Tiniest example of (co-) homology

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

For context, recall that in a category of G-modules for group G, the fixed-vector functor [1]

$$M \to M^G = \{ m \in M : g \cdot m = m \text{ for all } g \in G \}$$

and cofixed-vector functor [2]

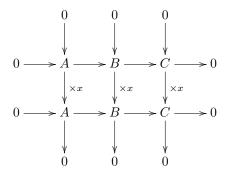
 $M \rightarrow M_G = \text{largest quotient of } M \text{ on which } G \text{ acts trivially}$

are mutual adjoints: $\operatorname{Hom}_G(M_G, N) \approx \operatorname{Hom}_G(M, N^G)$. Because left adjoints are right-exact, the G-cofixed-vector functor $M \to M_G$ is right-exact, so has left-derived functors, called $\operatorname{group\ homology}$. Similarly, because right adjoints are left-exact, the G-fixed-vector $M \to M^G$ is left-exact, so has right-derived functors, called $\operatorname{group\ cohomology}$. [3]

In the situation where G is a real Lie group, taking smooth vectors V^{∞} in G-representations gives representations of the Lie algebra \mathfrak{g} of G. Then G-fixed becomes \mathfrak{g} -annihilation, and G-cofixed becomes \mathfrak{g} -coannihilation. Lie algebra homology consists of the left-derived functors of the \mathfrak{g} -coannihilation functor, and Lie algebra cohomology consists of the right-derived functors of the \mathfrak{g} -annihilation functor. Many other classical (co-) homologies are also derived functors of fairly trivial functors, with homologies and cohomologies of mutually adjoint functors having further relationships.

In the category of $\mathbb{C}[x]$ -modules, similarly, the two simplest functors are $M \to M^x$ (the x-annihilated submodule) and $M \to M_x$ (the x-annihilated quotient). Of course, this is just saying "ker x" and "M/xM" in a fancier way. These are mutual adjoints: $\operatorname{Hom}(M_x, N) \approx \operatorname{Hom}(M, N^x)$. Thus, M_x is a 0^{th} homology we might denote by $H_0(M, x)$, and M^x is a 0^{th} cohomology $H^0(M, x)$.

To determine the higher (co-)homologies, from a short exact sequence of complexes



^[1] This familiar description is not the categorically-best characterization, but, rather, is a construction. The best description is that M^G is the largest subobject of M on which G acts trivially.

^[2] This is a categorically correct description of the cofixed quotient. In contrast to the fixed-vector characterization/construction, the construction of cofixed-vector modules depends more delicately on the ambient category. For example, in the category of G-modules, M_G is the quotient of M by the submodule generated by all elements $m-g \cdot m$ for $m \in M$ and $g \in G$. In the category of topological vector spaces with continuous G actions, the quotient must be by the topological closure of the subspace generated by such elements, so that the quotient is Hausdorff.

^[3] In some sources, group (co-) homology is defined *ad hoc* by specifying a particular (injective) projective resolution, without comment about the larger homological context.

Paul Garrett: Tiniest example of (co-) homology (June 3, 2019)

the Snake Lemma gives a long exact sequence

$$0 \longrightarrow A^x \longrightarrow B^x \longrightarrow C^x \stackrel{\eta}{\longrightarrow} A_x \longrightarrow B_x \longrightarrow C_x \longrightarrow 0$$

That is, A^x can be viewed as $H_1(A, x)$, with all higher homologies $H_i(A, x)$ vanishing, and A_x can be viewed as $H^1(A, x)$, with all higher cohomologies $H^i(A, x)$ vanishing. The short-long exact sequence is either/both a long (co-)homology sequence.