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Rigged Hilbert spaces attached to pairs of operators

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1. *Rigged Hilbert spaces from pairs*

A pair of symmetric, semi-bounded operators S, T on a Hilbert space $\mathfrak{B}^0$ gives rise to a *rigged Hilbert space* structure when the operators have a common domain $D = D_S = D_T$ dense in $\mathfrak{B}^0$ stabilized by them, that is, when $S(D) \subset D$ and $T(D) \subset D$, as follows.

Without loss of generality, suppose that S, T are non-negative and $S + T \geq 1$, in the sense that

$$\langle Sv, v\rangle_{\mathfrak{B}^0} \geq 0 \qquad \langle Tv, v\rangle_{\mathfrak{B}^0} \geq 0 \qquad \langle (S+T)v, v\rangle_{\mathfrak{B}^0} \geq \langle v, v\rangle_{\mathfrak{B}^0} \qquad (\text{for all } v \in D)$$

The $\mathfrak{B}^1$-norm relative to S, T is

$$\langle v, w\rangle_{B^1} \;=\; \langle (S+T)v, w\rangle_{\mathfrak{B}^0} \qquad (\text{for } v, w \in D)$$

and $\mathfrak{B}^1$ is the completion of D with respect to this norm. The $\mathfrak{B}^k$-norm is described inductively:

$$\langle v, w\rangle_{\mathfrak{B}^k} \;=\; \langle Sv, Sw\rangle_{\mathfrak{B}^{k-2}} + \langle Tv, Tw\rangle_{\mathfrak{B}^{k-2}} \qquad (\text{for } v, w \in D \text{ and } k \geq 2)$$

and $\mathfrak{B}^k$ is the Hilbert-space completion. Let $\mathfrak{B}^{+\infty}$ be the projective limit. The maps $\mathfrak{B}^k \to \mathfrak{B}^{k-1}$ induced by the denseness of D in every $\mathfrak{B}^k$ are continuous injections with dense images, thus giving a *rigged Hilbert-space*

$$\ldots \longrightarrow \mathfrak{B}^k \longrightarrow \mathfrak{B}^{k-1} \longrightarrow \ldots \longrightarrow \mathfrak{B}^2 \longrightarrow \mathfrak{B}^1 \longrightarrow \mathfrak{B}^0 = V$$

By design, S and T are continuous $D \to D$ with $\mathfrak{B}^k$-topology on the source and $\mathfrak{B}^{k-2}$-topology on the target:

$$|Sv|^2_{\mathfrak{B}^{k-2}} \;\leq\; |(Sv)|^2_{\mathfrak{B}^{k-2}} + |(Tv)|^2_{\mathfrak{B}^{k-2}} \;=\; |v|^2_{\mathfrak{B}^k} \qquad (\text{for } v \in D)$$

and similarly for T. Thus, S, T extend by continuity to continuous maps $S^\#, T^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \geq 2$, and, then, to continuous maps $\mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$. The triangle inequality shows continuity of $S + T$:

$$|(S+T)v|_{B^{k-2}} \;\leq\; |Sv|_{B^{k-2}} + |Tv|_{B^{k-2}} \;\leq\; 2|v|_{\mathfrak{B}^k} \qquad (\text{for } v \in D)$$

so $S + T$ likewise extends by continuity to $(S+T)^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \geq 2$, and then to $\mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$.

Non-commutative polynomials in S, T are to be understood as having domain D. Non-commutative monomials Q of total degree d are proven continuous $\mathfrak{B}^k \to \mathfrak{B}^{k-d}$ by induction on d, for Q of degree d giving a continuous linear map $\mathfrak{B}^k \to \mathfrak{B}^{k-d}$ for all $k \geq d$,

$$|(Q \cdot S)v|^2_{\mathfrak{B}^{k-d-1}} \;=\; |Q(Sv)|^2_{\mathfrak{B}^{k-d-1}} \;\ll_Q\; |Sv|^2_{\mathfrak{B}^{k-1}} \;\leq\; |v|^2_{\mathfrak{B}^k} \qquad (\text{for } v \in D)$$

and similarly for $Q \cdot T$. Symmetry of S, T shows that this induction gives the same outcome as induction by adding factors on the left. The triangle inequality gives an induction on the number of summands in Q to prove a similar continuity for all non-commutative polynomials: for a polynomial Q of total degree d, and M a monomial of total degree at most d,

$$|(Q+M)v|_{\mathfrak{B}^{k-d}} \;\leq\; |Qv|_{\mathfrak{B}^{k-d}} + |Mv|_{\mathfrak{B}^{k-d}} \;\ll_{Q,M}\; |v|_{\mathfrak{B}^k} \qquad (\text{for } v \in D)$$

Thus, all polynomials Q in S, T of total degree at most d extend by continuity to $Q^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-d}$, and to continuous maps of $\mathfrak{B}^{+\infty}$ to itself.

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2. *Large extensions of operators*

For Hilbert spaces with a *complex conjugation* stabilizing D, operators S, T commuting with the conjugation have *large extensions*, still denoted $S^\#, T^\#$, to the dual of $\mathfrak{B}^{+\infty}$.

For $k \geq 1$, let $\mathfrak{B}^{-k}$ be the complex-linear Hilbert-space dual of $\mathfrak{B}^k$, with hermitian inner product $\langle , \rangle_{\mathfrak{B}^{-k}}$ coming from the norm

$$|\lambda|_{-k} \;=\; \sup_{v \in \mathfrak{B}^k : |v|_k \leq 1} |\lambda v| \qquad\qquad (\text{for } \lambda \in \mathfrak{B}^{-k})$$

The natural *complex-bilinear* pairing on $\mathfrak{B}^k \times \mathfrak{B}^{-k}$ is

$$\langle , \rangle_{\mathfrak{B}^k \times \mathfrak{B}^{-k}} \;:\; \mathfrak{B}^k \times \mathfrak{B}^{-k} \longrightarrow \mathbb{C} \qquad \text{by} \qquad \langle v, \lambda \rangle_{\mathfrak{B}^k \times \mathfrak{B}^{-k}} \;=\; \lambda(v) \qquad\qquad (v \in \mathfrak{B}^k \text{ and } \lambda \in \mathfrak{B}^{-k})$$

The maps

$$\ldots \longrightarrow \mathfrak{B}^k \longrightarrow \mathfrak{B}^{k-1} \longrightarrow \ldots \longrightarrow \mathfrak{B}^2 \longrightarrow \mathfrak{B}^1 \longrightarrow \mathfrak{B}^0$$

give Hilbert-space adjoints

$$(\mathfrak{B}^0)^* \longrightarrow \mathfrak{B}^{-1} \longrightarrow \mathfrak{B}^{-2} \longrightarrow \ldots \longrightarrow \mathfrak{B}^{-(k-1)} \longrightarrow \mathfrak{B}^{-k} \longrightarrow \ldots$$

These two collections of maps can be spliced together, and the hermitian inner products compared with the complex-bilinear pairings, when when $\mathfrak{B}^0$ has a complex-conjugate-linear *conjugation* map, as follows. The conjugation $v \to \overline{v}$ should have expected properties: $\overline{\overline{v}} = v$, $\overline{\alpha \cdot v} = \overline{\alpha} \cdot \overline{v}$ for complex α, and $\langle v, \overline{w} \rangle_{\mathfrak{B}^0} = \langle w, \overline{v} \rangle_{\mathfrak{B}^0}$. Suppose D is stabilized by $v \to \overline{v}$, and that S and T commute with $v \to \overline{v}$.

A compatible conjugation map is induced on $\mathfrak{B}^k$ and $\mathfrak{B}^{-k}$, and $i : D \to \mathfrak{B}^{+1}$ and $j : \mathfrak{B}^{+1} \to \mathfrak{B}^0$ commute with the conjugation.

Using the conjugation on $\mathfrak{B}^0$, let $\Lambda : \mathfrak{B}^0 \to (\mathfrak{B}^0)^*$ be the complex-linear isomorphism of $\mathfrak{B}^0$ with its complex-linear dual by $\Lambda(x)(y) = \langle y, \overline{x} \rangle_{\mathfrak{B}^0} = \langle x, \overline{y} \rangle_{\mathfrak{B}^0}$. The continuous injection $j : \mathfrak{B}^{+1} \to \mathfrak{B}^0$ dualizes to $j^* : (\mathfrak{B}^0)^* \to \mathfrak{B}^{-1}$ by $(j^*\mu)(x) = \mu(jx)$ for $\mu \in (\mathfrak{B}^0)^*$ and $x \in \mathfrak{B}^{+1}$, and we have the splicing

$$\mathfrak{B}^{+\infty} \qquad \ldots \longrightarrow \mathfrak{B}^{+2} \longrightarrow \mathfrak{B}^{+1} \xrightarrow{\;j\;} \mathfrak{B}^0 \underset{\approx}{\xrightarrow{\;\Lambda\;}} (\mathfrak{B}^0)^* \xrightarrow{\;j^*\;} \mathfrak{B}^{-1} \longrightarrow \mathfrak{B}^{-2} \longrightarrow \ldots \qquad \mathfrak{B}^{-\infty}$$

with $\mathfrak{B}^{-\infty} = \text{colim}\,\mathfrak{B}^{-k}$ the strong dual of $\mathfrak{B}^{+\infty}$. [1]

[2.0.1] Note: Thus, for $k, \ell \geq 0$, letting $\varphi : \mathfrak{B}^k \to \mathfrak{B}^{-\ell}$ be the injective map induced by the identity map $D \to D$, the comparison of hermitian and complex-bilinear forms is essentially described by

$$\langle v, w \rangle_{\mathfrak{B}^k} \;=\; \langle v, \overline{\varphi v} \rangle_{\mathfrak{B}^k \times \mathfrak{B}^{-k}} \qquad\qquad (\text{for } v, w \in \mathfrak{B}^k)$$

[2.0.2] Note: Since D injects to $\mathfrak{B}^0$ and is dense in $\mathfrak{B}^0$, every $\mathfrak{B}^k \to \mathfrak{B}^{k-1}$ for $k \geq 1$ is injective with dense image. The injectivity and dense image of $\mathfrak{B}^{+1} \to \mathfrak{B}^0$ give injective adjoint $(\mathfrak{B}^0)^* \to \mathfrak{B}^{-1}$ with dense

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[1] For general categorical reasons, $\mathfrak{B}^{+\infty}$ is the dual of $\mathfrak{B}^{-\infty}$, but $(\mathfrak{B}^{+\infty})^* = \mathfrak{B}^{-\infty}$ needs the fact that a continuous linear map from a limit of Banach spaces to a normed space necessarily factors through a limitand.

image. Since S is symmetric and commutes with conjugation, the extensions $S^\#, T^\#$ are compatible with the complex-linear identification $\Lambda : \mathfrak{B}^0 \to (\mathfrak{B}^0)^*$.

[2.1] Large extensions of operators

The extended operators $S^\#, T^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for $k \geq 2$ have adjoints $(S^\#)^*$ and $(T^\#)^*$ mapping $\mathfrak{B}^{-(k-2)} \to \mathfrak{B}^{-k}$.

For even indices k, compatibility with conjugation and the complex-linear isomorphism $\Lambda : \mathfrak{B}^0 \approx (\mathfrak{B}^0)^*$ allows us to consider these adjoints as *extensions* of $S^\#, T^\#$, and denote them simply by the same symbols, $S^\#$ and $T^\#$.

To connect positive and negative *odd* indices k, the conjugation allows us to extend $S^\#, T^\#$ to maps $\mathfrak{B}^{+1} \to \mathfrak{B}^{-1}$, by

$$(S^\# x)(y) \;=\; \langle x, \overline{y} \rangle_{\mathfrak{B}^1} \qquad\qquad (T^\# x)(y) \;=\; \langle x, \overline{y} \rangle_{\mathfrak{B}^1} \qquad\qquad (x, y \in \mathfrak{B}^{+1})$$

Again, these extensions are indeed compatible with $\mathfrak{B}^{+1} \to \mathfrak{B}^0 \approx (\mathfrak{B}^0)^* \to \mathfrak{B}^{-1}$.

Thus, S, T extend to $S^\#, T^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \in \mathbb{Z}$, inducing $S^\#, T^\# : \mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$ and the *large extensions* $S^\#, T^\# : \mathfrak{B}^{-\infty} \to \mathfrak{B}^{-\infty}$, denoted by the same symbols. [2]

Then non-commutative polynomials Q in S, T with *real* coefficients are likewise compatible with conjugation, so have large extensions $Q^\#$. Writing a non-commutative polyomial's arguments as x, y, the compatibility of such polynomials with formation of large extensions is

$$Q(S, T)^\# \;=\; Q(S^\#, Q^\#)$$

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[2] Laplacians on test functions give the archetype for $S^\# : \mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$, and the extension to distributional differentiation is the archetype for the *large* extension $S^\# : \mathfrak{B}^{-\infty} \to \mathfrak{B}^{-\infty}$.