THIRD PROBLEM SET
Math 5615H: Honors Analysis

Due W 27 September, 2017.
10 points each; total 50 points.

1. Let $A := \{a_1, a_2, \dots\}$ be a set of real numbers defined as follows:

$$a_1 = 1, \qquad \text{and } a_{k+1} = 1 + \sqrt{a_k} \text{ for } k = 1, 2, \dots.$$

   Find $\sup A$.

2. Let z and w be complex numbers such that both the sum $z + w$ and the product zw are **real**. Show that $w = \overline{z}$, i.e. w is the **conjugate** of z.

3. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^k$, $\vec{x} \neq \vec{0}$. Show that $\vec{y}$ is **uniquely** represented in the form

$$\vec{y} = \vec{a} + \vec{b}, \text{ where } \vec{a}, \vec{b} \in \mathbb{R}^k \text{ satisfy } \vec{a} = \alpha\vec{x} \text{ for some real } \alpha, \text{ and } \vec{b} \cdot \vec{x} = 0.$$

   Also, verify this fact for $\vec{x} = (1, 1, 1)$ and $\vec{y} = (1, 2, 3)$ in $\mathbb{R}^3$.

4. Let A be a nonempty set in $\mathbb{R}^k$. For $\vec{x} \in \mathbb{R}^k$, define

$$d(\vec{x}, A) := \inf\{|\vec{x} - \vec{a}| : \ \vec{a} \in A\}.$$

   Show that

$$|d(\vec{x}, A) - d(\vec{y}, A)| \leq |\vec{x} - \vec{y}| \text{ for all } \vec{x}, \vec{y} \in \mathbb{R}^k.$$

5. Exercise 8 on p.43 of Rudin.