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Fujisaki's lemma, units theorem, class number

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Fujisaki's lemma asserts the compactness of $\mathbb{J}^1/k^\times$, where $\mathbb{J}^1$ is the ideles of idele-norm 1 of a number field k. This is basic in the harmonic analysis of automorphic forms. It is a corollary of existence and uniqueness of Haar measure. It *implies* finiteness of class numbers and the units theorem. We recall the standard argument, apparently first enunciated by Iwasawa. [1]

Let i be the *ideal map* from ideles to non-zero fractional ideals of the integers $\mathfrak{o}$ of k. That is,

$$i(\alpha) \;=\; \prod_{v<\infty} \mathfrak{p}_v^{\mathrm{ord}_v \alpha} \qquad\qquad (\text{for } \alpha \in \mathbb{J})$$

where $\mathfrak{p}_v$ is the prime ideal in $\mathfrak{o}$ attached to the place v. The subgroup $\mathbb{J}^1$ of $\mathbb{J}$ still surjects to the group of non-zero fractional ideals. The kernel in $\mathbb{J}$ of the ideal map is

$$H \;=\; \prod_{v|\infty} k_v^\times \;\times\; \prod_{v<\infty} \mathfrak{o}_v^\times$$

and the kernel on $\mathbb{J}^1$ is $H^1 \,=\, H \cap J^1$. The principal ideals are the image $i(k^\times)$. The map of $\mathbb{J}^1$ to the ideal class group factors through the idele class group $\mathbb{J}^1/k^\times$, noting that the product formula implies that $k^\times \subset \mathbb{J}^1$.

The intersection $H^1 = H \cap \mathbb{J}^1$ is open in $\mathbb{J}^1$, so its image K in the quotient $\mathbb{J}^1/k^\times$ is open. The cosets of K cover $\mathbb{J}^1/k^\times$, and by compactness there is a finite subcover. Thus, the quotient $\mathbb{J}^1/k^\times K$ is finite, and this finite group is the absolute ideal class group.

Since K is open, its cosets are open. Thus, K is closed. Since $\mathbb{J}^1/k^\times$ is Hausdorff and compact, K is compact. That is, we have compactness of

$$K \;=\; (H^1 \cdot k^\times)/k^\times \;\approx\; H^1/(k^\times \cap H^1) \;=\; H^1/\mathfrak{o}^\times$$

with the global units $\mathfrak{o}^\times$ imbedded on the diagonal. Since

$$\prod_{v<\infty} \mathfrak{o}_v^\times$$

is compact, its image U under the continuous map to $H^1/\mathfrak{o}^\times$ is compact. By Haudorff-ness, the image U is closed. Thus, we can take a further (Hausdorff) quotient by U,

$$H^1/(U \cdot \mathfrak{o}^\times) \;=\; (\text{compact})$$

Let

$$k^1_\infty \;=\; \{\alpha \in \prod_{v|infty} k_v^\times \;:\; \prod_v |\alpha_v|_v = 1\}$$

Then

$$k^1_\infty/\mathfrak{o}^\times \;\approx\; H^1/(U \cdot \mathfrak{o}^\times) \;=\; (\text{compact})$$

This compactness is the units theorem. ///

Obviously, the same arguments prove finiteness of generalized ideal class groups and prove generalized units theorems. One might want to prove the accompanying result that a discrete (closed) additive subgroup L of $\mathbb{R}^n$ such that $\mathbb{R}^n/L$ is *compact* is a free $\mathbb{Z}$-module on n generators.

http://www.math.umn.edu/˜garrett/m/v/fujisaki.pdf

[Lang 1970] S. Lang, *Algebraic Number Theory*, Addison-Wesley, 1970.

[Weil 1968] A. Weil, *Basic Number Theory*, Springer-Verlag, 1968.

[1] The compactness can be made a corollary of finiteness of class number and the units theorem.