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Symmetrization maps and differential operators

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The *symmetrization map* $s : S\mathfrak{g} \to U\mathfrak{g}$ is a linear surjection from the symmetric algebra $S\mathfrak{g}$ to the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, completely characterized by being the identity map on $\mathfrak{g}$ (and on the scalars). This map is peculiar, effectively attempting to parametrize a non-commutative algebra by a commutative one. It is linear, but cannot quite be a ring homomorphism. Nevertheless, beginning with Harish-Chandra's work, the symmetrization map plays an important role.

The intent is to have a commutative algebra be mapped as surjectively as possible to a non-commutative algebra by a linear map as much an algebra homomorphism as possible. These are conflicting requirements.

Given the technicality of this map, coordinate-free characterization is all the more important.

Throughout, k is a field of characteristic 0. All algebras are k-algebras, in particular requiring that k is in the center. Unless specifically designated as *Lie* algebras, all algebras are associative.

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1. *Symmetrization maps*

[1.1] What should a symmetrization map be? Of course, commutative algebras cannot linearly surject to non-commutative algebras without losing the algebra homomorphism property, leaving the mystery of what structure might remain.

For a *commutative* algebra S and an arbitrary associative algebra A, a requirement that a k-linear map $f : S \to A$ be an *algebra* homomorphism sharply restricts the image $f(S)$: it must lie inside a commutative sub-algebra of A.

On the other hand, for $A = U\mathfrak{g}$ the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, the non-commutativity is not severe, since $U\mathfrak{g}$ *is* commutative modulo lower-degree terms, as we will see in the proof of surjectivity below. In other words, the *associated graded algebra* of the filtration by degree on $U\mathfrak{g}$ *is* commutative, so is the universal commutative algebra $S\mathfrak{g}$ on the vector space $\mathfrak{g}$.

There is the natural algebra homomorphism $F : U\mathfrak{g} \to S\mathfrak{g}$, which reasonably-enough has a large kernel, generated by commutators $xy - yx$ for $x, y \in \mathfrak{g}$. So, again, it is unreasonable to hope for a two-sided inverse to F, but it is plausible to ask for a merely-linear *right* inverse $s : S\mathfrak{g} \to U\mathfrak{g}$.

Some further algebraic structure must be required, or such a map is certainly not unique, and, concommitantly, probably not useful.

The desired sort of linear map $s : S \to A$ from a commutative algebra S to a not-necessarily-commutative algebra A ought to be as much a algebra homomorphism as possible, meaning that whenever $s(x)$ and $s(y)$ *commute*, we should have $s(xy) = s(x)s(y)$. However, the only systematic thing that can be said is that $s(x)$ commutes with itself, so the only universally safe condition to impose is

$$s(x^n) \;=\; s(x)^n$$

This may seem very weak, but the multinomial theorem effectively exploits this, over a field of characteristic 0. For example,

$$s(2xy) \;=\; s\big((x+y)^2 - x^2 - y^2\big) \;=\; s\big((x+y)^2\big) - s(x)^2 - s(x)^2 \;=\; s(x+y)^2 - s(x)^2 - s(y)^2 \;=\; s(x)\,s(y) + s(y)\,s(x)$$

In fact, by slightly more elaborate identities (below) the condition $s(x^n) = s(x)^n$ for a collection $\{x\}$ of generators can be used to completely determine the linear map s.

Thus, a symmetrization map $s : S\mathfrak{g} \to U\mathfrak{g}$ is required to be the identity on $\mathfrak{g}$, to be linear, and to have the property $s(x^n) = s(x)^n$ for all $x \in \mathfrak{g}$.

Proof is required that such a map *exists*, is *unique*, and gives a *linear isomorphism*.

[1.2] Universal algebras

We will prove that *any* symmetrization map $s : S\mathfrak{g} \to U\mathfrak{g}$ satisfies

$$s(x_1 \ldots x_n) \;=\; \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)} \ldots x_{\pi(n)} \qquad\qquad (\text{for } x_1, \ldots, x_n \in \mathfrak{g})$$

where S_n is the permutation group on $\{1, 2, \ldots, n\}$. In fact, this identity has nothing to do with Lie algebras $\mathfrak{g}$, insofar as it holds for an over-lying symmetrization map $t : SV \to \bigotimes^\bullet V$ from the symmetric algebra to the universal associate algebra for *any* vector space V.

The characterization of $\bigotimes^\bullet V$ is that it has the following universal property: there is a linear map $V \to \bigotimes^\bullet V$ such that, for every *linear* map $V \to A$ to an associative algebra A, there is a unique *algebra* map $\bigotimes^\bullet V \to A$ through which the original $V \to A$ factors. The diagram is

$$\begin{tikzcd} \bigotimes^\bullet V \arrow[dr, dashed] & \\ V \arrow[u] \arrow[r] & A \end{tikzcd}$$

That is, the functor taking V to $\bigotimes^\bullet V$ is a left adjoint to the forgetful functor F that sends an associative algebra to the underlying vector space: for every associative algebra A,

$$\mathrm{Hom}_{\mathrm{algebras}}(\bigotimes^\bullet V, A) \;\approx\; \mathrm{Hom}_{\mathrm{vectorspaces}}(V, FA)$$

The *construction*, as proof of existence, of $\bigotimes^\bullet V$ is by tensors:

$$\bigotimes^\bullet V \;=\; \bigoplus_{n=0}^{\infty} \bigotimes^n V$$

where $\bigotimes^n V$ is the universal object for n-multi-linear maps from $V \times \ldots \times V$: there is a fixed n-multi-linear $V \times \ldots \times V \to \bigotimes^n V$ such that every *n-multi-linear* $V \times \ldots \times V \to W$ factors through a unique *linear* map $\bigotimes^n V \to W$. The diagram is

$$\begin{tikzcd} \bigotimes^n V \arrow[dr, dashed] & \\ V \times \ldots \times V \arrow[u] \arrow[r] & W \end{tikzcd}$$

The multiplication on $\bigotimes^\bullet V$ is given summand-wise

$$\bigotimes^m V \times \bigotimes^n V \;\longrightarrow\; \bigotimes^{m+n} V$$

by the innocuous

$$(u_1 \otimes \ldots \otimes u_m) \times (v_1 \otimes \ldots \otimes v_n) \;\longrightarrow\; u_1 \otimes \ldots \otimes u_m \otimes v_1 \otimes \ldots \otimes v_n$$

Similarly, SV is the universal *commutative* algebra over a vector space V: there is a linear map $V \to SV$ such that, for every *linear* map $V \to A$ to a *commutative* algebra A, there is a unique *algebra* map $\bigotimes^\bullet V \to A$ through which the original $V \to A$ factors. The diagram is

$$\begin{tikzcd} SV \arrow[dr, dashed] & \\ V \arrow[u] \arrow[r] & A \end{tikzcd}$$

That is, the functor taking V to SV is a left adjoint to the forgetful functor F that sends a commutative algebra to the underlying vector space: for every associative algebra A,

$$\mathrm{Hom}_{\mathrm{commutative}}(SV, A) \;\approx\; \mathrm{Hom}_{\mathrm{vectorspaces}}(V, FA)$$

[1.3] Universal symmetrization map

To avoid presuming well-definedness, and to avoid coordinate-dependency issues, we first define a *universal* symmetrization map.

Say that a linear map $f : SV \to A$ of SV to an associative algebra A is *a symmetrization map* if it is linear and if $f(x^n) = f(x)^n$ for all $x \in V$. The *universal* symmetrization map $j : SV \to Q$ is a symmetrization map to an associative algebra Q such that, given another symmetrization map $SV \to A$, there is a unique *algebra* homomorphism $Q \to A$ through which $SV \to A$ factors. That is, we have a diagram

$$\begin{tikzcd} Q \arrow[dr, dashed, "\mathrm{alg}"] & \\ SV \arrow[u, "\mathrm{sym}"] \arrow[r, "\mathrm{sym}"'] & A \end{tikzcd}$$

The usual categorical argument gives uniqueness up to unique isomorphism, assuming *existence.*

Since the identity map $SV \to SV$ is a symmetrization map, and is injective on V, necessarily the copy of V inside SV *injects* to Q.

Existence of the universal symmetrization is straighforward, as follows. For a vector space V, let $i : SV \to \bigotimes^\bullet SV$ be the natural (injective) linear map. Let I be the two-sided ideal in $\bigotimes^\bullet(SV)$ generated by all images $i\big(s(x^n) - s(x)^n\big)$ for $x \in V$, and let

$$Q \;=\; \bigotimes{}^\bullet SV \Big/ I$$

be the quotient, with $j : SV \to Q$ the natural linear map. The universal properties of $\bigotimes^\bullet SV$ yield the desired universal properties of j and Q.

[1.4] The universal formula

We can deduce formulas in Q for $j(x_1 \ldots x_n)$ in terms of the $j(x_\ell)$. For example, from

$$j((x+y)^2) \;=\; j(x+y)^2 \qquad\qquad (\text{for } x, y \in \mathfrak{g})$$

we obtain

$$j(x)^2 + 2j(xy) + j(y)^2 \;=\; (j(x)+j(y))^2 \;=\; j(x)^2 + j(x)j(y) + j(y)j(x) + j(y)^2$$

and then deduce

$$j(xy) \;=\; \tfrac{1}{2}\big(j(x)j(y) + j(y)j(x)\big)$$

Among many possible approaches to obtain the general expression, we can consider scalars $t_1, \ldots, t_n$ and $x_1, \ldots, x_n$ in $\mathfrak{g}$, and expand $j((t_1x_1 + \ldots + t_nx_n)^n)$ two different ways, with the exponent the same as the number of summands. Without writing out either expression entirely, over a field of characteristic 0 equality of the two sides for *all* scalars t_i implies equality of the two sides as *polynomials* in *indeterminates* t_i (with values in Q). Equating the coefficients of the *middle term* $t_1 \ldots t_n$ gives

$$\binom{n}{1\;1\;\ldots\;1} j(x_1 \ldots x_n) \;=\; \sum_{\pi \in S_n} j(x_{\pi(1)}) \ldots j(x_{\pi(n)})$$

which gives the *universal* formula for the *universal* symmetrization map:

$$j(x_1 \ldots x_n) \;=\; \frac{1}{n!} \sum_{\pi \in S_n} j(x_{\pi(1)}) \ldots j(x_{\pi(n)})$$

The formula shows that the map $\bigotimes^\bullet V \to Q$ induced from $V \to SV$, composed with the quotient $\bigotimes^\bullet SV \to Q$ is *surjective*, since the formula exhibits every element of Q as a linear combination of monomials in elements $j(x)$ with $x \in V$.

Again, since the identity $SV \to SV$ is injective on V, V injects to Q. Thus, since V generates $\bigotimes^\bullet V$ and any algebra homomorphism image thereof, the image $j(V)$ generates Q.

[1.5] Canonical symmetrization map to $\bigotimes^\bullet V$ There is *a* canonical symmetrization-like map $SV \to \bigotimes^\bullet V$. This also depends upon the underlying field being of characteristic 0.

From their characterizations as universal algebra and universal commutative algebra for V, there is a canonical surjection $\bigotimes^\bullet V \to SV$ with kernel the two-sided ideal generated by commutators $xy - yx$ with x, y in $\bigotimes^\bullet V$. This quotient respects the grading by degree, and is the direct sum of the canonical maps

$$q_n \;:\; \bigotimes{}^n V \longrightarrow \mathrm{Sym}^n V$$

For each n, we will construct a linear section $s_n : \mathrm{Sym}^n V \to \bigotimes^n V$, that is, a linear map such that $q_n \circ s_n$ is the identity map on $\mathrm{Sym}^n V$.

Each element π of the permutation group S_n on n things gives a multilinear map

$$\pi \;:\; \underbrace{V \times \ldots \times V}_{n} \longrightarrow \bigotimes{}^n V$$

by

$$\pi(x_1 \times \ldots \times x_n) \;=\; x_{\pi(1)} \otimes \ldots \otimes x_{\pi(n)}$$

and thus gives a unique map of $\bigotimes^n V$ to itself. The symmetric n^{th} power $\mathrm{Sym}^n V$ is the S_n *co-fixed* vectors in $\bigotimes^n V$, that is, the largest S_n-quotient of $\bigotimes^n V$ on which S_n acts trivially.

On the other hand, since the characteristic is 0, there is an *averaging map* α_n of $\bigotimes^n V$ to the S_n *fixed* vectors in $\bigotimes^n V$, by

$$\alpha_n(\beta) \;=\; \frac{1}{n!} \sum_{\pi \in S_n} \pi(\beta) \qquad\qquad (\text{for } \beta \in \textstyle\bigotimes^n V)$$

This map is visibly the identity map on fixed vectors. Thus, there is a direct sum decomposition

$$\bigotimes^n V \;=\; \ker\big(\alpha_n\big) \oplus \big(\text{fixed vectors}\big)$$

where $\ker(\alpha_n)$ contains no non-zero fixed vectors.

We claim that the S_n-fixed vectors in $\bigotimes^n V$ map isomorphically to the S_n-cofixed vectors $\mathrm{Sym}^n V$. On one hand, the map of $\bigotimes^n V$ to its own fixed vectors by α_n must factor through q_n, so the fixed vectors *inject* to the co-fixed vectors.

To prove *surjectivity*, we make the subordinate claim that the kernel K of the quotient map is generated by elements $m - \pi m$ for $m \in \bigotimes^n V$ and $\pi \in S_n$. Certainly this maps to 0 under any S_n-homomorphism to an S_n-module on which S_n acts trivially. On the other hand, S_n acts trivially on the quotient of $\bigotimes^n V$ by K, since

$$\pi(m + K) \;=\; m + (\pi m - m) + K \;=\; m + K$$

This is the sub-claim. Then

$$q_n\big(\alpha_n\, m\big) \;=\; q_n\big(\frac{1}{n!} \sum_{\pi} \pi\, m\big) \;=\; \frac{1}{n!} \sum_{\pi} q_n(\pi\, m) \;=\; \frac{1}{n!} \sum_{\pi} q_n(m) \;=\; q_n(m)$$

That is, every element $q_n(m)$ in the quotient $\mathrm{Sym}^n V$ is hit by a fixed vector, among them the fixed vector $\alpha_n(m)$. This proves that the fixed vectors *surject* to the co-fixed vectors, and, thus, map isomorphically.

Let $\sigma_n : \mathrm{Sym}^n V \to \bigotimes^n V$ be the *inverse* of the isomorphism of fixed to co-fixed vectors, and let $\sigma : SV \to \bigotimes^\bullet V$ be the direct sum of these isomorphisms.

We claim that σ is *a* symmetrization map, meaning that $\sigma(x^n) = \sigma(x)^n$ for $x \in V$. Note that for $x \in V$ the element $x^n \in SV$ is $q_n(x \otimes \ldots \otimes x)$. Conveniently, $x \otimes \ldots \otimes x$ is already a fixed vector, and the averaging map is the identity on it. Thus,

$$\sigma(x^n) \;=\; \alpha_n\big(q_n(x \otimes \ldots \otimes x)\big) \;=\; x \otimes \ldots \otimes x \;=\; \sigma(x) \otimes \ldots \otimes \sigma(x) \;=\; \sigma(x)^n$$

Thus, indeed, $\sigma : SV \to \bigotimes^\bullet V$ is *a* symmetrization map.

[1.6] The universal symmetrization identified

Now we will see that the canonical symmetrization $\sigma : SV \to \bigotimes^\bullet V$ is *universal*.

Among all the other symmetrization-like maps from SV, we have this canonical σ to $\bigotimes^\bullet V$. Thus, this σ must factor through a *unique* algebra homomorphism $\widetilde{\sigma} : Q \to \bigotimes^\bullet V$ from the *universal* symmetrization $j : SV \to Q$. And Q is a canonical image $f(\bigotimes^\bullet V)$ of $\bigotimes^\bullet V$. This fits into a diagram

$$\begin{tikzcd}
SV \arrow[r] \arrow[dr, "j"] \arrow[ddr, "\sigma"'] & \bigotimes^\bullet SV \arrow[d] & \\
& Q \arrow[d, dashed, "\widetilde{\sigma}"] & \bigotimes^\bullet V \arrow[l, "f"'] \arrow[ul, "\mathrm{inc}"'] \\
& \bigotimes^\bullet V &
\end{tikzcd}$$

Thus, $j : SV \to f(\bigotimes^\bullet V)$ and $\sigma : SV \to \bigotimes^\bullet V$ are both symmetrization maps, and σ factors through j.

As observed above, j is injective on the copy of V in SV, and $j(V)$ generates Q as an algebra. Likewise, σ is injective on V, and obviously $\sigma(V)$ generates $\bigotimes^\bullet V$ as an algebra. Therefore, the composite $\bigotimes^\bullet V \to Q \to \bigotimes^\bullet V$ must be the identity on $\bigotimes^\bullet V$.

That is, $Q = \bigotimes^\bullet V$, and the canonical symmetrization $\sigma : SV \to \bigotimes^\bullet V$ is the *universal* one.

In particular, σ is the unique symmetrization map $SV \to \bigotimes^\bullet V$ with the property of being the identity on the copies of V.

[1.7] Symmetrization $S\mathfrak{g} \to U\mathfrak{g}$

Now return to Lie algebras $V = \mathfrak{g}$. Note that any associative algebra A has a natural Lie algebra structure given by $[x, y] = xy - yx$. Then the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is an *associative* algebra characterized as follows. There is there is a *Lie algebra* map $i : \mathfrak{g} \to U\mathfrak{g}$ such that, for every *Lie algebra* map $\mathfrak{g} \to A$ to an *associative* algebra A, there is a unique *associative* algebra map $U\mathfrak{g} \to A$ through which the original $\mathfrak{g} \to A$ factors. The diagram is

$$\begin{tikzcd}
U\mathfrak{g} \arrow[dr, dashed, "\mathrm{assoc}"] & \\
\mathfrak{g} \arrow[u, "\mathrm{Lie}"] \arrow[r, "\mathrm{Lie}"'] & A
\end{tikzcd}$$

That is, the functor taking $\mathfrak{g}$ to $\mathcal{U}\mathfrak{g}$ is a left adjoint to the forgetful functor F that sends an associative algebra to the underlying vector space: for every associative algebra A,

$$\mathrm{Hom}_{\mathrm{assoc}}(U\mathfrak{g}, A) \;\approx\; \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, FA)$$

The Poincaré-Birkhoff-Witt theorem proves that i injects $\mathfrak{g}$ to $U\mathfrak{g}$.

This categorical characterization shows that $i : \mathfrak{g} \to U\mathfrak{g}$ is unique up to unique isomorphism, *if* it exists. To prove existence, *construct* $U\mathfrak{g}$ as the quotient of $\bigotimes^\bullet \mathfrak{g}$ by the two-sided ideal generated by all $(xy - yx) - [x, y]$ with $x, y \in \mathfrak{g}$, deducing the desired properties of $U\mathfrak{g}$ from those of $\bigotimes^\bullet \mathfrak{g}$.

We want a symmetrization map $s : S\mathfrak{g} \to U\mathfrak{g}$ with s the identity on the copies of $\mathfrak{g}$. Since $U\mathfrak{g}$ is generated by the image of $\mathfrak{g}$, there is *at most one* such symmetrization, given by the universal formula above, *if* a symmetrization map exists.

For *existence*, let $s : S\mathfrak{g} \to U\mathfrak{g}$ be the composition

$$\begin{tikzcd} S\mathfrak{g} \arrow[r, "\sigma"] \arrow[rr, bend left, "s"] & \bigotimes^\bullet \mathfrak{g} \arrow[r, "\text{quot}"] & U\mathfrak{g} \end{tikzcd}$$

As a corollary, this symmetrization $s : S\mathfrak{g} \to U\mathfrak{g}$ is the *universal* symmetrization map for Lie algebras $\mathfrak{g}$, in the following sense. For a *symmetrization* map $f : S\mathfrak{g} \to A$ to an associative algebra A which is a *Lie algebra* map on $\mathfrak{g}$, there is a unique *associative* algebra homomorphism $U\mathfrak{g} \to A$ through which f factors by $s : S\mathfrak{g} \to U\mathfrak{g}$. That is,

$$\begin{tikzcd} & U\mathfrak{g} \arrow[ddr, dashed, "\text{assoc}"] & \\ & S\mathfrak{g} \arrow[u, "\text{sym}"] \arrow[dr, "\text{sym}"] & \\ \mathfrak{g} \arrow[uur, bend left, "\text{Lie}"] \arrow[ur, "\text{linear}"] \arrow[rr, "\text{Lie}"] & & A \end{tikzcd}$$

[1.7.1] Remark: This definition produces the same outcome as attempting to define a symmetrization map $S\mathfrak{g} \to U\mathfrak{g}$ by taking the identity map on $\mathfrak{g}$ and extending it by the universal formula from above. However, doing this directly faces two possible difficulties. First, it is not clear *a priori* that there exists *any* $s : S\mathfrak{g} \to U\mathfrak{g}$ with $s(x^n) = s(x)^n$ for all $x \in \mathfrak{g}$, and s is the identity on $\mathfrak{g}$, because this might impose mutually conflicting conditions. Second, we might attempt to avoid potential conflicts by forgetting about the requirements $s(x^n) = s(x)^n$, instead merely choosing a *basis* $\{x_i\}$ for $\mathfrak{g}$, taking the corresponding basis for $S\mathfrak{g}$ consisting of elements

$$x_{i_1} \ldots x_{i_n} \qquad\qquad (\text{where } i_1 \le i_2 \le \ldots \le i_n)$$

and *defining* a linear map $s : S\mathfrak{g} \to U\mathfrak{g}$ on that basis by the (in fact, *universal*) formula

$$s(x_{i_1} \ldots x_{i_n}) \;=\; \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(i_1)} \ldots x_{\pi(i_n)} \;\in\; U\mathfrak{g} \qquad\qquad (\text{where } i_1 \le i_2 \le \ldots \le i_n)$$

Among other flaws, it is not clear that this is independent of the basis for $\mathfrak{g}$, and it is surely a fool's errand to try to prove it by choosing two bases for $\mathfrak{g}$ and comparing.

[1.8] Surjectivity of $S\mathfrak{g} \to U\mathfrak{g}$ It is mildly surprising that the symmetrization map $s : S\mathfrak{g} \to U\mathfrak{g}$ is *surjective*, so is a *linear isomorphism*. The surjectivity is not merely a curiosity.

Let

$$\bigotimes\nolimits^{\le n} \mathfrak{g} \;=\; \bigoplus\nolimits_{0 \le i \le n} \bigotimes\nolimits^i \mathfrak{g}$$

$$\bigotimes\nolimits^{< n} \mathfrak{g} \;=\; \bigoplus\nolimits_{0 \le i < n} \bigotimes\nolimits^i \mathfrak{g}$$

and $S^{\le n}\mathfrak{g}$, $S^{\le n}\mathfrak{g}$, $U^{\le n}\mathfrak{g}$, $U^{<n}\mathfrak{g}$ the corresponding images. In all cases, the parameter n is the *degree*. We prove that $S^{\le n}\mathfrak{g}$ surjects to $U^{\le n}\mathfrak{g}$, by induction on degree.

At degree 0 (the scalars, by convention), the symmetrization map is the identity. At degree 1, the symmetrization map is the identity, being the identity on $\mathfrak{g}$ by *definition*. Consider a monomial

$$u \;=\; x_1 \ldots x_n \;\in\; U^{\le n}\mathfrak{g}$$

Every permutation $\pi \in S_n$ is a product $\pi = \sigma_1 \ldots \sigma_\ell$ of *adjacent* transpositions $\sigma_{i,i+1}$, and

$$\sigma_{i,i+1}u \;=\; x_{\sigma_{i,i+1}(1)} \ldots x_{\sigma_{i,i+1}(n)} \;=\; x_1 \ldots x_{i-1}x_{i+1}x_i x_{i+2}x_n$$

$$=\; x_1 \ldots x_{i-1}[x_i+1, x_i]x_{i+2}x_n + x_1 \ldots x_{i-1}x_i x_{i+1}x_{i+2}x_n \;\in\; U^{<n}\mathfrak{g} + u$$

That is, by induction on the number of adjacent transpositions needed to express a permutation,

$$u - \pi u \;\in\; U^{<n}\mathfrak{g}$$

In particular,

$$u - \frac{1}{n!}\sum_\pi \pi u \;\in\; U^{<n}\mathfrak{g}$$

and the sum is in the image of the symmetrization map, by the universal formula. This proves the surjectivity. Injectivity follows from the Poincaré-Birkhoff-Witt theorem. Thus, the symmetrization map is a linear isomorphism $S\mathfrak{g} \to U\mathfrak{g}$.
