DENSITY ESTIMATES FOR MINIMAL SURFACES AND SURFACES FLOWING BY MEAN CURVATURE

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ABSTRACT. Let Σ be a two-dimensional immersed minimal surface in a manifold M^n, having a curve Γ as boundary. We do not assume that Σ has minimum area. It will be shown that the number of sheets of Σ passing through a point $p \in M$ (the density of Σ at p) will be bounded by geometric measures of the complexity of Γ. However, such an estimate must also depend on the geometry of the ambient manifold M.

Suppose that M is simply connected, and that the sectional curvatures of M are less than or equal to a nonpositive constant $-\kappa^2$. Let $\mathcal{A}(\Gamma)$ denote the minimum over $p \in M$ of the area of the geodesic cone over Γ with vertex p. If for some integer $m \geq 0$ the total absolute curvature of Γ satisfies

$$\int_\Gamma |\vec{k}|\, ds \leq 2\pi m + \kappa^2 \mathcal{A}(\Gamma),$$

then the number of sheets through one point is at most $m-1$. In particular, if this inequality holds with $m = 2$, then Σ must be embedded.

An analogous result holds if M is a hemisphere.

We shall also discuss conjectures about analogous estimates for a surface which evolves by its mean curvature vector.

The Euclidean case $M = R^n$ was proved by Eckholm, White and Wienholtz [EWW]. This report is based on joint work with Jaigyoung Choe [CG].

1. INTRODUCTION

The elliptic problem of finding minimal surfaces in three-dimensional space has a compelling geometric interest. The parametric theory pioneered by Radó and Douglas (see [D] and [R]) may be used to find a minimal surface of the type of the disk bounded by a given curve in $\mathbf{R}^3$, which must be immersed (see [G] or [A]) but may well intersect itself. Since self-intersections are unrealistic for such physical contexts as soap films or biological membranes, the question of whether a minimal surface is **embedded** carries great significance.

In a recent paper, Ekholm, White, and Wienholtz [EWW] ingeniously proved the embeddedness of any minimal surface bounded by a curve Γ in $\mathbf{R}^n$ with total curvature $\leq 4\pi$. Their result may be seen to follow from the following three observations. **(i)** The logarithm of the distance function $\rho(x) = d(x,p)$ in $\mathbf{R}^n$ is a fundamental solution of the Laplacian on a two-dimensional plane through p. Similarly, $G(x) = \log \rho(x)$ is harmonic on a cone $p \times\!\!\!\!\times \Gamma$ over Γ with vertex p. By contrast, $G(x)$ is strictly subharmonic on a nonplanar (branched) minimal surface Σ in $\mathbf{R}^n$. Further, at each point of Γ, the outward normal derivative of $G(x)$ in the cone $p \times\!\!\!\!\times \Gamma$ is greater than or equal to the outward normal derivative

Date: 21 March, 2003.

of $G(x)$ in the minimal surface Σ. As a consequence, the density of Σ at p is less than or equal to the density of the cone. This part of their proof is intimately related to the well-known monotonicity formula. **(ii)** By the Gauss-Bonnet theorem, 2π times the density at p of the cone $p⋇\Gamma$, which is intrinsically flat, is at most the total curvature of Γ. **(iii)** Since an immersed submanifold must have density at least two at a point of intersection, it follows that a branched minimal surface whose boundary has total curvature 4π or less must be embedded. The theorem of Fáry and Milnor, that a curve with total curvature at most 4π is unknotted [F], [M], follows as a consequence of the existence of a branched minimal surface of the type of the disk with boundary curve Γ (see [Mo] for the general case).

In contrast with the problem of minimal surfaces, the **parabolic** problem of flow by mean curvature has little difficulty with self-intersections. In fact, if a hypersurface Σ_t evolves according to

(1)
$$\frac{\partial \Sigma_t}{\partial t} = H\vec{\nu};$$

if Σ_0 is embedded; and if the boundary of Σ_t never touches the interior of Σ_t; then Σ_t will remain embedded for all positive time (assuming it exists). Here H denotes the mean curvature of the evolving surface Σ_t, and $\vec{\nu}$ is its unit normal vector.

One should therefore view the result of [EWW] as a density estimate for a nonplanar minimal surface Σ, depending only on the total curvature of its boundary curve Γ:

(2)
$$\Theta_\Sigma(p) < \frac{1}{2\pi}\int_\Gamma |\vec{k}|\, ds.$$

Here, $\vec{k}$ is the curvature vector of Γ. In this context, there are natural conjectures which suggest themselves for the problem of mean-curvature flow. Roughly speaking, one expects that the density of a surface evolving by mean curvature can be bounded by an average density at time $t = 0$ and the total curvature of the boundary of Σ_t at times between 0 and t. See section 4 below.

The paper [CG] extends the result of [EWW] to minimal surfaces in an n-dimensional Riemannian manifold M with sectional curvature K^M bounded above by a nonpositive constant $\widehat{K}$, or with constant positive sectional curvature. The two conclusions **(i), (ii)** above can be appropriately generalized for these purposes, and **(iii)** is unchanged. Thus, it is proved that if Γ is a Jordan curve in M^n with total curvature

(3)
$$\mathcal{C}_{\rm tot}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \le 4\pi + \inf_{p\in M}(-\widehat{K})\,\mathrm{Area}(p⋇\Gamma),$$

then every branched minimal surface bounded by Γ is embedded (see Theorem 2 and Theorem 3.) The cone $p⋇\Gamma$ is defined as the union of geodesic segments from p to points of Γ. Somewhat more precisely, in inequality (3), the infimum of area of cones $p⋇\Gamma$ is taken only over vertices p lying in the convex hull $\mathcal{H}_{\rm cvx}(\Gamma)$ of Γ. Even more precisely, we may restrict to points p which lie in the *mean-convex hull* of Γ, that is, the intersection of smooth closed domains in M^n whose boundaries have non-strictly inward mean curvature.

The embedding theorem is a consequence of the following density estimate: for any stationary, non-totally geodesic minimal surface Σ^2 in M^n with boundary Γ, the density of Σ at p satisfies

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\rm tot}(\Gamma) + \widehat{K}\,\mathrm{Area}(p \times\!\!\!\!\times \Gamma). \tag{4}$$

The same paper treats the case when M has positive sectional curvature K^M. With the additional assumptions that $K^M \equiv \widehat{K}$ is a positive *constant*, and that Σ lies in a ball of M of radius $\pi/(2\sqrt{\widehat{K}})$, the corresponding density estimate (4) is proved, along with the embeddedness of Σ if (3) holds. Note that in this case, the right-hand side of (3) involves the supremum of areas of cones $p \times\!\!\!\!\times \Gamma$ over Γ. with vertex lying in the convex hull of Γ. All these estimates are sharp.

Morrey proved that any closed curve in a manifold of bounded geometry is the boundary of a branched minimal surface of the type of the disk [Mo]. As a consequence, the results of [CG] give a new proof of the unknotting theorem for curves of total curvature at most 4π in a Hadamard-Cartan manifold [AB], [S]. In addition, a new proof is given of a slightly weaker version of the unknotting theorem in hyperbolic space, with sectional curvatures $\equiv -1$, for curves with total curvature at most 4π plus the area of the smallest cone $q \times\!\!\!\!\times \Gamma$ among $q \in \Gamma$ [BH]. Further, new unknotting results are proved, including one which is a simultaneous generalization of the two just mentioned: if Γ is a curve of total curvature at most $4\pi + \kappa^2\mathcal{A}(\Gamma)$ in a simply-connected manifold with sectional curvatures bounded above by $-\kappa^2 \le 0$, then Γ is unknotted (see Theorem 3).

2. Methods for Density Estimates: Flat Space

We shall first sketch the proof of the density estimate in the simplest case: $M = \mathbf{R}^n$. This proof is analogous to the proof given in [EWW], although it differs somewhat in the approach.

Let Γ be a smooth curve in $\mathbf{R}^n$ which bounds a minimal surface Σ, and consider a point $p \in \Sigma$. We will compare Σ with the Euclidean cone $C := p \times\!\!\!\!\times \Gamma$. The proof of the density estimate (4) at p, with $\widehat{K} = 0$, is broken up into the first two parts **(i)** (see Proposition 1 below) and **(ii)** (see Proposition 2 below), as indicated in the Introduction. Write $\rho(x) := |x - p|$ for $x \in \mathbf{R}^n$, and $G(x) := \log\rho(x)$.

Lemma 1. *Let N^2 be a two-dimensional manifold immersed in $\mathbf{R}^n$. Then except at p,*

$$\triangle_N G(\rho) = \frac{2}{\rho^2}\left(1 - |\nabla_N\rho|^2\right) + \frac{d\rho(\vec{H})}{\rho}$$

where $\vec{H}$ is the mean curvature vector of N.

Proof. Elementary calculations show that the Hessian in $\mathbf{R}^n$:

$$\overline{\nabla}^2\rho^2 = 2g,$$

where g is the Euclidean metric tensor.

The well-known trace formula states that

$$\triangle_N G = \sum_{\alpha=1}^{2} \overline{\nabla}^2 G(e_\alpha, e_\alpha) + dG(\vec{H}),$$

where $\{e_1, e_2\}$ is an orthonormal basis for the tangent plane to N.

This formula leads us by straightforward computations to the conclusion. ■

Note that the mean-curvature term $\frac{d\rho(\vec{H})}{\rho}$ vanishes in both cases $N = \Sigma$ and $N = C$. Since the gradient $\overline{\nabla}\rho$ in $\mathbf{R}^n$ has norm 1, the gradient on Σ has norm ≤ 1, implying $\triangle_\Sigma G(\rho) \ge 0$. Since $\overline{\nabla}\rho$ is tangent to the cone C, we find $\triangle_C G(\rho) = 0$.

In the next two propositions, we shall first assume that $C\backslash\{p\}$ is immersed in M. Results such as equation (5) and (6) below may be proved in the general case by approximation.

Proposition 1. *(Density Comparison) Let Γ be a C^2 immersed closed curve in $\mathbf{R}^n$. Choose $p \in \mathbf{R}^n\backslash\Gamma$. If Σ^2 is a branched minimal surface in $\mathbf{R}^n$ with boundary $\partial\Sigma = \Gamma$, and C is the cone $p \times \Gamma$ over p, then their densities at p satisfy the inequality*

$$\Theta_\Sigma(p) < \Theta_C(p),$$

unless Σ lies in a plane.

Proof. As we have just seen, $\triangle_\Sigma G \ge 0$ and $\triangle_C G \equiv 0$. For small $\varepsilon > 0$, write $C_\varepsilon := C\backslash B_\varepsilon(p)$, and similarly Σ_ε. Then the boundary of Σ_ε is $\Gamma \cup (\Sigma \cap \partial B_\varepsilon(p))$. Let ν_Σ (ν_C, respectively) be the outward unit normal vector tangent to Σ_ε at $\partial\Sigma_\varepsilon$ (to C_ε at ∂C_ε, resp.). Then

$$0 \le \int_{\Sigma_\varepsilon} \triangle_\Sigma G(\rho)\, dA = \int_{\partial\Sigma_\varepsilon} \nu_\Sigma \cdot \overline{\nabla} G\, ds = \int_{\Sigma\cap\partial B_\varepsilon(p)} \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\varepsilon}\, ds + \int_\Gamma \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\rho}\, ds.$$

As $\varepsilon \to 0$, along the small boundary component $\Sigma \cap \partial B_\varepsilon(p)$, $\nu_\Sigma \cdot \overline{\nabla}\rho \to -1$ uniformly, and

$$\frac{L(\Sigma \cap \partial B_\varepsilon(p))}{2\pi\varepsilon} \to \Theta_\Sigma(p).$$

Along Γ, $\nu_\Sigma \cdot \overline{\nabla}\rho \le \nu_C \cdot \overline{\nabla}\rho$. Hence as $\varepsilon \to 0$, we find

$$2\pi\Theta_\Sigma(p) \le \int_\Gamma \frac{\nu_C \cdot \overline{\nabla}\rho}{\rho}\, ds.$$

Similarly, along $C \cap \partial B_\varepsilon(p)$, we have $\nu_C \equiv -\overline{\nabla}\rho$. After applying the divergence theorem to the vector field $\nabla_C G(\rho)$ on C_ε, we find

$$2\pi\Theta_C(p) = \int_\Gamma \frac{\nu_C \cdot \overline{\nabla}\rho}{\rho}\, ds. \tag{5}$$

This implies $\Theta_\Sigma(p) \le \Theta_C(p)$. If equality holds, then $\triangle_\Sigma G \equiv 0$, which requires $|\nabla_\Sigma\rho| \equiv 1$ according to Lemma 1. This can only happen when the minimal surface Σ is flat. ■

Proposition 2. *(Gauss-Bonnet) Let Γ be a C^2 immersed closed curve in $\mathbf{R}^n$. Choose $p \in \mathbf{R}^n \backslash \Gamma$. If C is the cone $p ⋇ \Gamma$ over p, then its density at p is given by*

$$2\pi\Theta_C(p) = -\int_\Gamma \vec{k} \cdot \nu_C \, ds. \tag{6}$$

Proof. Recall that C is intrinsically flat: $K^C \equiv 0$. Also, its inner boundary component $C \cap \partial B_\varepsilon(p)$ has curvature vector $\vec{k} = \frac{1}{\varepsilon}\nu_C$. Finally, C_ε is a topological annulus, so $\chi(C_\varepsilon) = 0$. Therefore, by the Gauss-Bonnet Theorem,

$$0 = \int_{C_\varepsilon} K^C \, dA = \int_{C \cap \partial B_\varepsilon(p)} \vec{k} \cdot \nu_C \, ds + \int_\Gamma \vec{k} \cdot \nu_C \, ds = \frac{L(C \cap \partial B_\varepsilon(p))}{\varepsilon} + \int_\Gamma \vec{k} \cdot \nu_C \, ds.$$

But as $\varepsilon \to 0$, $\frac{L(C \cap \partial B_\varepsilon(p))}{\varepsilon} \to 2\pi\Theta_C(p)$. ∎

Theorem 1. *Let Σ^2 be a minimal surface in $\mathbf{R}^n$ with boundary curve Γ. For any point $p \in \mathbf{R}^n$,*

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\mathrm{tot}}(\Gamma)$$

unless Σ lies in a plane.

Proof. Follows immediately from Proposition 1 and Proposition 2, since at each point of Γ, $-\vec{k} \cdot \nu_C \leq |\vec{k}|$. ∎

3. Methods for Density Estimates: Curved Space

If the ambient manifold M has **constant sectional curvature** $K^M \equiv \widehat{K}$, then results analogous to Lemma 1, Proposition 1 and Proposition 2 above may be proved in a similar fashion to the Euclidean case $\widehat{K} = 0$ of Section 2. More precisely, when $\widehat{K} > 0$, we replace the Green's function $G(x) = \log \rho(x)$ of $\mathbf{R}^2$ with the Green's function of the 2-sphere of constant Gauss curvature $\widehat{K}$:

$$G(x) = \log \tan \frac{1}{2}\rho(x)\sqrt{\widehat{K}};$$

or, in the case $\widehat{K} =: -\kappa^2 < 0$, with the Green's function of the 2-dimensional hyperbolic plane of constant Gauss curvature $\widehat{K}$:

$$G(x) = \log \tanh \frac{1}{2}\kappa\rho(x).$$

In this more general case, the conclusion of Proposition 1 is unchanged, and the conclusion of Proposition 2 becomes

$$2\pi\Theta_C(p) = -\int_\Gamma \vec{k} \cdot \nu_C \, ds + \widehat{K}\,\mathrm{Area}(p ⋇ \Gamma).$$

The density estimate is a consequence:

Theorem 2. *Suppose M^n is a complete, simply connected Riemannian manifold with constant sectional curvatures $K^M \equiv \widehat{K}$. Let Σ^2 be a minimal surface in M^n with boundary curve Γ. In case $\widehat{K} > 0$, we assume that $\Sigma \cup \Gamma$ lies in an open hemisphere of the sphere M^n. Then for any point $p \in \Sigma$,*

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\mathrm{tot}}(\Gamma) + \widehat{K}\,\mathrm{Area}(p ⋇ \Gamma).$$

unless Σ is totally geodesic.

Remark 1. In the case of constant positive sectional curvatures, it is not completely necessary to require that $\Sigma \cup \Gamma$ lie in an open hemisphere. The proof continues to hold if, instead, it is assumed that the mean-convex hull of Γ in M^n is compact and does not contain two antipodal points. It is not difficult to construct such examples which do not lie in a hemisphere, using, for example, certain unstable minimal hypersurfaces constructed in [PR].

On the other hand, when M has **variable sectional curvature**, it should be observed that the cone $p ⋇ \Gamma$ may lie in a region of M whose geometry is unrelated to the geometry of M near Σ. Therefore, it is essential to consider a second Riemannian metric $(C, \widehat{g})$ on the geodesic cone (C, g), where g is the metric on C induced from M. We shall also write $\widehat{C}$ for the singular Riemannian manifold $(C, \widehat{g})$. We choose the metric $\widehat{g}$ of $\widehat{C}$ characterized by the properties that the radial unit-speed geodesics which generate $C = p ⋇ \Gamma$ remain unit-speed geodesics in $\widehat{C}$; that the curve Γ has the same arc length in either C or $\widehat{C}$ (or in M); that each radial geodesic meets Γ in the same angle measured in C or in $\widehat{C}$; and that $\widehat{C}$ has constant Gauss curvature $\widehat{K}$ away from the singular point p. This metric was introduced by Choe in his study of isoperimetric inequalities on minimal surfaces ([C].)

In the rest of this section, we shall treat explicitly only the case $\widehat{K} = -\kappa^2 < 0$. Some of the corresponding formulas for $\kappa = 0$ have already been given in section 2 above; others follow from limits of standard functions as $\kappa \to 0$. For the case $\widehat{K} > 0$, see [CG].

Lemma 2. *Let N^2 be a two-dimensional manifold immersed in a complete, simply connected Riemannian manifold M whose sectional curvature is bounded above by $-\kappa^2,\ \kappa > 0$. Then except at p,*

$$\triangle_N G(\rho) \geq 2\kappa^2 \frac{\cosh \kappa\rho}{\sinh^2 \kappa\rho} \left(1 - |\nabla_N \rho|^2\right) + \kappa \frac{d\rho(\vec{H})}{\sinh \kappa\rho}.$$

Proof. The Hessian comparison theorem shows that

$$\overline{\nabla}^2 \rho \geq \kappa \coth \kappa\rho (g - \overline{\nabla}\rho \otimes \overline{\nabla}\rho)$$

where $\rho(x)$ is the distance from p to x in M. We apply the trace theorem as in the proof of Lemma 1. ■

As a consequence of Lemma 2, we see that $G(x) = \log \tanh \frac{1}{2}\kappa\rho(x)$ is subharmonic on the minimal surface Σ and harmonic on the hyperbolic cone $\widehat{C}$.

In the next four propositions, as in section 2 above, we shall first assume that $C \setminus \{p\}$ is immersed in M. The key results obtained in the proofs of the propositions may be proved in the general case by approximation.

Proposition 3. *(Density Comparison) Let Σ^2 be a branched minimal surface in an n-dimensional simply connected Riemannian manifold M with sectional curvature $\leq -\kappa^2$.*

If $\widehat{C}$ is the hyperbolic cone defined above, then $\Theta_\Sigma(p) \leq \Theta_{\widehat{C}}(p)$, with strict inequality unless Σ is totally geodesic with constant Gauss curvature $-\kappa^2$.

Proof. As we have just shown, $\triangle_\Sigma G(x) \geq 0$ and $\triangle_{\widehat{C}} G(x) \equiv 0$.

Then with the notation of Proposition 1,

$$0 \leq \int_{\Sigma_\varepsilon} \triangle_\Sigma G\, dA = \int_{\partial\Sigma_\varepsilon} \nu_\Sigma \cdot \overline{\nabla} G\, ds = \int_{\Sigma\cap\partial B_\varepsilon(p)} \kappa \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sinh \kappa\varepsilon}\, ds + \int_\Gamma \kappa \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sinh \kappa\rho}\, ds.$$

Note that

$$\kappa \frac{L(\Sigma \cap \partial B_\varepsilon(p))}{2\pi \sinh \kappa\varepsilon} \to \Theta_\Sigma(p).$$

Also, it should be observed that

$$\nu_\Sigma \cdot \overline{\nabla}\rho \leq \nu_C \cdot \overline{\nabla}\rho \text{ along } \Gamma.$$

Thus, we find that the inequality above implies

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \kappa \frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh \kappa\rho}\, ds. \tag{7}$$

Note here that ν_C, considered as a tangent vector to C, is also the outward unit normal vector in the metric $\widehat{g}$. Along the intrinsic distance sphere $\partial\widehat{B}_\varepsilon(p) \subset \widehat{C}$, $-\nabla\rho$ is the outward unit normal vector. Hence since G is harmonic on $\widehat{C}$, as $\varepsilon \to 0$,

$$0 = \int_{\widehat{C}_\varepsilon} \triangle_{\widehat{C}} G(x)\, dA \to -2\pi\Theta_{\widehat{C}}(p) + \int_\Gamma \kappa \frac{\nu_C \cdot \nabla\rho}{\sinh \kappa\rho}\, ds.$$

Therefore, by inequality (7),

$$2\pi\Theta_{\widehat{C}}(p) = \int_\Gamma \kappa \frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh \kappa\rho}\, ds \geq 2\pi\Theta_\Sigma(p),$$

which is the desired estimate.

If equality holds, then $\triangle_\Sigma G \equiv 0$, which requires $|\nabla_\Sigma \rho| \equiv 1$ according to Lemma 2. But this means that Σ is a cone over p, as well as being minimal, which can only occur when Σ is totally geodesic. Moreover, $\triangle_\Sigma G \equiv 0$ now implies that $\triangle_\Sigma \rho \equiv \kappa \coth \kappa\rho$, which, along with $K_\Sigma \leq K^M \leq -\kappa^2$, implies that Σ has constant Gauss curvature $K_\Sigma \equiv -\kappa^2$. ∎

Proposition 4. *(Geodesic Curvature Comparison) Let Γ be a C^2 curve in M^n, a manifold with sectional curvatures $\leq -\kappa^2$, and let C be the cone $p ⋇ \Gamma$. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as defined above, then $k(q) \geq \widehat{k}(q)$ for almost all $q \in \Gamma$, where k and $\widehat{k}$ denote the inward geodesic curvatures of Γ in C and $\widehat{C}$, respectively.*

Proof. The proof follows from comparison of the Jacobi equations along a radial geodesic γ through p:

$$f''(t) + K^C(\gamma(t))f(t) = 0 \quad\text{and}\quad \widehat{f}''(t) + \widehat{K}\widehat{f}(t) = 0, \tag{8}$$

with initial conditions $f(0) = 0 = \widehat{f}(0)$ and $f'(0) = a_0 > 0$, $\widehat{f}'(0) = \widehat{a}_0 > 0$, where $K^C = K^M \leq -\kappa^2 = \widehat{K}$ from the Gauss equations for C as a submanifold of M. The geodesic curvature of $C \cap \partial B_{\rho_0}(p)$ (as a curve in C) is $f'(\rho_0)/f(\rho_0)$, and similarly the geodesic curvature of $\widehat{C} \cap \partial B_{\rho_0}(p)$ is $\widehat{f}'(\rho_0)/\widehat{f}(\rho_0)$. The comparison theorem shows that

$$f'/f \geq \widehat{f}'/\widehat{f}. \tag{9}$$

Since C and $\widehat{C}$ have the same metric at points of Γ, one may show that

$$k - \widehat{k} = (\frac{f'}{f} - \frac{\widehat{f}'}{\widehat{f}}) \cos\varphi \geq 0,$$

where at each point of Γ, φ is the angle in either metric between $\nu_C = \nu_{\widehat{C}}$ and $\nabla\rho = \widehat{\nabla}\rho$. (For details see [CG], Proposition 4.) Thus $k \geq \widehat{k}$. ∎

Proposition 5. *(Area Comparison) Let Γ be a C^2 curve in M^n, and let $C = p \times\!\!\!\!\times \Gamma$. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as defined above, then the areas* $\mathrm{Area}(C) \leq \mathrm{Area}(\widehat{C})$.

Proof. We continue to use the notation $f(t), \widehat{f}(t), \varphi(q)$ as in the proof of Proposition 4, with the following refinement. Along the radial unit-speed geodesic $\gamma_q : [0, \rho(q)] \to C = p \times\!\!\!\!\times \Gamma$, where $\gamma_q(0) = p$, $\gamma_q(\rho(q)) = q \in \Gamma$, which is a geodesic in both metrics g and $\widehat{g}$, let $f(t) = f_q(t)$ or $\widehat{f}(t) = \widehat{f}_q(t)$ denote solutions of the Jacobi equation (8). We may choose the normalizations $f_q'(0) = a_0(q)$ and $\widehat{f}_q'(0) = \widehat{a}_0(q)$ so that $f_q(\rho(q)) = \cos\varphi(q) = \widehat{f}_q(\rho(q))$ as well as $f_q(0) = 0 = \widehat{f}_q(0)$, since K^C and $\widehat{K}$ are nonpositive. Then

$$\mathrm{Area}(C) = \int_\Gamma \int_0^{\rho(q)} f_q(t)\, dt\, ds(q),$$

and similarly for $\mathrm{Area}(\widehat{C})$. But $f_q(t),\ \widehat{f}_q(t) > 0$ for $t > 0$, and $f_q(t)/\widehat{f}_q(t)$ is nondecreasing according to inequality (9). Since $f_q(\rho(q))/\widehat{f}_q(\rho(q)) = 1$, we find $f_q(t) \leq \widehat{f}_q(t)$ for all $0 \leq t \leq \rho(q)$, which implies $\mathrm{Area}(C) \leq \mathrm{Area}(\widehat{C})$. ∎

Proposition 6. *(Gauss-Bonnet) For any cone $\widehat{C}$ over an immersed C^2 curve Γ in M^n, with vertex $p \notin \Gamma$, and endowed with constant Gauss curvature $-\kappa^2$,*

$$2\pi\Theta_{\widehat{C}}(p) + \kappa^2\,\mathrm{Area}(\widehat{C}) = \int_\Gamma \widehat{k}\, ds,$$

where $\widehat{k}$ is the geodesic curvature of Γ in $\widehat{C}$.

Proof. By the Gauss-Bonnet formula on $\widehat{C}_\varepsilon := \widehat{C} \backslash B_\varepsilon(p)$,

$$\int_{\widehat{C}_\varepsilon} \widehat{K}\, dA + \int_\Gamma \widehat{k}\, ds + \int_{\widehat{C} \cap \partial B_\varepsilon(p)} \widehat{k}\, ds = 2\pi\chi(\widehat{C}_\varepsilon) = 0, \tag{10}$$

since $\widehat{C}_\varepsilon$ is an immersed annulus, implying the Euler number $\chi(\widehat{C}_\varepsilon) = 0$.

The inward geodesic curvature $\widehat{k}$ of $\widehat{C} \cap \partial B_\varepsilon(p)$ equals $-\kappa \coth \kappa\varepsilon$. Thus,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\widehat{C} \cap \partial B_\varepsilon(p)} \widehat{k}\, ds &= -\lim_{\varepsilon \to 0} (\kappa \coth \kappa\varepsilon) L(\widehat{C} \cap \partial B_\varepsilon(p)) \\ &= -2\pi\Theta_{\widehat{C}}(p). \end{aligned}$$

Since $\text{Area}(\widehat{C}_\varepsilon) \to \text{Area}(\widehat{C})$, the formula (10) now implies

$$-\kappa^2 \,\text{Area}(\widehat{C}) + \int_\Gamma \widehat{k}\, ds - 2\pi\Theta_{\widehat{C}}(p) = 0. \tag{11}$$

■

Theorem 3. *Let Σ^2 be a branched minimal surface (of arbitrary topological type) in an n-dimensional complete, simply connected Riemannian manifold M whose sectional curvature is bounded above by a nonpositive constant $\widehat{K} = -\kappa^2$. Write $\Gamma = \partial\Sigma$, which we assume to be a C^2 embedding of the circle S^1. Then the density of Σ at any point $p \notin \Gamma$ satisfies*

$$2\pi\Theta_\Sigma(p) \le \mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2 \;\text{Area}(p ⋇ \Gamma); \tag{12}$$

moreover, equality can hold only if Σ is totally geodesic.

Proof. We sketch the proof only for an immersed minimal surface; see [CG] for branch points.

Consider any $p \in \Sigma\backslash\Gamma$, and let $C = p ⋇ \Gamma$ be the geodesic cone over Γ with vertex p. If Σ is totally geodesic, then Σ is embedded, since there are no compact totally geodesic surfaces in M. Otherwise, by Proposition 3 and Proposition 6, we have

$$2\pi\Theta_\Sigma(p) < 2\pi\Theta_{\widehat{C}}(p) = \int_\Gamma \widehat{k}\, ds - \kappa^2 \,\text{Area}(\widehat{C}).$$

Since $\widehat{k} \le k \le |\vec{k}|$ almost everywhere along Γ by Proposition 4, and using the area comparison of Proposition 5, we find

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2 \,\text{Area}(C).$$

■

4. Towards Density Estimates for Flow by Mean Curvature

Motivated by the elliptic results of sections 2 and 3 above, we consider the question of density estimates for two-dimensional surfaces evolving by mean-curvature flow (1). This brief section is the result of ongoing discussions with Mu-Tao Wang and Mao-Pei Tsui.

Step **(i)** of the program carried out above, for example in Proposition 1, finds a sharp upper bound $\Theta_\Sigma(p) \le \Theta_C(p)$ on the density of a minimal surface Σ in $\mathbf{R}^n$ with boundary Γ, where C is the cone $p ⋇ \Gamma$ with vertex p. For the theory of minimal surfaces in $\mathbf{R}^n$, the cone has special properties which make it appropriate for such an estimate: **(a)** The cone is self-similar for the family of homotheties of $\mathbf{R}^n$ which preserve p; **(b)** the function $G(x) = \log \rho(x)$ is harmonic on C, even though C itself is not minimal; finally, **(c)** the normal derivative $\nu_C \cdot \nabla\rho$ of $\rho(x)$ is the maximum at each point of Γ among all surfaces with boundary Γ. Property **(a)** is very useful in constructing a "soliton" and investigating its properties. However, properties **(b)** and **(c)** are more directly relevant to deriving results for minimal surfaces.

When we turn to the problem of a surface Σ_t in $\mathbf{R}^n$ evolving by its mean curvature:

$$\frac{\partial \Sigma_t}{\partial t} = H\vec{\nu} =: \vec{H}, \tag{13}$$

we observe that there is a strong analog of the monotonicity inequality of minimal surfaces, which has now become a basic tool [H]. In $\mathbf{R}^n \times [0, t_0]$, let $G(x,t)$ be the fundamental solution for the backwards heat flow on $\mathbf{R}^2$:

$$G(x,t) = \frac{1}{2\pi(t_0 - t)} \exp\left[-|x - x_0|^2/4(t_0 - t)\right].$$

Then

$$\frac{d}{dt}\int_{\Sigma_t} G(x,t)\,dA = -\int_{\Sigma_t} G(x,t)\left|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 dA$$

plus boundary terms, where $(x - x_0)^\perp$ is the normal component of the vector $x - x_0$. Equivalently,

$$\frac{\partial G}{\partial t} + \Delta_{\Sigma_t} G(x,t) = 2\langle \vec{H}, \nabla G\rangle - \frac{\nabla^\perp G}{G}.$$

This leads to an inequality, for which equality holds on the parabolically self-similar surface having the same evolving boundary Γ_t as does Σ_t.

The backwards heat kernel $G(x,t)$ is closely related to the parabolic density of the evolving surface $\Sigma_. := \{\Sigma_t : 0 \le t \le t_0\}$ at the point (x_0, t_0), where $x_0 \in \Sigma_{t_0}$:

$$\Theta_{\Sigma_.}(x_0, t_0) = \lim_{t \to t_0^-} \int_{\Sigma_t} G(x,t)dA.$$

If $\Theta_{\Sigma_.}(x_0, t_0)$ is sufficiently close to 1, then a varifold solution of mean-curvature flow (1) will be smooth (see e.g. [W]).

In space forms such as hyperbolic space, there are no similarity transformations other than isometries, so the self-similar "soliton" suggested by **(a)** does not exist. Nonetheless, analogues of **(b)** and **(c)** are possible and may well form the basis of a density estimate parallel to the result of [CG].

In the general case where the ambient manifold M^n has variable sectional curvatures, methods for proving density bounds will become more involved. In particular, the reader will note from the model of section 3 above that the evolving comparison surfaces in the general variable-curvature case will need to be endowed with metrics other than the induced metric from M. Instead, one should aim to find evolving surfaces with artificial Riemannian metrics which are extremal in the sense of **(b)** over all ambient manifolds with a given upper bound on sectional curvatures.

REFERENCES

[AB] S. Alexander and R. L. Bishop, *The Fáry-Milnor theorem in Hadamard manifolds*, Proc. Amer. Math. Soc. **126** (1998), 3427–3436.

[A] H. W. Alt, *Verzweigungspunkte von H-Flächen:* I, Math. Z. **127** (1972), 333–362; II, Math. Ann. **201** (1973), 33–56.

[BH] F. Brickell and C. C. Hsiung, *The total absolute curvature of closed curves in Riemannian manifolds*, J. Differential Geom. **9** (1974), 177–193.

[C] J. Choe, *The isoperimetric inequality for minimal surfaces in a Riemannian manifold*, J. reine angewandte Mathematik **506** (1999), 205-214.

[CG] J. Choe and R. Gulliver, *Embedded Minimal Surfaces and Total Curvature of Curves in a Manifold*, to appear in Math Research Letters.

[D] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), 263–321.

[EWW] T. Ekholm, B. White, and D. Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most* 4π, Annals of Mathematics **155** (2002), 109-234.

[F] M. I. Fáry, *Sur la courbure totale d'une courbe gauche faisant un noeud*, Bull. Soc. Math. France **77** (1949), 128–138.

[G] R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Annals of Math. **97** (1973), 275–305.

[H] G. Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Differential Geom. **31**, (1990), 285–299.

[M] J. Milnor, *On the total curvature of knots*, Annals of Math. **52** (1950), 248–257.

[Mo] C. B. Morrey, Jr., *The problem of Plateau on a Riemannian manifold*, Annals of Math. **49** (1948), 807–851.

[PR] Pitts, Jon T. and Rubinstein, J. Hyam, *Equivariant minimax and minimal surfaces in geometric three-manifolds*, Bull. Amer. Math. Soc. **19** (1988), 303–309.

[R] T. Radó, *The problem of least area and the problem of Plateau*, Math. Z. **32** (1930), 763–796.

[S] C. Schmitz, *The theorem of Fáry-Milnor for Hadamard manifolds*, Geom. Dedicata **71** (1998), 83–90.

[W] B. White, *Stratification of minimal surfaces, mean curvature flows and harmonic maps*, J. reine angew. Math. **488** (1997), 1–35.

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