$= \sum_{x\in P_k\backslash G_k/H_k} \int_{H_k\backslash H_{\mathbb{A}}} \sum_{\eta\in(x^{-1}P_kx\cap H_k)\backslash H_k} \varepsilon(x\eta \cdot hg_{v_0})\, \overline{F}(h)\, dh$$

$$= \sum_{x\in P_k\backslash G_k/H_k} \int_{Z_{\mathbb{A}}(x^{-1}P_kx\cap H_k)\backslash H_{\mathbb{A}}} \varepsilon(x \cdot hg_{v_0})\, \overline{F}(h)\, dh = \int_{\Theta_k\backslash H_{\mathbb{A}}} \varepsilon(\xi \cdot hg_{v_0})\, \overline{F}(h)\, dh$$

In fact, in this example, $h \to \varepsilon(\xi hg_{v_0})$ is left $\Theta_{\mathbb{A}}$-invariant. Thus, the integral can be rewritten as

$$\langle g_{v_0} \cdot E,\, F\rangle_H = \int_{\Theta_{\mathbb{A}}\backslash H_{\mathbb{A}}} \varepsilon(\xi \cdot hg_{v_0}) \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} \overline{F}(\theta h)\, d\theta\, dh = \int_{\Theta_{\mathbb{A}}\backslash H_{\mathbb{A}}} \varepsilon(\xi \cdot hg_{v_0}) \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} \overline{F}(\theta h)\, d\theta\, dh$$

Write $\overline{F} = f_1 \otimes f_2$ with f_i on Sp_{2n}, and take representatives $\{x \times 1 : x \in Sp_{2n}\}$ for $\Theta_{\mathbb{A}}\backslash H_{\mathbb{A}}$. This is

$$\langle g_{v_0} \cdot E,\, F\rangle_H = \int_{Sp_{2n}(\mathbb{A})} \varepsilon(\xi(x\times 1)g_{v_0}) \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} f_1(\theta x)\, f_2(\theta)\, d\theta\, dx$$

The order of integration can be reversed, giving

$$\langle g_{v_0} \cdot E,\, F\rangle_H = \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} f_2(\theta) \cdot \Big(\int_{Sp_{2n}(\mathbb{A})} \varepsilon(\xi(x\times 1)g_{v_0})\, f_1(\theta x)\, dx\Big)\, d\theta$$

Since g_{v_0} has non-trivial component only at the archimedean place v_0, the inner integral is a product of local operators coming from the functions [15]

$$\eta_v(x) = \varepsilon_v(\xi(x\times 1))$$

on $Sp_{2n}(k_v)$ for $v \neq v_0$. At almost all places v, the function ε_v is right $Sp_{4n}(\mathfrak{o}_v)$-invariant. Then the left invariance by Θ_v implies that

$$\eta_v(a\cdot x\cdot b) = \eta_v(x) \qquad \text{(for all } a, b \in Sp_{2n}(\mathfrak{o}_v)\text{, for } x \in Sp_{2n}(k_v)\text{, for almost all } v)$$

At such places v, if $\overline{F} = f_1 \otimes f_2$ has irreducible right K_v-type other than *spherical*, then $\langle g_{v_0} \cdot E,\, F\rangle_H = 0$, while spherical F generating spherical representation π_v at v gives

$$\int_{Sp_{2n}(k_v)} \varepsilon_v(\xi(x_v\times 1))\, f_1(\theta x_v)\, dx_v = \lambda_v \cdot f_1(\theta)$$

[14] When $P \cap \xi H\xi^{-1}$ has a normal subgroup N a unipotent radical of a k-parabolic in G, the corresponding integral of a cuspform vanishes, by Gelfand's condition $\int_{N_k\backslash N_{\mathbb{A}}} F(ng)\, dn = 0$.

[15] These functions are not compactly supported, so are not in the usual spherical Hecke algebra. Nevertheless, in the region of convergence of the Eisenstein series E they give convergent integrals.

with eigenvalue [16] λ_v depending only upon the isomorphism class of π_v. By contrast, at v_0, the right translation by g_{v_0} disrupts the right K-types at v_0 of either the Eisenstein series E or the spectral components $F = \overline{f}_1 \otimes \overline{f}_2$. Nevertheless, in the worst case, these integral operators cannot move f_1 outside the irreducible representation we assume it generates.

In particular, if $\overline{f}_2$ is not inside the representation generated by f_1, then the integral is 0. If there were no right translation by g_{v_0}, we could control this, and effectively take $f_2 = \overline{f}_1$. However, with the right translation by g_{v_0} present we must include an infinite sum over (probably) all the K-types in the archimedean representation generated by f_1.

That is, with f_1 taken K_{v_0}-finite, certainly $f_2 \in U(\mathfrak{g}) \cdot \overline{f}_1$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, the Lie algebra of $Sp_{2n}(k_{v_0})$.

For convenience, we continue to suppose that there is a single archimedean place v_0 at which the deformation takes place, and, further, that at all finite places the Eisenstein series E is *spherical*. [17] Then the previous remarks show that the only cuspforms $F = \overline{f}_1 \otimes \overline{f}_2$ appearing are spherical at finite places. So far,

$$\langle g_{v_0} \cdot E,\, F\rangle_H \;=\; \prod_{v<\infty} \lambda_v \;\cdot\; \int_{\Theta_k\backslash\Theta_{\mathbb{A}}} f_2(\theta) \cdot \int_{Sp_{2n}(k_{v_0})} \varepsilon_{v_0}(\xi(x_{v_0} \times 1) g_{v_0})\; f_1(\theta x_{v_0})\; dx_{v_0}\; d\theta$$

where the local integrals for λ_v are computed for $n = 1$ in the appendix. That is, except for the ${v_0}^{th}$ local integral, this is the integral of f_1 against f_2.

Replacing the initial distribution u by the λ^{th} deformation Pé^λ, evaluating in two ways,

$$(\text{... moment side...}) \;=\; \langle \text{Pé}^\lambda,\, |E|^2\rangle_G \;=\; \sum\!\!\!\!\!\!\int_{F \text{ on } G} \frac{F_H}{\lambda_F - \lambda} \;\cdot\; \langle F, |E|^2\rangle_G$$

Note that the right-hand side has singularities [18] at eigenvalues of elements of the discrete spectrum that have non-vanishing periods along H.

The example $H = GL_{n-1} \times GL_1$ inside $G = GL_n$ is anomalously simpler, in that H_v is a Levi component of a parabolic in G_v, so a standard Iwasawa decomposition itself already gives useful transverse coordinates along which to deform an initial distribution supported on H_v.

1. *Appendix: normalization of L-functions*

The classical description of the L-function attached to a holomorphic modular form

$$f_0(z) \;=\; \sum_{n\geq 1} a_n\, e^{2\pi i n z}$$

of level 1 and of weight $\kappa \in 2\mathbb{Z}$ on the upper half-plane is

$$\Lambda(s, f_0) \;=\; \int_0^\infty y^s\; f_0(iy)\, \frac{dy}{y} \;=\; (2\pi)^{-s}\, \Gamma(s)\, \sum_{n\geq 1} \frac{a_n}{n^s}$$

[16] The v^{th} eigenvalue is the v^{th} local factor of the standard Sp_{2n} L-function for f_1. We verify this explicitly for $n = 1$ in an appendix.

[17] The local data defining the Eisenstein series E on Sp_{4n} can be arranged to accommodate or detect any fixed right K_v-type on Sp_{2n}, but this is not the point here. Similarly, the possibilities for differing deformations at various archimedean places are not the point.

[18] It is easy, almost inevitable in any serious situation, for these eigenvalues to have many accumulation points. Proof of this presumably requires some trace-formula considerations.

The functional equation $f_0(-1/z) = z^\kappa \cdot f_0(z)$ of f_0 gives the corresponding functional equation

$$\Lambda(\kappa - s, f_0) \;=\; \Lambda(s, f_0)$$

For various reasons, a normalization that gives a functional equation $s \longleftrightarrow 1-s$ is more convenient. This is *almost* accomplished by thinking in terms of the associated automorphic form f on the Lie group, in this case given by

$$f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; y^{\kappa/2} \cdot f_0(x+iy)$$

If we were to take the Mellin transform of this, the functional equation would be with respect to $s \longleftrightarrow -s$, which would be better, in that it would depend less upon the specific local data, but still would obscure the notion of *critical strip* for the L-function. Therefore, the modern normalization is

$$\Lambda(s,f) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\, f(iy)\, \frac{dy}{y} \;=\; \int_0^\infty y^{s-\frac{1}{2}+\frac{\kappa}{2}}\, f_0(iy)\, \frac{dy}{y} \;=\; (2\pi)^{s-\frac{1}{2}+\frac{\kappa}{2}}\, \Gamma(s-\tfrac{1}{2}+\frac{\kappa}{2}) \sum_{n]ge1} \frac{a_n}{n^{s-\frac{1}{2}+\frac{\kappa}{2}}}$$

The *standard* L-function[19] attached to a cuspform f on GL_2 over a number field k, including the gamma factor, is given by the Mellin transform

$$\Lambda(s,f) \;=\; \int_{\mathbb{J}/k^\times} |y|^{s-\frac{1}{2}}\, f\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^\times y \;=\; \int_{\mathbb{J}} |y|^{s-\frac{1}{2}}\, W_f \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^\times y$$

where W_f is the global Whittaker function for f, namely,

$$W_f(g) \;=\; \int_{\mathbb{A}} f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\, dx$$

In this normalization, the L-function has a functional equation under $s \longleftrightarrow 1-s$. Uniqueness of local Whittaker models implies that W_f factors over primes $W_f = \bigotimes_v W_v$. Thus, letting π_v denote the (irreducible) representation of $GL_2(k_v)$ generated by f, the v^{th} Euler factor of $\Lambda(s,f)$ is given by the local Mellin transform

$$L_v(s,\pi_v) \;=\; \int_{k_v^\times} |y|^{s-\frac{1}{2}}\, W_v \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^\times y$$

For example, for $k_v \approx \mathbb{R}$, for a holomorphic discrete series representation π_v of weight $\kappa \in 2\mathbb{Z}$, the Whittaker function for the lowest K_v-type is

$$W_v \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; y^{\kappa/2}\, e^{-2\pi y} \qquad\qquad (\text{for } y > 0)$$

Thus, the local L-function (actually a gamma factor) in this normalization is

$$L_v(s,\pi_v) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\, y^{\kappa/2}\, e^{-2\pi y} \frac{dy}{y} \;=\; \int_0^\infty y^{s+\frac{\kappa-1}{2}}\, e^{-2\pi y} \frac{dy}{y} \;=\; (2\pi)^{-(s+\frac{\kappa-1}{2})}\, \Gamma(s+\frac{\kappa-1}{2})$$

At spherical finite places v, the local Whittaker function is given by the easiest case of the Shintani-Saito-Casselman-Shalika formula,

$$W_v(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) \;=\; \begin{cases} \psi(x) \cdot \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & (\text{for } n = \operatorname{ord} y \geq 0) \\ 0 & (\text{for } n = \operatorname{ord} y < 0) \end{cases}$$

[19] This integral is most properly termed a *zeta integral*, rather than *L-function*, since only an optimal choice of cuspform within an irreducible gives good local factors, especially at bad primes. The discussion of finite bad primes is not the point here.

where $\alpha\beta = 1/q$, with q the residue field cardinality, where ψ is the fixed additive character specifying the Whittaker model, and we assume that W_v has trivial central character. Thus, at good finite primes

$$L_v(s, \pi_v) \;=\; \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} q^{-n(s-\frac{1}{2})} (\alpha^{n+1} - \beta^{n+1})$$

$$= \; \frac{1}{\alpha - \beta} \cdot \Big(\frac{\alpha}{1 - \alpha q^{-(s-\frac{1}{2})}} - \frac{\beta}{1 - \beta q^{-(s-\frac{1}{2})}} \Big) \;=\; \frac{1}{\big(1 - \alpha q^{-(s-\frac{1}{2})}\big)\big(1 - \beta q^{-(s-\frac{1}{2})}\big)}$$

If we write the local L-factor in the form

$$L_v(s, \pi_v) \;=\; \frac{1}{\big(1 - A\, q^{-s}\big)\big(1 - B\, q^{-s}\big)}$$

then it must be that

$$A \;=\; q^{\frac{1}{2}} \alpha \qquad\qquad B \;=\; q^{\frac{1}{2}} \beta \qquad\qquad \text{(up to permutations)}$$

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2. *Appendix: computation of local integrals*

This appendix verifies that the non-archimedean local integrals of cuspforms $f \otimes f^\vee$ on $SL_2 \times SL_2$ against the restriction of a Siegel-type Eisenstein series on Sp_4 are the local factors of $L(s, \mathrm{Sym}^2 f)$, up to more-elementary normalizing factors. A similar computation is done for Eisenstein series, to be sure of normalizations.

Let v be a non-archimedean place of k. The naive normalization I_s^{nf} of the s^{th} degenerate principal series of $G_v = Sp_{2n}(k_v)$ includes smooth functions f with the left equivariance

$$f\Big(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \cdot g \Big) \;=\; \chi_s \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \cdot f(g) \qquad\qquad \Big(\text{where } \chi \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} = |\deg a|^s \Big)$$

Let ε be the spherical function in I_s^{nf}. That is, ε is right $K_v = Sp_4(\mathfrak{o}_v)$-invariant, and $\varepsilon(1) = 1$. For f on G_v generating a spherical (irreducible) representation π_v of G_v, the integral

$$\int_{G_v} f(xh) \cdot \varepsilon(\xi(h \times 1))\, dh$$

with

$$\xi \;=\; \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(when convergent) is necessarily a constant $\lambda_v(s, \pi_v)$ (depending upon s and π_v) multiple of $f(x)$, since the subspace of spherical vectors in the spherical representation π_v is one-dimensional. This constant is *intrinsic*, in that it depends only upon the isomorphism class of π_v, so it can be computed via any model of the spherical representation π_v.

Take $f = W$. Since $f(1) = 1$, the constant $\lambda_v(s, \pi_v)$ is

$$\lambda_v(s, \pi_v) \;=\; \int_{G_v} f(h) \cdot \varepsilon(\xi(h \times 1))\, dh$$

Compute this integral via Iwasawa coordinates in $SL_2(k_v)$

$$h \;=\; n_x m_y \theta \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \theta \qquad\qquad (\text{with } \theta \in SL_2(\mathfrak{o}_v))$$

By Witt's theorem, with the Siegel parabolic P_v in $Sp_4(k_v)$,

$$P_v\backslash G_v \;\approx\; \{\text{maximal totally isotropic subspaces}\} \;\approx\; GL_2(k_v)\backslash\{\text{lower halves of elements of } Sp_4(k_v)\}$$

Thus, compute with the lower half of ξ, namely

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot (n_x m_y \times 1) \;=\; \begin{pmatrix} y & 1 & x/y & 0 \\ 0 & 0 & 1/y & -1 \end{pmatrix}$$

For $W(m_y)$ to be non-zero, $\operatorname{ord} y \geq 0$. Thus,

$$\begin{pmatrix} 1 & 0 & & \\ -y & 1 & & \\ & & 1 & y \\ & & 0 & 1 \end{pmatrix} \;\in\; Sp_4(\mathfrak{o}_v)$$

Right multiplication by this changes neither the value of the spherical Whittaker function nor the value of ε, and puts the lower half of $\xi(n_x m_y \times 1)$ into the form

$$\begin{pmatrix} 0 & 1 & x/y & x \\ 0 & 0 & 1/y & 0 \end{pmatrix}$$

Left multiplication by

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \;\in\; SL_2(k_v)$$

(effectively in the kernel of the character defining the degenerate principal series) gives

$$\begin{pmatrix} 0 & 1 & 0 & x \\ 0 & 0 & 1/y & 0 \end{pmatrix}$$

After a further permutation of rows and columns, we see

$$\xi \cdot (n_x m_y \times 1) \;=$$

Thus,

$$\ker(\chi_s) \cdot \xi \cdot (n_x m_y \times 1) \cdot Sp_{4n}(\mathfrak{o}_v) \;\ni\; \begin{cases} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/y \end{pmatrix} & (\text{for } x \in \mathfrak{o}_v) \\[2em] \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1/y \end{pmatrix} & (\text{for } x \notin \mathfrak{o}_v) \end{cases}$$

Since the lower right 2-by-2 block determines the upper left by inverting and taking transpose,

$$\varepsilon(\xi \cdot (n_x m_y \times 1)) \;=\; \begin{cases} |y|^s & (\text{for } x \in \mathfrak{o}_v) \\[1em] |y/x|^s & (\text{for } x \notin \mathfrak{o}_v) \end{cases}$$

Thus, the local integral is

$$\int_{\operatorname{ord} y \geq 0} |y|^s \cdot W\begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix} \frac{1}{|y|}\, d^\times y \;\;\cdot\;\; \Big(1 + \int_{\operatorname{ord} x < 0} \psi(x) \cdot |x|^{-s}\, dx\Big)$$

The additive character ψ is trivial on the local integers $\mathfrak{o}_v$ and non-trivial on $\varpi^{-1}\mathfrak{o}_v$ where ϖ is a local parameter at v. Let q be the cardinality of the residue field, and let $\mathfrak{o}_v$ have total measure 1. Note that for $\operatorname{ord} x < -1$, the function $u \to \psi(x \cdot (1 + \varpi u))$ is a non-trivial character on $u \in \mathfrak{o}_v$, while $|x(1 + \varpi u)| = |x|$. Thus, the integral in x can be computed on $\operatorname{ord} x = -1$. The integral of ψ over $\varpi^{-1}\mathfrak{o}_v$ would be 0, but we are missing $\mathfrak{o}_v$, so the integral of ψ over $\operatorname{ord} x = -1$ is -1. Thus, the integral over x is $-q^{-s}$. Since $\alpha\beta = 1/q$, the whole local integral is

$$\sum_{n\geq 0} q^{-n(s-1)} \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \cdot \Big(1 - q^{-s}\Big) = \frac{1}{\alpha - \beta}\Big(\frac{\alpha}{1 - \alpha^2 q^{-(s-1)}} - \frac{\beta}{1 - \beta^2 q^{-(s-1)}}\Big) \cdot \Big(1 - q^{-s}\Big)$$

$$= \frac{1 + \alpha\beta q^{-(s-1)}}{(1 - \alpha^2 q^{-(s-1)})\,(1 - \beta^2 q^{-(s-1)})} \cdot (1 - q^{-s}) = \frac{(1 - \alpha^2\beta^2 q^{-2(s-1)})\,(1 - q^{-s})}{(1 - \alpha^2 q^{-(s-1)})\,(1 - \alpha\beta q^{-(s-1)})\,(1 - \beta^2 q^{-(s-1)})}$$

$$= \frac{(1 - q^{-2s})\,(1 - q^{-s})}{(1 - A^2 q^{-s})\,(1 - AB q^{-s})\,(1 - B^2 q^{-s})}$$

with $A = q^{\frac{1}{2}}\alpha$ and $B = q^{\frac{1}{2}}\beta$ as above. This is

$$L_v(s, \operatorname{Sym}^2 f) \cdot \frac{1}{\zeta_v(2s) \cdot \zeta_v(s)}$$

The naively normalized Siegel-type Eisenstein series E_s on Sp_{2n} (where the index indicates the size of the matrices) attached to the s^{th} degenerate principal series has functional equation relating E_s and $E_{(n+1)-s}$. Thus, for $n = 2$, the relation is between E_s and E_{3-s}. That is, apart from the renormalization by the zeta factors, $L(s, \operatorname{Sym}^2 f)$ is related to $L(3 - s, \operatorname{Sym}^2 f)$.

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3. *Appendix: local integrals for Eisenstein series*

A similar local computation arises in computation of the continuous spectrum components on $SL_2 \times SL_2$ of suitably adjusted [20] Eisenstein series on Sp_4, but the Whittaker function W^E of Eisenstein series is normalized differently, as follows. Suppress subscripts by letting k be a non-archimedean local field with ring of integers $\mathfrak{o}$. The global Eisenstein series is locally an image of the naively normalized principal series consisting of functions φ on $GL_2(k)$ with the equivariance

$$\varphi(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g) = |a/d|^\mu \cdot \varphi(g)$$

Take the normalized spherical φ, namely, also right $K = GL_2(\mathfrak{o})$-invariant and $\varphi(1) = 1$. Then the natural normalization of the local factor of the Fourier coefficient (Whittaker function) of the Eisenstein series is the integral transform [21]

$$W^E(g) = \int_N \overline{\psi}(n)\, \varphi(w \cdot n \cdot g)\, dn \qquad \text{(where } N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}\text{)}$$

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[20] A naively normalized Eisenstein series E_s on Sp_4 can be adjusted so that on $\operatorname{Re}(s) > 3/2$ its restriction to $SL_2 \times SL_2$ has decay at infinity, without changing the *level*. This device allows computation of continuous-spectrum components by integration against Eisenstein series. An equivalent effect is achieved, with somewhat different details, by subtracting the Eisenstein series $E_s \otimes E_s$ on $SL_2 \times SL_2$ from the restriction, before computing the spectral projection. More generally, decomposition of the restricted Eisenstein series as a *tempered distribution* legitimizes and shows the essential equivalence of all such devices.

[21] This is literally a naively normalized version of computation of the spherical Whittaker function for an unramified principal series.

It suffices to evaluate

$$W^E(m_y) \;=\; \int_k \overline{\psi}(x)\,\varphi(w\cdot n_x\cdot m_y)\,dx \qquad\qquad \left(\text{where } n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

The plan of the computation is as follows. Unless y is integral, local cancellation due to ψ will cause the integrand to vanish entirely. For y integral, there is still a local cancellation effect for $\operatorname{ord} x$ large negative. At the edge of this regime, some cancellation occurs without annihilating the integrand entirely. Thus, the integral will be equal to a finite geometric series with altered edge terms.

First,

$$w\cdot n_x\cdot m_y \;=\; w\cdot m_y\cdot n_{x/y} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\cdot w\cdot \begin{pmatrix} 1 & x/y \\ 0 & 1 \end{pmatrix}$$

Thus, because of the trivial central character,

$$\varphi(w\cdot n_x\cdot m_y) \;=\; |1/y|^{\mu}\cdot\varphi(w\cdot n_{x/y})$$

and

$$W(m_y) \;=\; |y|^{-\mu}\cdot\int_k \overline{\psi}(x)\,\varphi(w\cdot n_{x/y})\,dx \;=\; |y|^{1-\mu}\cdot\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx$$

by replacing x by xy. For $y\notin\mathfrak{o}$, the character $x\to\overline{\psi}(xy)$ is non-trivial on $\mathfrak{o}$. On the other hand, $n_t\in K$ for $t\in\mathfrak{o}$, and φ is right K-invariant, so

$$\varphi(w\cdot n_x\cdot n_t) \;=\; \varphi(w\cdot n_x) \qquad\qquad (\text{for } t\in\mathfrak{o})$$

We have a standard vanishing argument by change of variables:

$$\int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x)\,dx \;=\; \int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x{\cdot}n_t)\,dx \;=\; \int_k \overline{\psi}((x{-}t)y)\,\varphi(w{\cdot}n_x)\,dx \;=\; \psi(ty)\int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x)\,dx$$

by replacing x by $x-t$. Since $y\notin\mathfrak{o}$, there is $t\in\mathfrak{o}$ such that $\psi(ty)\neq 1$. Thus,

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; 0 \qquad\qquad (\text{for } y\notin\mathfrak{o})$$

For $y\in\mathfrak{o}$ compute $\varphi(w\cdot n_x)$ via the p-adic Iwasawa decomposition of wn_x: right modulo K,

$$w\cdot n_x \;=\; \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \;=\; \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \;=\; \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} & (\text{for } \operatorname{ord} x\leq 0) \\ \;=\; 1 & (\text{for } \operatorname{ord} x\geq 0) \end{cases}$$

Using triviality of the central character, the convention that $\mathfrak{o}$ has measure 1, break the integral over $k-\mathfrak{o}$ into $\mathfrak{o}^\times$ orbits:

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy)\cdot 1\,dx + \int_{k-\mathfrak{o}} \overline{\psi}(xy)\,|x^{-2}|^{\mu}\,dx$$

For fixed $y\in\mathfrak{o}$, for $\operatorname{ord} xy<0$, the map

$$x\to x\cdot(1+\varpi u) \qquad\qquad (\text{with } u\in\mathfrak{o})$$

leaves $\varphi(wn_x)$ invariant, but

$$\overline{\psi}(x(1+\varpi u)y) \;=\; \overline{\psi}(xy)\cdot\overline{\psi}(xy\varpi\cdot u)$$

Since $xy\varpi\notin\mathfrak{o}$, the character

$$u\to\overline{\psi}(xy\varpi\cdot u) \qquad\qquad (\text{for } u\in\mathfrak{o})$$

is non-trivial, so the integral in x over such an $(1+\varpi\mathfrak{o})$-orbit must vanish. Thus,

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy)\cdot 1\,dx + \int_{0>\mathrm{ord}x\geq -1-\mathrm{ord}\,y} \overline{\psi}(xy)\,|x|^{-2\mu}\,dx$$

There is no cancellation due to ψ except when $\mathrm{ord}\,xy=-1$, so

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; \int_{\mathfrak{o}} 1\,dx + \int_{-\mathrm{ord}\,y\leq \mathrm{ord}x<0} |x|^{-2\mu}\,dx + \int_{\mathrm{ord}x=-(1+\mathrm{ord}\,y)} \overline{\psi}(xy)\,|x|^{-2\mu}\,dx$$

Let $n=\mathrm{ord}\,y$ and q the residue field cardinality. In the last integral, $|x|^{-2\mu}$ is constant, and

$$\int_{\mathrm{ord}x=-(1+\mathrm{ord}\,y)} \overline{\psi}(xy)\,dx \;=\; \int_{\mathrm{ord}x\geq-(1+\mathrm{ord}\,y)} \overline{\psi}(xy)\,dx - \int_{\mathrm{ord}x\geq -\mathrm{ord}\,y} \overline{\psi}(xy)\,dx \;=\; 0-\mathrm{meas}\,(y^{-1}\mathfrak{o}) \;=\; -q^n$$

since the first integral is the integral of a non-trivial character. That is,

$$\int_{\mathrm{ord}x=-(1+\mathrm{ord}\,y)} \overline{\psi}(xy)\,dx \;=\; -q^n\cdot (q^{(1+n)})^{-2\mu}$$

Using the comparison

$$\mathrm{meas}\,(\varpi^{-n}\mathfrak{o}^\times) \;=\; q^n\cdot \frac{q-1}{q}$$

of additive and multiplicative measures,

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; 1+\frac{q-1}{q}\sum_{n=1}^{\mathrm{ord}y} q^n\cdot|\varpi^{-n}|^{-2\mu} \;-\; q^n\cdot q^{-2\mu(n+1)}$$

$$=\; 1+\frac{q-1}{q}\sum_{n=1}^{\mathrm{ord}y}(q^{1-2\mu})^n \;-\; q^n\cdot q^{-2\mu(n+1)}$$

Summing the finite geometric series, this is

$$1+\frac{q-1}{q}\cdot\frac{q^{1-2\mu}-(q^{1-2\mu})^{n+1}}{1-q^{1-2\mu}} \;-\; q^n\cdot q^{-2\mu(n+1)}$$

To see how this should simplify, let $X=q^{1-2\mu}$. The whole is

$$1+\frac{q-1}{q}\cdot\frac{X-X^{n+1}}{1-X} \;-\; \frac{X^{n+1}}{q}$$

$$=\; \frac{q(1-X)+(q-1)(X-X^{n+1})-(1-X)X^{n+1}}{q(1-X)}$$

$$=\; \frac{q-qX+qX-X-qX^{n+1}+X^{n+1}-X^{n+1}+X^{n+2}}{q(1-X)} \;=\; \frac{q-X-qX^{n+1}+X^{n+2}}{q(1-X)}$$

$$=\; \frac{1-\frac{1}{q}X-X^{n+1}+\frac{1}{q}X^{n+2}}{1-X} \;=\; \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X}$$

Also, express $|y|^{1-\mu}$ in terms of X:

$$|y|^{1-\mu} \;=\; (q^{-n})^{1-\mu} \;=\; (q^{-\frac{n}{2}})^{2-2\mu} \;=\; q^{-\frac{n}{2}}\cdot(q^{-\frac{n}{2}})^{1-2\mu} \;=\; q^{-\frac{n}{2}}\cdot X^{-\frac{n}{2}}$$

Thus, for $\mathrm{ord}\, y \geq 0$,

$$W^E(m_y) \;=\; |y|^{1-\mu} \cdot \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X} \;=\; q^{-\frac{n}{2}} \cdot X^{-\frac{n}{2}} \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X}$$

$$=\; (1-\tfrac{1}{q}X) \cdot q^{-\frac{n}{2}} \cdot \frac{X^{-\frac{n+1}{2}} - X^{\frac{n+1}{2}}}{X^{-\frac{1}{2}} - X^{\frac{1}{2}}} \;=\; (1-\tfrac{1}{q}X) \cdot \frac{(1/qX)^{\frac{n+1}{2}} - (X/q)^{\frac{n+1}{2}}}{(1/qX)^{\frac{1}{2}} - (X/q)^{\frac{1}{2}}}$$

$$=\; (1-q^{-2\mu}) \cdot \frac{(q^{\mu-1})^{n+1} - (q^{-\mu})^{n+1}}{q^{\mu-1} - q^{-\mu}}$$

That is, up to switching the two, $\alpha = q^{\mu-1}$ and $\beta = q^{-\mu}$, and there is an extra leading factor of $(1-q^{2\mu})$.

Clearly $\alpha\beta = 1/q$. Then the integral against the restriction of the s^{th} Siegel Eisenstein series gives local integrals at finite places v of the form

$$(1-q^{2\mu}) \cdot \frac{(1-(\alpha\beta)^2 \cdot q^{-2(s-1)})\,(1-q^{-s})}{(1-\alpha^2 q^{-(s-1)})\,(1-\alpha\beta q^{-(s-1)})\,(1-\beta^2 q^{-(s-1)})}$$

$$=\; \frac{1}{\zeta_v(2\mu)\,\zeta_v(s)\,\zeta_v(2s)} \cdot \frac{1}{(1-q^{2\mu-2-(s-1)})\,(1-q^{-s})\,(1-q^{(-2\mu)-(s-1)})}$$

$$=\; \frac{\zeta_v(s+1-2\mu)\,\zeta_v(s)\,\zeta_v(s-1-2\mu)}{\zeta_v(2\mu)\,\zeta_v(s)\,\zeta_v(2s)}$$

In fact, for purposes of spectral decomposition, $\mu = \frac{1}{2} + i\nu$ with $\nu \in \mathbb{R}$, so this becomes

$$\frac{\zeta_v(s-2i\nu)\,\zeta_v(s)\,\zeta_v(s-2-2i\nu)}{\zeta(1+2i\nu)\,\zeta_v(s)\,\zeta_v(2s)}$$

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4. *Appendix: normalization of Eisenstein series*

We recall the normalization of Siegel-type Eisenstein series giving control over poles to the right of the critical line.

Let k be a number field, and $G = Sp_{2n}$. Let v be a non-archimedean place of k. The naive normalization I_s^{nf} of the degenerate principal series of $G_v = Sp_{2n}(k_v)$ consists of f with left equivariance

$$f\Big(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \cdot g\Big) \;=\; \chi_s \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \cdot f(g) \qquad \text{(where } \chi \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} = |\deg a|^s\text{)}$$

Let ε_v be the spherical function in I_s^{nf}. That is, ε_v is right $K_v = Sp_{2n}(\mathfrak{o}_v)$-invariant, and $\varepsilon_v(1) = 1$.

The naively normalized Siegel-type Eisenstein series E_s on Sp_{2n} (where the index indicates the size of the matrices) attached to the s^{th} degenerate principal series has functional equation relating E_s and $E_{(n+1)-s}$.
