TOTAL CURVATURE OF GRAPHS AFTER MILNOR AND EULER

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Abstract. We define a new notion of total curvature, called *net total curvature,* for finite graphs embedded in $\mathbb{R}^n$, and investigate its properties. Two guiding principles are given by Milnor's way of measuring the local crookedness of a Jordan curve via a Crofton-type formula, and by considering the double cover of a given graph as an Eulerian circuit. The strength of combining these ideas in defining the curvature functional is (1) it allows us to interpret the singular/non-eulidean behavior at the vertices of the graph as a superposition of vertices of a 1-dimensional manifold, and thus (2) one can compute the total curvature for a wide range of graphs by contrasting local and global properties of the graph utilizing the integral geometric representation of the curvature. A collection of results on upper/lower bounds of the total curvature on isotopy/homeomorphism classes of embeddings is presented, which in turn demonstrates the effectiveness of net total curvature as a new functional

measuring complexity of spatial graphs in differential-geometric terms.

1. INTRODUCTION: CURVATURE OF A GRAPH

The celebrated Fáry-Milnor theorem states that a curve in $\mathbb{R}^n$ of total curvature at most 4π is unknotted.

As a key step in his 1950 proof, John Milnor showed that for a smooth Jordan curve Γ in $\mathbb{R}^3$, the total curvature equals half the integral over $e \in S^2$ of the number $\mu(e)$ of local maxima of the linear "height" function $\langle e, \cdot\rangle$ along Γ [M]. This equality can be regarded as a Crofton-type representation formula of total curvature where the order of integrations over the curve and the unit tangent sphere (the space of directions) are reversed. The Fáry-Milnor theorem follows, since total curvature less than 4π implies there is a unit vector $e_0 \in S^2$ so that $\langle e_0, \cdot\rangle$ has a unique local maximum, and therefore that this linear function is increasing on an interval of Γ and decreasing on the complement. Without changing the pointwise value of this "height" function, Γ can be topologically untwisted to a standard embedding of S^1 into $\mathbb{R}^3$. The Fenchel theorem, that any curve in $\mathbb{R}^3$ has total curvature at least 2π, also follows from Milnor's key step, since for all $e \in S^2$, the linear function $\langle e, \cdot\rangle$ assumes its maximum somewhere along Γ, implying $\mu(e) \geq 1$. Milnor's proof is independent of the proof of Istvan Fáry, published earlier, which takes a different approach [Fa].

We would like to extend the methods of Milnor's seminal paper, replacing the simple closed curve by a finite *graph* Γ in $\mathbb{R}^3$. Γ consists of a finite number of points, the *vertices*, and a finite number of simple arcs, the *edges*, each of which has as its endpoints one or two of the vertices. We shall assume Γ is connected.

Date: December 31, 2010.
Supported in part by JSPS Grant-in-aid for Scientific Research No.17740030.
Thanks to the Korea Institute for Advanced Study for invitations.

The *degree* of a vertex q is the number $d(q)$ of edges which have q as an endpoint. (Another word for degree is "valence".) We remark that it is technically not needed that the dimension n of the ambient space equals three. All the arguments can be generalized to higher dimensions, although in higher dimensions ($n \geq 4$) there are no nontrivial knots. Moreover, any two homeomorphic graphs are isotopic.

The key idea in generalizing total curvature for knots to total curvature for graphs is to consider the Euler circuits of the given graph, namely, parameterizations by S^1, of the *double* cover of the graph. We note that given a graph of even degree, there can be several Euler circuits, or ways to "trace it without lifting the pen." A topological vertex of a graph of degree d is a singularity, in that the graph is not locally Euclidean. However by considering an Euler circuit of the double of the graph, the vertex becomes locally the intersection point of d paths. We will show (Corollary 2) that at the vertex, each path through it has a (signed) measure-valued curvature, and the absolute value of the sum of those measures is well-defined, independent of the choice of the Euler circuit of the double cover. We define (Definition 1) the *net total curvature* (NTC) of a piecewise C^2 graph to be the sum of the total curvature of the smooth arcs and the contributions from the vertices as described.

This notion of net total curvature is substantially different from the total curvature, denoted TC, as defined by Taniyama [T]. (Taniyama writes τ for TC.) See section 2 below.

This is consistent with known results for the vertices of degree $d = 2$; with vertices of degree three or more, this definition helps facilitate a new Crofton-type representation formula (Theorem 1) for total curvature of graphs, where the total curvature is represented as an integral over the unit sphere. Recall that the vertex is now seen as d distinct points on an Euler circuit. The way we pick up the contribution of the total curvature at the vertices identifies the d distinct points, and thus the $2d$ unit tangent spheres on a circuit. As Crofton's formula in effect reverses the order of integrations — one over the circuit, the other over the space of tangent directions — the sum of the d exterior angles at the vertex is incorporated in the integral over the unit sphere. On the other hand the integrand of the integral over the unit sphere counts the number of net local maxima of the height function along an axis, where net local maximum means the number of local maxima minus the number of local minima at these d points of the Euler circuit. This establishes a correspondence between the differential geometric quantity (net total curvature) and the differential topological quantity (average number of maxima) of the graph, as stated in Theorem 1 below.

In section 2, we compare several definitions for total curvature of graphs which have appeared in the recent literature. In section 3, we introduce the main tool (Lemma 1) which in a sense reduces the computation of NTC to counting intersections with planes.

Milnor's treatment [M] of total curvature also contained an important topological extension. Namely, in order to define total curvature, the knot needs only to be *continuous*. This makes the total curvature a geometric quantity defined on any homeomorphic image of S^1. In this article, we first define net total curvature (Definition 1) on piecewise C^2 graphs, and then extend the definition to continuous graphs (Definition 3.) In analogy to Milnor, we approximate a given continuous graph by a sequence of polygonal graphs. In showing the monotonicity of the total curvature (Proposition 2) under the refining process of approximating graphs we use our representation formula (Theorem 1) applied to the polygonal graphs.

Consequently the Crofton-type representation formula is also extended (Theorem 2) to cover continuous graphs. Additionally, we are able to show that continuous graphs with finite total curvature (NTC or TC) are tame. We say that a graph is *tame* when it is isotopic to an embedded polyhedral graph.

In sections 5 through 8, we characterize NTC with respect to the geometry and the topology of the graph. Proposition 5 shows the subadditivity of NTC under the union of graphs which meet in a finite set. In section 6, the concept of bridge number is extended from knots to graphs, in terms of which the minimum of NTC can be explicitly computed, provided the graph has at most one vertex of degree > 3. In section 7, Theorem 6 gives a lower bound for NTC in terms of the width of an isotopy class. The infimum of NTC is computed for specific graph types: the two-vertex graphs θ_m, the "ladder" L_m, the "wheel" W_m, the complete graph K_m on m vertices and the complete bipartite graph $K_{m,n}$.

Finally we prove a result (Theorem 7) which gives a Fenchel type lower bound ($\geq 3\pi$) for total curvature of a theta graph (an image of the graph consisting of a circle with an arc connecting a pair of antipodal points), and a Fáry-Milnor type upper bound ($< 4\pi$) to imply the theta graph is isotopic to the standard embedding. A similar result was given by Taniyama [T], referring to TC. In contrast, for graphs of the type of K_m ($m \geq 4$), the infimum of NTC in the isotopy class of a polygon on m vertices is also the infimum for a sequence of distinct isotopy classes.

Many of the results in our earlier preprint [GY2] have been incorporated into the present paper.

We thank Yuya Koda for his comments regarding Proposition 7, and Jaigyoung Choe and Rob Kusner for their comments about Theorem 7, especially about the sharp case NTC(Γ) = 3π of the lower bound estimate.

2. DEFINITIONS OF TOTAL CURVATURE

The first difficulty, in extending the results of Milnor's classic paper, is to understand the contribution to total curvature at a vertex of degree $d(q) \geq 3$. We first consider the well-known case:

Definition of Total Curvature for Knots

For a smooth closed curve Γ, the total curvature is

$$C(\Gamma) = \int_\Gamma |\vec{k}|\, ds,$$

where s denotes arc length along Γ and $\vec{k}$ is the curvature vector. If $x(s) \in \mathbb{R}^3$ denotes the position of the point measured at arc length s along the curve, then

$\vec{k} = \frac{d^2x}{ds^2}$. For a piecewise smooth curve, that is, a graph with vertices $q_1, \ldots, q_N$ having always degree $d(q_i) = 2$, the total curvature is readily generalized to

$$C(\Gamma) = \sum_{i=1}^{N} \mathrm{c}(q_i) + \int_{\Gamma_{\mathrm{reg}}} |\vec{k}|\, ds, \tag{1}$$

where the integral is taken over the separate C^2 edges of Γ without their endpoints; and where $\mathrm{c}(q_i) \in [0, \pi]$ is the exterior angle formed by the two edges of Γ which meet at q_i. That is, $\cos(\mathrm{c}(q_i)) = \langle T_1, -T_2 \rangle$, where $T_1 = \frac{dx}{ds}(q_i^+)$ and $T_2 = -\frac{dx}{ds}(q_i^-)$ are the unit tangent vectors at q_i pointing into the two edges which meet at q_i. The exterior angle $\mathrm{c}(q_i)$ is the correct contribution to total curvature, since any sequence of smooth curves converging to Γ in C^0, with C^1 convergence on compact subsets of each open edge, includes a small arc near q_i along which the tangent vector changes from near $\frac{dx}{ds}(q_i^-)$ to near $\frac{dx}{ds}(q_i^+)$. The greatest lower bound of the contribution to total curvature of this disappearing arc along the smooth approximating curves equals $\mathrm{c}(q_i)$.

Note that $C(\Gamma)$ is well defined for an *immersed* knot Γ.

Definitions of Total Curvature for Graphs

When we turn our attention to a *graph* Γ, we find the above definition for curves (degree $d(q) = 2$) does not generalize in any obvious way to higher degree (see [G]). The ambiguity of the general formula (1) is resolved if we specify the replacement for c(0) when Γ is the cone over a finite set $\{T_1, \ldots, T_d\}$ in the unit sphere S^2.

The earliest notion of total curvature of a graph appears in the context of the first variation of length of a graph, which we call **variational total curvature**, and is called the *mean curvature* of the graph in [AA]: we shall write VTC. The contribution to VTC at a vertex q of degree 2, with unit tangent vectors T_1 and T_2, is $\mathrm{vtc}(q) = |T_1 + T_2| = 2\sin(c(q)/2)$. At a non-straight vertex q of degree 2, $\mathrm{vtc}(q)$ is less than the exterior angle $\mathrm{c}(q)$. For a vertex of degree d, the contribution is $\mathrm{vtc}(q) = |T_1 + \cdots + T_d|$.

A rather natural definition of total curvature of graphs was given by Taniyama in [T]. We have called this **maximal total curvature** $\mathrm{TC}(\Gamma)$ in [G]. The contribution to total curvature at a vertex q of degree d is

$$\mathrm{tc}(q) := \sum_{1 \le i < j \le d} \arccos\langle T_i, -T_j \rangle.$$

In the case $d(q) = 2$, the sum above has only one term, the exterior angle $\mathrm{c}(q)$ at q. Since the length of the Gauss image of a curve in S^2 is the total curvature of the curve, $\mathrm{tc}(q)$ may be interpreted as adding to the Gauss image in $\mathbb{R}P^2$ of the edges, a complete great-circle graph on $T_1(q), \ldots, T_d(q)$, for each vertex q of degree d. Note that the edge between two vertices does not measure the distance in $\mathbb{R}P^2$ but its supplement.

In our earlier paper [GY1] on the density of an area-minimizing two-dimensional rectifiable set Σ spanning Γ, we found that it was very useful to apply the Gauss-Bonnet formula to the cone over Γ with a point p of Σ as vertex. The relevant notion of total curvature in that context is **cone total curvature** $\mathrm{CTC}(\Gamma)$, defined using $\mathrm{ctc}(q)$ as the replacement for $\mathrm{c}(q)$ in equation (1):

$$\mathrm{ctc}(q) := \sup_{e \in S^2} \left\{ \sum_{i=1}^{d} \left(\frac{\pi}{2} - \arccos\langle T_i, e\rangle \right) \right\}. \tag{2}$$

Note that in the case $d(q) = 2$, the supremum above is assumed at vectors e lying in the smaller angle between the tangent vectors T_1 and T_2 to Γ, so that $\mathrm{ctc}(q)$ is then the exterior angle $\mathrm{c}(q)$ at q. The main result of [GY1] is that 2π times the area density of Σ at any of its points is at most equal to $\mathrm{CTC}(\Gamma)$. The same result had been proven by Eckholm, White and Wienholtz for the case of a simple closed curve [EWW]. Taking Σ to be the branched immersion of the disk given by Douglas [D1] and Radó [R], it follows that if $C(\Gamma) \leq 4\pi$, then Σ is embedded, and therefore Γ is unknotted. Thus [EWW] provided an independent proof of the Fáry-Milnor theorem. However, $\mathrm{CTC}(\Gamma)$ may be small for graphs which are far from the simplest isotopy types of a graph Γ.

In this paper, we introduce the notion of **net total curvature** $\mathrm{NTC}(\Gamma)$, which is the appropriate definition for generalizing — *to graphs* — Milnor's approach to isotopy and total curvature of *curves*. For each unit tangent vector T_i at q, $1 \leq i \leq d = d(q)$, let $\chi_i : S^2 \to \{-1, +1\}$ be equal to -1 on the hemisphere with center at T_i, and $+1$ on the opposite hemisphere (modulo sets of zero Lebesgue measure). We then define

$$\mathrm{ntc}(q) := \frac{1}{4} \int_{S^2} \left[\sum_{i=1}^{d} \chi_i(e) \right]^+ dA_{S^2}(e). \tag{3}$$

We note that the function $\sum_{i=1}^{d} \chi_i(e)$ is odd, hence the quantity above can be written as

$$\mathrm{ntc}(q) := \frac{1}{8} \int_{S^2} \left| \sum_{i=1}^{d} \chi_i(e) \right| dA_{S^2}(e).$$

as well. In the case $d(q) = 2$, the integrand of (3) is positive (and equals 2) only on the set of unit vectors e which have negative inner products with both T_1 and T_2, ignoring e in sets of measure zero. This set is bounded by semi-great circles orthogonal to T_1 and to T_2, and has spherical area equal to twice the exterior angle. So in this case, $\mathrm{ntc}(q)$ is the exterior angle. Thus, in the special case where Γ is a piecewise smooth curve, the following quantity $\mathrm{NTC}(\Gamma)$ coincides with total curvature, as well as with $\mathrm{TC}(\Gamma)$ and $\mathrm{CTC}(\Gamma)$:

Definition 1. *We define the* net total curvature *of a piecewise* C^2 *graph* Γ *with vertices* $\{q_1, \ldots, q_N\}$ *as*

$$\mathrm{NTC}(\Gamma) := \sum_{i=1}^{N} \mathrm{ntc}(q_i) + \int_{\Gamma_{\mathrm{reg}}} |\vec{k}|\, ds. \tag{4}$$

For the sake of simplicity, elsewhere in this paper, we consider the ambient space to be $\mathbb{R}^3$. However the definition of the net total curvature can be generalized for a graph in $\mathbb{R}^n$ by defining the vertex contribution in terms of an average over S^{n-1}:

$$\mathrm{ntc}(q) := \pi\Big(\fint_{S^{n-1}} \Big[\sum_{i=1}^{d} \chi_i(e) \Big]^+ \, dA_{S^{n-1}}(e) \Big),$$

which is consistent with the definition (3) of ntc when $n = 3$.

Recall that Milnor [M] defines the total curvature of a continuous simple closed curve C as the supremum of the total curvature of all polygons inscribed in C. By analogy, we define net total curvature of a *continuous* graph Γ to be the supremum of the net total curvature of all polygonal graphs P suitably inscribed in Γ as follows.

Definition 2. *For a given continuous graph* Γ*, we say a polygonal graph* $P \subset \mathbb{R}^3$ *is* Γ-approximating*, provided that its topological vertices (those of degree* $\neq 2$*) are exactly the topological vertices of* Γ*, and having the same degrees; and that the arcs of P between two topological vertices correspond one-to-one to the edges of* Γ *between those two vertices.*

Note that if P is a Γ-approximating polygonal graph, then P is homeomorphic to Γ. According to the statement of Proposition 2, whose proof will be given in the next section, if P and $\widetilde{P}$ are Γ-approximating polygonal graphs, and $\widetilde{P}$ is a refinement of P, then $\mathrm{NTC}(\widetilde{P}) \geq \mathrm{NTC}(P)$. Here $\widetilde{P}$ is said to be a refinement of P provided the set of vertices of P is a subset of the vertices of $\widetilde{P}$. Assuming Proposition 2 for the moment, we can generalize the definition of the total curvature to non-smooth graphs.

Definition 3. *Define the* net total curvature *of a continuous graph* Γ *by*

$$\mathrm{NTC}(\Gamma) := \sup_P \mathrm{NTC}(P)$$

where the supremum is taken over all Γ*-approximating polygonal graphs P.*

For a polygonal graph P, applying Definition 1,

$$\mathrm{NTC}(P) := \sum_{i=1}^{N} \mathrm{ntc}(q_i),$$

where $q_1, \ldots, q_N$ are the vertices of P.

Definition 3 is consistent with Definition 1 in the case of a piecewise C^2 graph Γ. Namely, as Milnor showed, the total curvature $C(\Gamma_0)$ of a smooth curve Γ_0 is the supremum of the total curvature of inscribed polygons ([M], p. 251), which gives the required supremum for each edge. At a vertex q of the piecewise-C^2 graph Γ, as a sequence P_k of Γ-approximating polygons become arbitrarily fine, a vertex q of P_k (and of Γ) has unit tangent vectors converging in S^2 to the unit tangent vectors to Γ at q. It follows that for $1 \le i \le d(q)$, $\chi_i^{P_k} \to \chi_i^{\Gamma}$ in measure on S^2, and therefore $\mathrm{ntc}_{P_k}(q) \to \mathrm{ntc}_{\Gamma}(q)$.

3. CROFTON-TYPE REPRESENTATION FORMULA FOR TOTAL CURVATURE

We would like to explain how the net total curvature $\mathrm{NTC}(\Gamma)$ of a graph is related to more familiar notions of total curvature. Recall that a graph Γ has an Euler circuit if and only if its vertices all have even degree, by a theorem of Euler. An Euler circuit is a closed, connected path which traverses each edge of Γ exactly once. Of course, we do not have the hypothesis of even degree. We can attain that hypothesis by passing to the *double* $\widetilde{\Gamma}$ of Γ: $\widetilde{\Gamma}$ is the graph with the same vertices as Γ, but with two copies of each edge of Γ. Then at each vertex q, the degree as a vertex of $\widetilde{\Gamma}$ is $\widetilde{d}(q) = 2\,d(q)$, which is even. By Euler's theorem, there is an Euler circuit Γ' of $\widetilde{\Gamma}$, which may be thought of as a closed path which traverses each edge of Γ exactly *twice*. Now at each of the points $\{q_1, \ldots, q_d\}$ along Γ' which are mapped to $q \in \Gamma$, we may consider the exterior angle $\mathrm{c}(q_i)$. The sum of these exterior angles, however, depends on the choice of the Euler circuit Γ'. For example, if Γ is the union of the x-axis and the y-axis in Euclidean space $\mathbb{R}^3$, then one might choose Γ' to have four right angles, or to have four straight angles, or something in between, with completely different values of total curvature. In order to form a version of total curvature at a vertex q which only depends on the original graph Γ and not on the choice of Euler circuit Γ', it is necessary to consider some of the exterior angles as partially balancing others. In the example just considered, where Γ is the union of two orthogonal lines, two opposing right angles will be considered to balance each other completely, so that $\mathrm{ntc}(q) = 0$, regardless of the choice of Euler circuit of the double.

It will become apparent that the connected character of an Euler circuit of $\widetilde{\Gamma}$ is not required for what follows. Instead, we shall refer to a *parameterization* Γ' of the double $\widetilde{\Gamma}$, which is a mapping from a 1-dimensional manifold without boundary, not necessarily connected; the mapping is assumed to cover each edge of $\widetilde{\Gamma}$ once.

The nature of $\mathrm{ntc}(q)$ is clearer when it is localized on S^2, analogously to [M]. In the case $d(q) = 2$, Milnor observed that the exterior angle at the vertex q equals half the area of those $e \in S^2$ such that the linear function $\langle e, \cdot \rangle$, restricted to Γ, has a local maximum at q. In our context, we may describe $\mathrm{ntc}(q)$ as one-half the integral over the sphere of the number of *net local maxima*, which is half the difference of local maxima and local minima. Along the parameterization Γ' of the double of Γ, the linear function $\langle e, \cdot \rangle$ may have a local maximum at some of the vertices $q_1, \ldots, q_d$ over q, and may have a local minimum at others. In our construction, each local minimum balances against one local maximum. If there are more local minima than local maxima, the number $\mathrm{nlm}(e, q)$, the net number of local maxima, will be negative; however, our definition uses only the positive part $[\mathrm{nlm}(e, q)]^+$.

We need to show that

$$\int_{S^2} [\mathrm{nlm}(e,q)]^+ \, dA_{S^2}(e)$$

is independent of the choice of parameterization, and in fact is equal to $2\,\mathrm{ntc}(q)$; this will follow from another way of computing $\mathrm{nlm}(e,q)$ (see Corollary 2 below).

Definition 4. *Let a parameterization* Γ' *of the double of* Γ *be given. Then a vertex* q *of* Γ *corresponds to a number of vertices* $q_1, \ldots, q_d$ *of* Γ'*, where* d *is the degree* $d(q)$ *of* q *as a vertex of* Γ*. Choose* $e \in S^2$*. If* $q \in \Gamma$ *is a local extremum of* $\langle e, \cdot \rangle$*, then we consider* q *as a vertex of degree* $d(q) = 2$*. Let* $\mathrm{lmax}(e,q)$ *be the number of local maxima of* $\langle e, \cdot \rangle$ *along* Γ' *at the points* $q_1, \ldots, q_d$ *over* q*, and similarly let* $\mathrm{lmin}(e,q)$ *be the number of local minima. We define the number of* net local maxima *of* $\langle e, \cdot \rangle$ *at* q *to be*

$$\mathrm{nlm}(e,q) = \frac{1}{2}[\mathrm{lmax}(e,q) - \mathrm{lmin}(e,q)]$$

.

Remark 1. *The definition of* $\mathrm{nlm}(e,q)$ *appears to depend not only on* Γ *but on a choice of the parameterization* Γ' *of the double of* Γ*:* $\mathrm{lmax}(e,q)$ *and* $\mathrm{lmin}(e,q)$ *may depend on the choice of* Γ'*. However, we shall see in Corollary 1 below that the number of* **net** *local maxima* $\mathrm{nlm}(e,q)$ *is in fact independent of* Γ'*.*

Remark 2. *We have included the factor* $\frac{1}{2}$ *in the definition of* $\mathrm{nlm}(e,q)$ *in order to agree with the difference of the numbers of local maxima and minima along a parameterization of* Γ *itself, if* $d(q)$ *is even.*

We shall **assume** for the rest of this section that a unit vector e has been chosen, and that the linear "height" function $\langle e, \cdot \rangle$ has only a finite number of critical points along Γ; this excludes e belonging to a subset of S^2 of measure zero. We shall also assume that the graph Γ is subdivided to include among the vertices all critical points of the linear function $\langle e, \cdot \rangle$, with degree $d(q) = 2$ if q is an interior point of one of the topological edges of Γ.

Definition 5. *Choose a unit vector* e*. At a point* $q \in \Gamma$ *of degree* $d = d(q)$*, let the* up-degree $d^+ = d^+(e,q)$ *be the number of edges of* Γ *with endpoint* q *on which* $\langle e, \cdot \rangle$ *is greater ("higher") than* $\langle e, q \rangle$*, the "height" of* q*. Similarly, let the* down-degree $d^-(e,q)$ *be the number of edges along which* $\langle e, \cdot \rangle$ *is less than its value at* q*. Note that* $d(q) = d^+(e,q) + d^-(e,q)$*, for almost all* e *in* S^2*.*

Lemma 1. (Combinatorial Lemma) *For all* $q \in \Gamma$ *and for a.a.* $e \in S^2$*,* $\mathrm{nlm}(e,q) = \frac{1}{2}[d^-(e,q) - d^+(e,q)]$.

Proof. Let a parameterization Γ' of the double of Γ be chosen, with respect to which $\mathrm{lmax}(e,q)$ and $\mathrm{lmin}(e,q)$ are defined. Recall the assumption above, that Γ has been subdivided so that along each edge, the linear function $\langle e, \cdot \rangle$ is strictly monotone.

Consider a vertex q of Γ, of degree $d = d(q)$. Then Γ' has $2d$ edges with an endpoint among the points $q_1, \ldots, q_d$ which are mapped to $q \in \Gamma$. On $2d^+$, resp. $2d^-$ of these edges, $\langle e, \cdot \rangle$ is greater resp. less than $\langle e, q \rangle$. But for each $1 \le i \le d$, the parameterization Γ' has exactly two edges which meet at q_i. Depending on the

up/down character of the two edges of Γ' which meet at q_i, $1 \leq i \leq d$, we can count:
(+) If $\langle e, \cdot\rangle$ is greater than $\langle e, q\rangle$ on both edges, then q_i is a local minimum point; there are $\mathrm{lmin}(e, q)$ of these among $q_1, \ldots, q_d$.
(-) If $\langle e, \cdot\rangle$ is less than $\langle e, q\rangle$ on both edges, then q_i is a local maximum point; there are $\mathrm{lmax}(e, q)$ of these.
(0) In all remaining cases, the linear function $\langle e, \cdot\rangle$ is greater than $\langle e, q\rangle$ along one edge and less along the other, in which case q_i is not counted in computing $\mathrm{lmax}(e, q)$ nor $\mathrm{lmax}(e, q)$; there are $d(q) - \mathrm{lmax}(e, q) - \mathrm{lmin}(e, q)$ of these.

Now count the individual edges of Γ':
(+) There are $\mathrm{lmin}(e, q)$ pairs of edges, each of which is part of a local minimum, both of which are counted among the $2d^+(e, q)$ edges of Γ' with $\langle e, \cdot\rangle$ greater than $\langle e, q\rangle$.
(-) There are $\mathrm{lmax}(e, q)$ pairs of edges, each of which is part of a local maximum; these are counted among the number $2d^-(e, q)$ of edges of Γ' with $\langle e, \cdot\rangle$ less than $\langle e, q\rangle$. Finally,
(0) there are $d(q) - \mathrm{lmax}(e, q) - \mathrm{lmin}(e, q)$ edges of Γ' which are not part of a local maximum or minimum, with $\langle e, \cdot\rangle$ greater than $\langle e, q\rangle$; and an equal number of edges with $\langle e, \cdot\rangle$ less than $\langle e, q\rangle$.

Thus, the total number of these edges of Γ' with $\langle e, \cdot\rangle$ greater than $\langle e, q\rangle$ is

$$2d^+ = 2\,\mathrm{lmin} + (d - \mathrm{lmax} - \mathrm{lmin}) = d + \mathrm{lmin} - \mathrm{lmax}.$$

Similarly,

$$2d^- = 2\,\mathrm{lmax} + (d - \mathrm{lmax} - \mathrm{lmin}) = d + \mathrm{lmax} - \mathrm{lmin}.$$

Subtracting gives the conclusion:

$$\mathrm{nlm}(e, q) := \frac{\mathrm{lmax}(e, q) - \mathrm{lmin}(e, q)}{2} = \frac{d^-(e, q) - d^+(e, q)}{2}.$$

■

Corollary 1. *The number of net local maxima* $\mathrm{nlm}(e, q)$ *is independent of the choice of parameterization* Γ' *of the double of* Γ.

Proof. Given a direction $e \in S^2$, the up-degree and down-degree $d^\pm(e, q)$ at a vertex $q \in \Gamma$ are defined independently of the choice of Γ'. ■

Corollary 2. *For any* $q \in \Gamma$*, we have* $\mathrm{ntc}(q) = \frac{1}{2}\int_{S^2}\left[\mathrm{nlm}(e, q)\right]^+ dA_{S^2}$.

Proof. Consider $e \in S^2$. In the definition (3) of $\mathrm{ntc}(q)$, $\chi_i(e) = \pm 1$ whenever $\pm\langle e, T_i\rangle < 0$. But the number of $1 \leq i \leq d$ with $\pm\langle e, T_i\rangle < 0$ equals $d^\mp(e, q)$, so that

$$\sum_{i=1}^{d} \chi_i(e) = d^-(e, q) - d^+(e, q) = 2\,\mathrm{nlm}(e, q)$$

by Lemma 1, for almost all $e \in S^2$. ■

Definition 6. *For a graph* Γ *in* $\mathbb{R}^3$ *and* $e \in S^2$*, define the* multiplicity at e *as*

$$\mu(e) = \mu_\Gamma(e) = \sum\{\mathrm{nlm}^+(e, q) : q \text{ a vertex of } \Gamma \text{ or a critical point of } \langle e, \cdot\rangle\}.$$

Note that $\mu(e)$ is a half-integer. Note also that in the case when Γ is a knot, or equivalently, when $d(q) \equiv 2$, $\mu(e)$ is exactly the integer $\mu(\Gamma, e)$, the number of local maxima of $\langle e, \cdot \rangle$ along Γ as defined in [M], p. 252.

Corollary 3. *For almost all $e \in S^2$ and for any parameterization Γ' of the double of Γ, $\mu_\Gamma(e) \le \frac{1}{2}\mu_{\Gamma'}(e)$.*

Proof. We have $\mu_\Gamma(e) = \frac{1}{2}\sum_q [\mathrm{lmax}_{\Gamma'}(e,q) - \mathrm{lmin}_{\Gamma'}(e,q)], \le \frac{1}{2}\sum_q \mathrm{lmax}_{\Gamma'}(e,q) = \frac{1}{2}\mu_{\Gamma'}$. ■

If, in place of the positive part, we sum $\mathrm{nlm}(e,q)$ itself over q located above a plane orthogonal to e, we find a useful quantity:

Corollary 4. *For almost all $s_0 \in \mathbb{R}$ and almost all $e \in S^2$,*

$$2\sum\{\mathrm{nlm}(e,q) : \langle e,q\rangle > s_0\} = \#(e, s_0),$$

the cardinality of the fiber $\{p \in \Gamma : \langle e,p\rangle = s_0\}$.

Proof. If $s_0 > \max_{p\in\Gamma}\langle e,p\rangle$, then $\#(e,s_0) = 0$. Now proceed downward, using Lemma 1 by induction. ■

Note that the fiber cardinality of Corollary 4 is also the value obtained for knots, where the more general nlm may be replaced by the number of local maxima [M].

Remark 3. *In analogy with Corollary 4, we expect that an appropriate generalization of* NTC *to curved polyhedral complexes of dimension ≥ 2 will in the future allow computation of the homology of level sets and sub-level sets of a (generalized) Morse function in terms of a generalization of* $\mathrm{nlm}(e,q)$*.*

Corollary 5. *The multiplicity of a graph in direction $e \in S^2$ may also be computed as $\mu(e) = \frac{1}{2}\sum_{q\in\Gamma} |\mathrm{nlm}(e,q)|$.*

Proof. It follows from Corollary 4 with $s_0 < \min_\Gamma\langle e,\cdot\rangle$ that $\sum_{q\in\Gamma}\mathrm{nlm}(e,q) = 0$, which is the difference of positive and negative parts. The sum of these parts is $\sum_{q\in\Gamma}|\mathrm{nlm}(e,q)| = 2\mu(e)$. ■

It was shown in Theorem 3.1 of [M] that, in the case of knots, $C(\Gamma) = \frac{1}{2}\int_{S^2}\mu(e)\,dA_{S^2}$, where Milnor refers to Crofton's formula. We may now extend this result to *graphs*:

Theorem 1. *For a (piecewise C^2) graph Γ mapped into $\mathbb{R}^3$, the net total curvature has the following representation:*

$$\mathrm{NTC}(\Gamma) = \frac{1}{2}\int_{S^2}\mu(e)\,dA_{S^2}(e).$$

Proof. We have $\mathrm{NTC}(\Gamma) = \sum_{j=1}^N \mathrm{ntc}(q_j) + \int_{\Gamma_{\mathrm{reg}}}|\vec{k}|\,ds$, where $q_1,\dots,q_N$ are the vertices of Γ, including local extrema as vertices of degree $d(q_j) = 2$, and where $\mathrm{ntc(q)} := \frac{1}{4}\int_{\mathrm{S}^2}\left[\sum_{\mathrm{i}=1}^{\mathrm{d}}\chi_{\mathrm{i}}(\mathrm{e})\right]^+ \mathrm{dA}_{\mathrm{S}^2}(\mathrm{e})$ by the definition (3) of $\mathrm{ntc}(q)$. Applying Milnor's result to each C^2 edge, we have $C(\Gamma_{\mathrm{reg}}) = \frac{1}{2}\int_{S^2}\mu_{\Gamma_{\mathrm{reg}}}(e)\,dA_{S^2}$. But

$\mu_\Gamma(e) = \mu_{\Gamma_{\rm reg}}(e) + \sum_{j=1}^{N} \mathrm{nlm}^+(\mathrm{e}, \mathrm{q_j})$, and the theorem follows. ■

Corollary 6. *If $f : \Gamma \to \mathbb{R}^3$ is piecewise C^2 but is not an embedding, then the net total curvature* NTC(Γ) *is well defined, using the right-hand side of the conclusion of Theorem 1. Moreover,* NTC(Γ) *has the same value when points of self-intersection of* Γ *are redefined as vertices.*

For $e \in S^2$, we shall use the notation $p_e : \mathbb{R}^3 \to e\mathbb{R}$ for the orthogonal projection $\langle e, \cdot \rangle$. We shall sometimes identify $\mathbb{R}$ with the one-dimensional subspace $e\mathbb{R}$ of $\mathbb{R}^3$.

Corollary 7. *For any homeomorphism type* {Γ} *of graphs, the infimum* NTC({Γ}) *of net total curvature among mappings $f : \Gamma \to \mathbb{R}^n$ is assumed by a mapping $f_0 : \Gamma \to \mathbb{R}$.*

For any isotopy class [Γ] *of embeddings $f : \Gamma \to \mathbb{R}^3$, the infimum* NTC([Γ]) *of net total curvature is assumed by a mapping $f_0 : \Gamma \to \mathbb{R}$ in the closure of the given isotopy class.*

Conversely, if $f_0 : \Gamma \to \mathbb{R}$ is in the closure of a given isotopy class [Γ] *of embeddings into $\mathbb{R}^3$, then for all $\delta > 0$ there is an embedding $f : \Gamma \to \mathbb{R}^3$ in that isotopy class with* $\mathrm{NTC}(f) \leq \mathrm{NTC}(f_0) + \delta$.

Proof. Let $f : \Gamma \to \mathbb{R}^3$ be any piecewise smooth mapping. By Corollary 6 and Corollary 4, the net total curvature of the projection $p_e \circ f : \Gamma \to \mathbb{R}$ of f onto the line in the direction of almost any $e \in S^2$ is given by $2\pi\mu(e) = \pi(\mu(e) + \mu(-e))$. It follows from Theorem 1 that NTC(Γ) is the average of $2\pi\mu(e)$ over e in S^2. But the half-integer-valued function $\mu(e)$ is lower semi-continuous almost everywhere, as may be seen using Definition 4. Let $e_0 \in S^2$ be a point where μ attains its essential infimum. Then $\mathrm{NTC}(\Gamma) \geq \pi\mu(e_0) = \mathrm{NTC}(p_{e_0} \circ f)$. But $(p_{e_0} \circ f)e_0$ is the limit as $\varepsilon \to 0$ of the map f_ε whose projection in the direction e_0 is the same as that of f and is multiplied by ε in all orthogonal directions. Since f_ε is isotopic to f, $(p_{e_0} \circ f)e_0$ is in the closure of the isotopy class of f.

Conversely, given $f_0 : \Gamma \to \mathbb{R}$ in the closure of a given isotopy class, let f be an embedding in that isotopy class uniformly close to $f_0\, e_0$; f_ε as constructed above converges uniformly to f_0 as $\varepsilon \to 0$, and $\mathrm{NTC}(f_\varepsilon) \to \mathrm{NTC}(f_0)$. ■

Definition 7. *We call a mapping $f : \Gamma \to \mathbb{R}^n$* flat *(or* NTC-*flat) if* $\mathrm{NTC}(f) = \mathrm{NTC}(\{\Gamma\})$*, the minimum value for the topological type of* Γ*, among all ambient dimensions n.*

In particular, Corollary 7 above shows that for any Γ, there is a flat mapping $f : \Gamma \to \mathbb{R}$.

Proposition 1. *Consider a piecewise C^2 mapping $f_1 : \Gamma \to \mathbb{R}$. There is a mapping $f_0 : \Gamma \to \mathbb{R}$ which is monotonic along the topological edges of* Γ*, has values at topological vertices of* Γ *arbitrarily close to those of f_1, and has* $\mathrm{NTC}(f_0) \leq \mathrm{NTC}(f_1)$.

Proof. Any piecewise C^2 mapping $f_1 : \Gamma \to \mathbb{R}$ may be approximated uniformly by mappings with a finite set of local extreme points, using the compactness of

Γ. Thus, we may assume without loss of generality that f_1 has only finitely many local extreme points. Note that for a mapping $f : \Gamma \to \mathbb{R} = \mathbb{R}e$, $\mathrm{NTC}(f) = 2\pi\mu(e)$: hence, we only need to compare $\mu_{f_0}(e)$ with $\mu_{f_1}(e)$.

If f_1 is not monotonic on a topological edge E, then it has a local extremum at a point z in the interior of E. For concreteness, we shall assume z is a local maximum point; the case of a local minimum is similar. Write v, w for the endpoints of E. Let v_1 be the closest local minimum point to z on the interval of E from z to v (or $v_1 = v$ if there is no local minimum point between), and let w_1 be the closest local minimum point to z on the interval from z to w (or $w_1 = w$). Let $E_1 \subset E$ denote the interval between v_1 and w_1. Then E_1 is an interval of a topological edge of Γ, having end points v_1 and w_1 and containing an interior point z, such that f_1 is monotone increasing on the interval from v_1 to z, and monotone decreasing on the interval from z to w_1. By switching v_1 and w_1 if needed, we may assume that $f_1(v_1) < f_1(w_1) < f_1(z)$.

Let f_0 be equal to f_1 except on the interior of the interval E_1, and map E_1 monotonically to the interval of $\mathbb{R}$ between $f_1(v_1)$ and $f_1(w_1)$. Then for $f_1(w_1) < s < f_1(z)$, the cardinality $\#(e, s)_{f_0} = \#(e, s)_{f_1} - 2$. For s in all other intervals of $\mathbb{R}$, this cardinality is unchanged. Therefore, $\mathrm{nlm}_{f_1}(w_1) = \mathrm{nlm}_{f_0}(w_1) - 1$, by Lemma 1. This implies that $\mathrm{nlm}^+_{f_1}(w_1) \geq \mathrm{nlm}^+_{f_0}(w_1) - 1$. Meanwhile, $\mathrm{nlm}_{f_1}(z) = 1$, a term which does not appear in the formula for μ_{f_0} (see Definition 6).Thus $\mu_{f_0} \leq \mu_{f_1}$, and $\mathrm{NTC}(f_0) \leq \mathrm{NTC}(f_1)$.

Proceeding inductively, we remove each local extremum in the interior of any edge of Γ, without increasing NTC. ■

4. REPRESENTATION FORMULA FOR NOWHERE-SMOOTH GRAPHS

Recall, while defining the total curvature for continuous graphs in section 2 above, we needed the monotonicity of $\mathrm{NTC}(P)$ under refinement of *polygonal* graphs P. We are now ready to prove this.

Proposition 2. *Let P and $\widetilde{P}$ be polygonal graphs in $\mathbb{R}^3$, having the same topological vertices, and homeomorphic to each other. Suppose that every vertex of P is also a vertex of $\widetilde{P}$: $\widetilde{P}$ is a* refinement *of P. Then for almost all $e \in S^2$, the multiplicity $\mu_{\widetilde{P}}(e) \geq \mu_P(e)$. As a consequence,* $\mathrm{NTC}(\widetilde{P}) \geq \mathrm{NTC}(P)$.

Proof. We may assume, as an induction step, that $\widetilde{P}$ is obtained from P by replacing the edge having endpoints q_0, q_2 with two edges, one having endpoints q_0, q_1 and the other having endpoints q_1, q_2. Choose $e \in S^2$. We consider various cases:

If the new vertex q_1 satisfies $\langle e, q_0\rangle < \langle e, q_1\rangle < \langle e, q_2\rangle$, then $\mathrm{nlm}_{\widetilde{P}}(e, q_i) = \mathrm{nlm}_P(e, q_i)$ for $i = 0, 2$ and $\mathrm{nlm}_{\widetilde{P}}(e, q_1) = 0$, hence $\mu_{\widetilde{P}}(e) = \mu_P(e)$.

If $\langle e, q_0\rangle < \langle e, q_2\rangle < \langle e, q_1\rangle$, then $\mathrm{nlm}_{\widetilde{P}}(e, q_0) = \mathrm{nlm}_P(e, q_0)$ and $\mathrm{nlm}_{\widetilde{P}}(e, q_1) = 1$. The vertex q_2 requires more careful counting: the up- and down-degree $d^{\pm}_{\widetilde{P}}(e, q_2) = d^{\pm}_P(e, q_2) \pm 1$, so that by Lemma 1, $\mathrm{nlm}_{\widetilde{P}}(e, q_2) = \mathrm{nlm}_P(e, q_2) - 1$. Meanwhile, for each of the polygonal graphs, $\mu(e)$ is the sum over q of $\mathrm{nlm}^+(e, q)$, so the change

from $\mu_P(e)$ to $\mu_{\widetilde{P}}(e)$ depends on the value of $\mathrm{nlm}_P(e, q_2)$:
(a) if $\mathrm{nlm}_P(e, q_2) \le 0$, then $\mathrm{nlm}^{+}_{\widetilde{P}}(e, q_2) = \mathrm{nlm}^{+}_{P}(e, q_2) = 0$;
(b) if $\mathrm{nlm}_P(e, q_2) = \frac{1}{2}$, then $\mathrm{nlm}^{+}_{\widetilde{P}}(e, q_2) = \mathrm{nlm}^{+}_{P}(e, q_2) - \frac{1}{2}$;
(c) if $\mathrm{nlm}_P(e, q_2) \ge 1$, then $\mathrm{nlm}^{+}_{\widetilde{P}}(e, q_2) = \mathrm{nlm}^{+}_{P}(e, q_2) - 1$.
Since the new vertex q_1 does not appear in P, recalling that $\mathrm{nlm}_{\widetilde{P}}(e, q_1) = 1$, we have $\mu_{\widetilde{P}}(e) - \mu_P(e) = +1, +\frac{1}{2}$ or 0 in the respective cases (a), (b) or (c). In any case, $\mu_{\widetilde{P}}(e) \ge \mu_P(e)$.

The reverse inequality $\langle e, q_1\rangle < \langle e, q_2\rangle < \langle e, q_0\rangle$ may be reduced to the case just above by replacing $e \in S^2$ with $-e$, since $\mu_P(-e) = -\mu_P(e)$ for any polhedral graph P. Then, depending whether $\mathrm{nlm}_P(e, q_2)$ is ≤ -1, $= -\frac{1}{2}$ or ≥ 0, we find that $\mu_{\widetilde{P}}(e) - \mu_P(e) = \mathrm{nlm}^{+}_{\widetilde{P}}(e, q_2) - \mathrm{nlm}^{+}_{P}(e, q_2) = 0, \frac{1}{2}$, or 1. In any case, $\mu_{\widetilde{P}}(e) \ge \mu_P(e)$.

These arguments are unchanged if q_0 is switched with q_2. This covers all cases except those in which equality occurs between $\langle e, q_i\rangle$ and $\langle e, q_j\rangle$ $(i \ne j)$. The set of such unit vectors e form a set of measure zero in S^2. The conclusion $\mathrm{NTC}(\widetilde{P}) \ge \mathrm{NTC}(P)$ now follows from Theorem 1. ■

We remark here that this step of proving the monotonicity for the nowhere-smooth case differs from Milnor's argument for the knot total curvature, where it was shown by two applications of the triangle inequality for spherical triangles.

Milnor extended his results for piecewise smooth knots to continuous knots in [M]; we shall carry out an analogous extension to continuous graphs.

Definition 8. *We say a point $q \in \Gamma$ is* critical *relative to $e \in S^2$ when q is a topological vertex of Γ or when $\langle e, \cdot\rangle$ is not monotone in any open interval of Γ containing q.*

Note that at some points of a differentiable curve, $\langle e, \cdot\rangle$ may have derivative zero but still not be considered a critical point relative to e by our definition. This is appropriate to the C^0 category. For a continuous graph Γ, when $\mathrm{NTC}(\Gamma)$ is finite, we shall show that the number of critical points is finite for almost all e in S^2 (see Lemma 4 below).

Lemma 2. *Let Γ be a continuous, finite graph in $\mathbb{R}^3$, and choose a sequence $\widehat{P_k}$ of Γ-approximating polygonal graphs with $\mathrm{NTC}(\Gamma) = \lim_{k\to\infty} \mathrm{NTC}(\widehat{P_k})$. Then for each $e \in S^2$, there is a refinement P_k of $\widehat{P_k}$ such that $\lim_{k\to\infty} \mu_{P_k}(e)$ exists in $[0, \infty]$.*

Proof. First, for each k in sequence, we refine $\widehat{P_k}$ to include all vertices of $\widehat{P_{k-1}}$. Then for all $e \in S^2$, $\mu_{\widehat{P_k}}(e) \ge \mu_{\widehat{P_{k-1}}}(e)$, by Proposition 2. Second, we refine $\widehat{P_k}$ so that the arc of Γ corresponding to each edge of $\widehat{P_k}$ has diameter $\le 1/k$. Third, given a particular $e \in S^2$, for each edge $\widehat{E_k}$ of $\widehat{P_k}$, we add $0, 1$ or 2 points from Γ as vertices of $\widehat{P_k}$ so that $\max_{\widehat{E_k}}\langle e, \cdot\rangle = \max_E\langle e, \cdot\rangle$ where E is the closed arc of Γ corresponding to $\widehat{E_k}$; and similarly so that $\min_{\widehat{E_k}}\langle e, \cdot\rangle = \min_E\langle e, \cdot\rangle$. Write P_k for the result of this three-step refinement. Note that all vertices of P_{k-1} appear among the vertices of P_k. Then by Proposition 2,

$$\mathrm{NTC}(\widehat{P_k}) \le \mathrm{NTC}(P_k) \le \mathrm{NTC}(\Gamma),$$

so we still have $\mathrm{NTC}(\Gamma) = \lim_{k\to\infty} \mathrm{NTC}(P_k)$.

Now compare the values of $\mu_{P_k}(e) = \sum_{q\in P_k} \mathrm{nlm_{P_k}}^+(e,q)$ with the same sum for P_{k-1}. Since P_k is a refinement of P_{k-1}, we have $\mu_{P_k}(e) \geq \mu_{P_{k-1}}(e)$ by Proposition 2.

Therefore the values $\mu_{P_k}(e)$ are non-decreasing in k, which implies they are either convergent or properly divergent; in the latter case we write $\lim_{k\to\infty} \mu_{P_k}(e) = \infty$. ∎

Definition 9. *For a continuous graph* Γ*, define the* multiplicity *at* $e \in S^2$ *as* $\mu_\Gamma(e) := \lim_{k\to\infty} \mu_{P_k}(e) \in [0,\infty]$*, where* P_k *is a sequence of* Γ*-approximating polygonal graphs, refined with respect to e, as given in Lemma 2.*

Remark 4. *Note that any two* Γ*-approximating polygonal graphs have a common refinement. Hence, from the proof of Lemma 2, any two choices of sequences* $\{\widehat{P_k}\}$ *of* Γ*-approximating polygonal graphs lead to the same value* $\mu_\Gamma(e)$*.*

Lemma 3. *Let* Γ *be a continuous, finite graph in* $\mathbb{R}^3$*. Then* $\mu_\Gamma : S^2 \to [0,\infty]$ *takes its values in the half-integers, or* $+\infty$*. Now assume* $\mathrm{NTC}(\Gamma) < \infty$*. Then* μ_Γ *is integrable, hence finite almost everywhere on* S^2*, and*

$$\mathrm{NTC}(\Gamma) = \frac{1}{2}\int_{S^2} \mu_\Gamma(e)\, dA_{S^2}(e). \tag{5}$$

For almost all $e \in S^2$*, a sequence* P_k *of* Γ*-approximating polygonal graphs, converging uniformly to* Γ*, may be chosen (depending on e) so that each local extreme point q of* $\langle e,\cdot\rangle$ *along* Γ *occurs as a vertex of* P_k *for sufficiently large k.*

Proof. Given $e \in S^2$, let $\{P_k\}$ be the sequence of Γ-approximating polygonal graphs from Lemma 2. If $\mu_\Gamma(e)$ is finite, then $\mu_{P_k}(e) = \mu_\Gamma(e)$ for k sufficiently large, a half-integer.

Suppose $\mathrm{NTC}(\Gamma) < \infty$. Then the half-integer-valued functions μ_{P_k} are non-negative, integrable on S^2 with bounded integrals since $\mathrm{NTC}(P_k) < \mathrm{NTC}(\Gamma) < \infty$, and monotone increasing in k. Thus for almost all $e \in S^2$, $\mu_{P_k}(e) = \mu_\Gamma(e)$ for k sufficiently large.

Since the functions μ_{P_k} are non-negative and pointwise non-decreasing almost everywhere on S^2, it now follows from the Monotone Convergence Theorem that

$$\int_{S^2} \mu_\Gamma(e)\, dA_{S^2}(e) = \lim_{k\to\infty} \int_{S^2} \mu_{P_k}(e)\, dA_{S^2}(e) = 2\mathrm{NTC}(\Gamma).$$

Finally, the polygonal graphs P_k have maximum edge length $\to 0$. For almost all $e \in S^2$, $\langle e,\cdot\rangle$ is not constant along any open arc of Γ, and $\mu_\Gamma(e)$ is finite. Given such an e, choose $\ell = \ell(e)$ sufficiently large that $\mu_{P_k}(e) = \mu_\Gamma(e)$ and $\mu_{P_k}(-e) = \mu_\Gamma(-e)$ for all $k \geq \ell$. Then for $k \geq \ell$, along any edge E_k of P_k with corresponding arc E of Γ, the maximum and minimum values of $\langle e,\cdot\rangle$ along E occur at the endpoints, which are also the endpoints of E_k. Otherwise, as P_k is further refined, new interior local maximum resp. local minimum points of E would contribute a new, positive value to $\mu_{P_k}(e)$ resp. to $\mu_{P_k}(-e)$ as k increases. Since the diameter of the corresponding arc E of Γ tends to zero as $k \to \infty$, any local maximum or local minimum of $\langle e,\cdot\rangle$ must become an endpoint of some edge of P_k for k sufficiently large, and for $k \geq \ell$

in particular. ∎

Our next lemma focuses on the regularity of a graph Γ, originally only assumed continuous, provided it has finite net total curvature, or another notion of total curvature of a graph which includes the total curvature of the edges.

Lemma 4. *Let Γ be a continuous, finite graph in $\mathbb{R}^3$, with $\mathrm{NTC}(\Gamma) < \infty$. Then Γ has continuous one-sided unit tangent vectors $T_1(p)$ and $T_2(p)$ at each point p, not a topological vertex. If p is a vertex of degree d, then each of the d edges which meet at p have well-defined unit tangent vectors at p: $T_1(p), \ldots, T_d(p)$. For almost all $e \in S^2$,*

$$\mu_\Gamma(e) = \sum_q \{\mathrm{nlm}(e,q)\}^+, \tag{6}$$

where the sum is over the finite *number of topological vertices of Γ and critical points q of $\langle e, \cdot\rangle$ along Γ. Further, for each q,* $\mathrm{nlm}(e,q) = \frac{1}{2}[d^-(e,q) - d^+(e,q)]$. *All of these critical points which are not topological vertices are local extrema of $\langle e, \cdot\rangle$ along Γ.*

Proof. We have seen in the proof of Lemma 3 that for almost all $e \in S^2$, the linear function $\langle e, \cdot\rangle$ is not constant along any open arc of Γ, and by Lemma 2 there is a sequence $\{P_k\}$ of Γ-approximating polygonal graphs with $\mu_\Gamma(e) = \mu_{P_k}(e)$ for k sufficiently large. We have further shown that each local maximum point of $\langle e, \cdot\rangle$ is a vertex of P_k, possibly of degree two, for k large enough. Recall that $\mu_{P_k}(e) = \sum_q \mathrm{nlm}^+_{P_k}(e,q)$. Thus, each local maximum point q for $\langle e, \cdot\rangle$ along Γ provides a non-negative term $\mathrm{nlm}^+_{P_k}(e,q)$ in the sum for $\mu_{P_k}(e)$. Fix such an integer k.

Consider a point $q \in \Gamma$ which is not a topological vertex of Γ but is a critical point of $\langle e, \cdot\rangle$. We shall show, by an argument similar to one used by van Rooij in [vR], that q must be a local extreme point. As a first step, we show that $\langle e, \cdot\rangle$ is monotone on a sufficiently small interval on either side of q. Choose an ordering of the closed edge E of Γ containing q, and consider the interval E_+ of points $\geq q$ with respect to this ordering. Suppose that $\langle e, \cdot\rangle$ is not monotone on any subinterval of E_+ with q as endpoint. Then in any interval (q, r_1) there are points $p_2 > q_2 > r_2$ so that the numbers $\langle e, p_2\rangle, \langle e, q_2\rangle, \langle e, r_2\rangle$ are not monotone. It follows by an induction argument that there exist decreasing sequences $p_n \to q$, $q_n \to q$, and $r_n \to q$ of points of E_+ such that for each n, $r_{n-1} > p_n > q_n > r_n > q$, but the value $\langle e, q_n\rangle$ lies outside of the closed interval between $\langle e, p_n\rangle$ and $\langle e, r_n\rangle$. As a consequence, there is a local extremum $s_n \in (r_n, p_n)$. Since $r_{n-1} > p_n$, the s_n are all distinct, $1 \leq n < \infty$. But by Lemma 3, all local extreme points, specifically s_n, of $\langle e, \cdot\rangle$ along Γ occur among the *finite* number of vertices of P_k, a contradiction. This shows that $\langle e, \cdot\rangle$ is monotone on an interval to the right of q. An analogous argument shows that $\langle e, \cdot\rangle$ is monotone on an interval to the left of q.

Recall that for a *critical point* q relative to e, $\langle e, \cdot\rangle$ is not monotone on any neighborhood of q. Since $\langle e, \cdot\rangle$ is monotone on an interval on either side, the sense of monotonicity must be opposite on the two sides of q. Therefore every critical point q along Γ for $\langle e, \cdot\rangle$, which is not a topological vertex, is a local extremum.

We have chosen k large enough that $\mu_\Gamma(e) = \mu_{P_k}(e)$. Then for any edge E_k of P_k, the function $\langle e, \cdot\rangle$ is monotone along the corresponding arc E of Γ, as well as along E_k. Also, E and E_k have common end points. It follows that for each $t \in \mathbb{R}$, the cardinality $\#(e,t)$ of the fiber $\{q \in \Gamma : \langle e, q\rangle = t\}$ is the same for P_k as for Γ. We may see from Lemma 1 applied to P_k that for each vertex or critical point q, $\mathrm{nlm}_{P_k}(e,q) = \frac{1}{2}[d^-_{P_k}(e,q) - d^+_{P_k}(e,q)]$; but $\mathrm{nlm}(e,q)$ and $d^\pm(e,q)$ have the *same* values for Γ as for P_k. The formula $\mu_\Gamma(e) = \sum_q\{\mathrm{nlm}_\Gamma(e,q)\}^+$ now follows from the corresponding formula for P_k, for almost all $e \in S^2$.

Consider an open interval E of Γ with endpoint q. We have just shown that for a.a. $e \in S^2$, $\langle e, \cdot\rangle$ is monotone on a subinterval with endpoint q. Choose a sequence p_ℓ from E, $p_\ell \to q$, and write $T_\ell := \frac{p_\ell - q}{|p_\ell - q|} \in S^2$. Then $\lim_{\ell\to\infty} T_\ell$ exists. Otherwise, since S^2 is compact, there are subsequences $\{T_{m_n}\}$ and $\{T_{k_n}\}$ with $T_{m_n} \to T'$ and $T_{k_n} \to T'' \neq T'$. But for an open set of $e \in S^2$, $\langle e, T'\rangle < 0 < \langle e, T''\rangle$. For such e, $\langle e, q_{m_n}\rangle < \langle e, q\rangle < \langle e, q_{k_n}\rangle$ for $n >> 1$. That is, as $p \to q$, $p \in E$, $\langle e, p\rangle$ assumes values above and below $\langle e, q\rangle$ infinitely often, contradicting monotonicity on an interval starting at q for a.a. $e \in S^2$.

This shows that Γ has one-sided tangent vectors $T_1(q), \dots, T_d(q)$ at each point $q \in \Gamma$ of degree $d = d(q)$ ($d = 2$ if q is not a topological vertex). Further, as $k \to \infty$, $T_i^{P_k}(q) \to T_i^\Gamma(q)$, $1 \le i \le d(q)$, since edges of P_k have diameter $\le \frac{1}{k}$.

The remaining conclusions follow readily.

■

Corollary 8. *Let Γ be a continuous, finite graph in $\mathbb{R}^3$, with* $\mathrm{NTC}(\Gamma) < \infty$. *Then for each point q of Γ, the contribution at q to net total curvature is given by equation* (3)*, where for $e \in S^2$, $\chi_i(e)$ = the sign of $\langle -T_i(q), e\rangle$, $1 \le i \le d(q)$. (Here, if q is not a topological vertex, we understand $d = 2$.)*

Proof. According to Lemma 4, for $1 \le i \le d(q)$, $T_i(q)$ is defined and tangent to an edge E_i of Γ, which is continuously differentiable at its end point q. If P_n is a sequence of Γ-approximating polygonal graphs with maximum edge length tending to 0, then the corresponding unit tangent vectors $T_i^{P_n}(q) \to T_i^\Gamma(q)$ as $n \to \infty$. For each P_n, we have

$$\mathrm{ntc}^{P_n}(q) = \frac{1}{4}\int_{S^2}\left[\sum_{i=1}^{d}\chi_i{}^{P_n}(e)\right]^+ dA_{S^2}(e),$$

and $\chi_i{}^{P_n} \to \chi_i{}^\Gamma$ in measure on S^2. Hence, the integrals for P_n converge to those for Γ, which is equation (3). ■

We are ready to state the formula for net total curvature, by localization on S^2, a generalization of Theorem 1:

Theorem 2. *For a continuous graph* Γ, *the net total curvature* $\mathrm{NTC}(\Gamma) \in (0, \infty]$ *has the following representation:*

$$\mathrm{NTC}(\Gamma) = \frac{1}{4}\int_{S^2} \mu(e)\, dA_{S^2}(e),$$

where, for almost all $e \in S^2$, *the multiplicity* $\mu(e)$ *is a positive half-integer or* $+\infty$, *given as the finite sum* (6).

Proof. If $\mathrm{NTC}(\Gamma)$ is finite, then the theorem follows from Lemma 3 and Lemma 4.

Suppose $\mathrm{NTC}(\Gamma) = \sup \mathrm{NTC}(P_k)$ is infinite, where P_k is a refined sequence of polygonal graphs as in Lemma 2. Then $\mu_\Gamma(e)$ is the non-decreasing limit of $\mu_{P_k}(e)$ for all $e \in S^2$. Thus $\mu_\Gamma(e) \geq \mu_{P_k}(e)$ for all e and k, and $\mu_\Gamma(e) = \mu_{P_k}(e)$ for $k \geq \ell(e)$. This implies that $\mu_\Gamma(e)$ is a positive half-integer or ∞. Since $\mathrm{NTC}(\Gamma) = \infty$, the integral

$$\mathrm{NTC}(P_k) = \frac{1}{2}\int_{S^2} \mu_{P_k}(e)\, dA_{S^2}(e)$$

is arbitrarily large as $k \to \infty$, but for each k is less than or equal to

$$\frac{1}{2}\int_{S^2} \mu_\Gamma(e)\, dA_{S^2}(e).$$

Therefore this latter integral equals ∞, and thus equals $\mathrm{NTC}(\Gamma)$. ■

We turn our attention next to the tameness of graphs of finite total curvature.

Proposition 3. *Let n be a positive integer, and write Z for the set of n-th roots of unity in* $\mathbb{C} = \mathbb{R}^2$. *Given a continuous one-parameter family* S_t, $0 \leq t < 1$, *of sets of n points in* $\mathbb{R}^2$, *there exists a continuous one-parameter family* $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ *of homeomorphisms with compact support such that* $\Phi_t(S_t) = Z$, $0 \leq t < 1$.

Proof. It is well known that there is an isotopy $\Phi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi_0(S_0) = Z$ and $\Phi_0 = \mathrm{id}$ outside of a compact set. This completes the case $t_0 = 0$ of the following continuous induction argument.

Suppose that $[0, t_0] \subset [0, 1)$ is a subinterval such that there exists a continuous one-parameter family $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of homeomorphisms with compact support, with $\Phi_t(S_t) = Z$ for all $0 \leq t \leq t_0$. We shall extend this property to an interval $[0, t_0 + \delta]$. Write $B_\varepsilon(Z)$ for the union of balls $B_\varepsilon(\zeta_i)$ centered at the n roots of unity $\zeta_1, \ldots \zeta_n$. For $\varepsilon < \sin\frac{\pi}{n}$, these balls are disjoint. We may choose $0 < \delta < 1 - t_0$ such that $\Phi_{t_0}(S_t) \subset B_\varepsilon(Z)$ for all $t_0 \leq t \leq t_0 + \delta$. Write the points of S_t as $x_i(t)$, $1 \leq i \leq n$, where $\Phi_{t_0}(x_i(t)) \in B_\varepsilon(\zeta_i)$. For each $t \in [t_0, t_0 + \delta]$, each of the balls $B_\varepsilon(\zeta_i)$ may be mapped onto itself by a homeomorphism ψ_t, varying continuously with t, such that ψ_{t_0} is the identity, ψ_t is the identity near the boundary of $B_\varepsilon(\zeta_i)$ for all $t \in [t_0, t_0 + \delta]$, and $\psi_t(\Phi_{t_0}(x_i(t))) = \zeta_i$ for all such t. For example, we may construct ψ_t so that for each $y \in B_\varepsilon(\zeta_i)$, $y - \psi_t(y)$ is parallel to $\Phi_{t_0}(x_i(t)) - \zeta_i$. We now define $\Phi_t = \psi_t \circ \Phi_{t_0}$ for each $t \in [t_0, t_0 + \delta]$.

As a consequence, we see that there is no maximal interval $[0, t_0] \subset [0, 1)$ such that there is a continuous one-parameter family $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of homeomorphisms with compact support with $\Phi_t(S_t) = Z$, for all $0 \le t \le t_0$. Thus, this property holds for the entire interval $0 \le t < 1$. ■

In the following theorem, the total curvature of a graph may be understood in terms of any definition which includes the total curvature of edges and which is continuous as a function of the unit tangent vectors at each vertex. This includes net total curvature, TC of [T] and CTC of [GY1].

Theorem 3. *Suppose $\Gamma \subset \mathbb{R}^3$ is a continuous graph with finite total curvature. Then for any $\varepsilon > 0$, Γ is isotopic to a Γ-approximating polygonal graph P with edges of length at most ε, whose total curvature is less than or equal to that of Γ.*

Proof. Since Γ has finite total curvature, by Lemma 4, at each topological vertex of degree d the edges have well-defined unit tangent vectors $T_1, \dots, T_d$, which are each the limit of the unit tangent vectors to the corresponding edges. If at each vertex the unit tangent vectors $T_1, \dots, T_d$ are distinct, then any sufficiently fine Γ-approximating polygonal graph will be isotopic to Γ; this easier case is proven.

We consider therefore n edges $E_1, \dots, E_n$ which end at a vertex q with common unit tangent vectors $T_1 = \cdots = T_n$. Choose orthogonal coordinates (x, y, z) for $\mathbb{R}^3$ so that this common tangent vector $T_1 = \cdots = T_n = (0, 0, -1)$ and $q = (0, 0, 1)$. For some $\varepsilon > 0$, in the slab $1 - \varepsilon \le z \le 1$, the edges $E_1, \dots, E_n$ project one-to-one onto the z-axis. After rescaling about q by a factor $\ge \frac{1}{\varepsilon}$, $E_1, \dots, E_n$ form a braid B of n strands in the slab $0 \le z < 1$ of $\mathbb{R}^3$, plus the point $q = (0, 0, 1)$. Each strand E_i has q as an endpoint, and the coordinate z is strictly monotone along E_i, $1 \le i \le n$. Write $S_t = B \cap \{z = t\}$. Then S_t is a set of n distinct points in the plane $\{z = t\}$ for each $0 \le t < 1$. According to Proposition 3, there are homeomorphisms Φ_t of the plane $\{z = t\}$ for each $0 \le t < 1$, isotopic to the identity in that plane, continuous as a function of t, such that $\Phi_t(S_t) = Z \times \{t\}$, where Z is the set of nth roots of unity in the (x, y)-plane, and Φ_t is the identity outside of a compact set of the plane $\{z = t\}$.

We may suppose that S_t lies in the open disk of radius $a(1 - t)$ of the plane $\{z = t\}$, for some (arbitrarily small) constant $a > 0$. We modify Φ_t, first replacing its values with $(1 - t)\Phi_t$ inside the disk of radius $a(1 - t)$. We then modify Φ_t outside the disk of radius $a(1 - t)$, such that Φ_t is the identity outside the disk of radius $2a(1 - t)$.

Having thus modified the homeomorphisms Φ_t of the planes $\{z = t\}$, we may now define an isotopy Φ of $\mathbb{R}^3$ by mapping each plane $\{z = t\}$ to itself by the homeomorphism $\Phi_0^{-1} \circ \Phi_t$, $0 \le t < 1$; and extend to the remaining planes $\{z = t\}$, $t \ge 1$ and $t < 0$, by the identity. Then the closure of the image of the braid B is the union of line segments from $q = (0, 0, 1)$ to the n points of S_0 in the plane $\{z = 0\}$. Since each Φ_t is isotopic to the identity in the plane $\{z = t\}$, Φ is isotopic to the identity of $\mathbb{R}^3$.

This procedure may be carried out in disjoint sets of $\mathbb{R}^3$ surrounding each unit vector which occurs as tangent vector to more than one edge at a vertex of Γ. Outside these sets, we inscribe a polygonal arc in each edge of Γ to obtain a Γ-approximating polygonal graph P. By Definition 3, P has total curvature less than or equal to the total curvature of Γ. ■

Artin and Fox [AF] introduced the notion of *tame* and *wild* knots in $\mathbb{R}^3$; the extension to graphs is the following

Definition 10. *We say that a graph in* $\mathbb{R}^3$ *is* tame *if it is isotopic to a polyhedral graph; otherwise, it is* wild.

Milnor proved in [M] that knots of finite total curvature are tame. More generally, we have

Corollary 9. *A continuous graph* $\Gamma \subset \mathbb{R}^3$ *of finite total curvature is tame.*

Proof. This is an immediate consequence of Theorem 3, since the Γ-approximating polygonal graph P is isotopic to Γ. ■

Observation 1. *Tameness does not imply finite total curvature.*

For a well-known example, consider $\Gamma \subset \mathbb{R}^2$ to be the continuous curve $\{(x, h(x)) : x \in [-1, 1]\}$ where the function

$$h(x) = -\frac{x}{\pi} \sin \frac{\pi}{x},$$

$h(0) = 0$, has a sequence of zeroes $\pm\frac{1}{n} \to 0$ as $n \to \infty$. Then the total curvature of Γ between $(0, \frac{1}{n})$ and $(0, \frac{1}{n+1})$ converges to π as $n \to \infty$. Thus $C(\Gamma) = \infty$.

On the other hand, $h(x)$ is continuous on $[-1, 1]$, from which it readily follows that Γ is tame.

5. ON VERTICES OF SMALL DEGREE

We are now in a position to illustrate some properties of net total curvature NTC(Γ) in a few relatively simple cases, and to make some observations regarding NTC({Γ}), the minimum net total curvature for the homeomorphism type of a graph $\Gamma \subset \mathbb{R}^n$ (see Definition 7 above).

5.1. **Minimum curvature for given degree.**

Proposition 4. *If a vertex q has* **odd** *degree, then* ntc(q) $\geq \pi/2$. *If* $d(q) = 3$, *then equality holds if and only if the three tangent vectors* T_1, T_2, T_3 *at q are coplanar but do not lie in any open half-plane. If q has* **even** *degree* $2m$, *then the minimum value of* ntc(q) *is* 0. *Moreover, the equality* ntc(q) $= 0$ *only occurs when* $T_1(q), \ldots, T_{2m}(q)$ *form m opposite pairs.*

Proof. Let q have odd degree $d(q) = 2m+1$. Then from Lemma 1, for any $e \in S^2$, we see that $\mathrm{nlm}(e, q)$ is a half-integer $\pm\frac{1}{2}, \ldots, \pm\frac{2m+1}{2}$. In particular, $|\mathrm{nlm}(e, q)| \geq \frac{1}{2}$. Corollary 2 and the proof of Corollary 5 show that

$$\mathrm{ntc}(q) = \frac{1}{4} \int_{S^2} \left|\mathrm{nlm}(e, q)\right| dA_{S^2}.$$

Therefore $\mathrm{ntc}(q) \geq \pi/2$.

If the degree $d(q) = 3$, then $|\mathrm{nlm}(e, q)| = \frac{1}{2}$ if and only if both $d^+(q)$ and $d^-(q)$ are nonzero, that is, q is not a local extremum for $\langle e, \cdot \rangle$. If ntc(q) $= \pi/2$, then this must be true for almost every direction $e \in S^2$. Thus, the three tangent vectors must be coplanar, and may not lie in an open half-plane.

If $d(q) = 2m$ is even and equality ntc(q) = 0 holds, then the formula above for ntc(q) in terms of $|\mathrm{nlm}(e, q)|$ would require $\mathrm{nlm}(e, q) \equiv 0$, and hence $d^+(e, q) = d^-(e, q) = m$ for almost all $e \in S^2$: whenever e rotates so that the plane orthogonal to e passes T_i, another tangent vector T_j must cross the plane in the opposite direction, for a.a. e, which implies $T_j = -T_i$. ■

Observation 2. *If a vertex q of odd degree $d(q) = 2p + 1$, has the minimum value* ntc(q) $= \pi/2$, *and a hyperplane $P \subset \mathbb{R}^n$ contains an even number of the tangent vectors at q, and no others, then these tangent vectors form opposite pairs.*

The proof is seen by fixing any $(n-2)$-dimensional subspace L of P and rotating P by a small positive or negative angle δ to a hyperplane P_δ containing L. Since P_δ must have k of the vectors $T_1, \ldots, T_{2p+1}$ on one side and $k + 1$ on the other side, for some $0 \le k \le p$, by comparing $\delta > 0$ with $\delta < 0$ it follows that exactly half of the tangent vectors in P lie nonstrictly on each side of L. The proof may be continued as in the last paragraph of the proof of Proposition 4. In particular, any two independent tangent vectors T_i and T_j share the 2-plane they span with a third, the three vectors not lying in any open half-plane: in fact, the third vector needs to lie in any hyperplane containing T_i and T_j.

For example, a flat $K_{5,1}$ in $\mathbb{R}^3$ must have five straight segments, two being opposite; and the remaining three being coplanar but not in any open half-plane. This includes the case of four coplanar line segments, since the four must be in opposite pairs, and either opposing pair may be considered as coplanar with the fifth segment.

5.2. **Non-monotonicity of** NTC **for subgraphs.**

Observation 3. *If Γ_0 is a subgraph of a graph Γ, then* NTC(Γ_0) *might* **not** *be* $\le$ NTC(Γ).

For a simple polyhedral example, we may consider the "butterfly" graph Γ in the plane with six vertices: $q_0^\pm = (0, \pm 1), q_1^\pm = (1, \pm 3)$, and $q_2^\pm = (-1, \pm 3)$. Γ has seven edges: three vertical edges L_0, L_1 and L_2 are the line segments L_i joining q_i^- to q_i^+. Four additional edges are the line segments from $q_0^\pm$ to $q_1^\pm$ and from $q_0^\pm$ to $q_2^\pm$, which form the smaller angle 2α at $q_0^\pm$, where $\tan\alpha = 1/2$, so that $\alpha < \pi/4$.

The subgraph Γ_0 will be Γ minus the interior of L_0. Then $\mathrm{NTC}(\Gamma_0) = C(\Gamma_0) = 6\pi - 8\alpha$. However, $\mathrm{NTC}(\Gamma) = 4(\pi - \alpha) + 2(\pi/2) = 5\pi - 4\alpha$, which is $< \mathrm{NTC}(\Gamma_0)$. ■

The monotonicity property, which is shown in Observation 3 to fail for NTC(Γ), is a virtue of Taniyama's total curvature TC(Γ).

5.3. **Net total curvature ≠ cone total curvature ≠ Taniyama's total curvature.** It is not difficult to construct three unit vectors T_1, T_2, T_3 in $\mathbb{R}^3$ such that the values of ntc(q), ctc(q) and tc(q), with these vectors as the $d(q) = 3$ tangent vectors to a graph at a vertex q, have different values. For example, we may take T_1, T_2 and T_3 to be three unit vectors in a plane, making equal angles

$2\pi/3$. According to Proposition 4, we have the contribution to net total curvature $\mathrm{ntc}(q) = \pi/2$. But the contribution to cone total curvature is $\mathrm{ctc}(q) = 0$. Namely, $\mathrm{ctc}(q) := \sup_{e\in S^2} \sum_{i=1}^{3} \left(\frac{\pi}{2} - \arccos\langle T_i, e\rangle\right)$. In this supremum, we may choose e to be normal to the plane of T_1, T_2 and T_3, and $\mathrm{ctc}(q) = 0$ follows. Meanwhile, $\mathrm{tc}(q)$ is the sum of the exterior angles formed by the three pairs of vectors, each equal to $\pi/3$, so that $\mathrm{tc}(q) = \pi$.

A similar computation for degree d and coplanar vectors making equal angles gives $\mathrm{ctc}(q) = 0$, and $\mathrm{tc}(q) = \frac{\pi}{2}\left[\frac{(d-1)^2}{2}\right]$ (brackets denoting integer part), while $\mathrm{ntc}(q) = \pi/2$ for d odd, $\mathrm{ntc}(q) = 0$ for d even. This example indicates that $\mathrm{tc}(q)$ may be significantly larger than $\mathrm{ntc}(q)$. In fact, we have

Observation 4. *If a vertex q of a graph Γ has degree $d = d(q) \geq 2$, then* $\mathrm{tc}(q) \geq (d-1)\mathrm{ntc}(q)$.

This follows from the definition (3) of $\mathrm{ntc}(q)$. Let $T_1, \ldots, T_d$ be the unit tangent vectors at q. The exterior angle between T_i and T_j is

$$\arccos\langle -T_i, T_j\rangle = \frac{1}{4}\int_{S^2} (\chi_i + \chi_j)^+ \, dA_{S^2}.$$

The contribution $\mathrm{tc}(q)$ at q to total curvature $\mathrm{TC}(\Gamma)$ equals the sum of these integrals over all $1 \leq i < j \leq d$. The sum of the integrands is

$$\sum_{1\leq i<j\leq d} (\chi_i + \chi_j)^+ \geq \left[\sum_{1\leq i<j\leq d} (\chi_i + \chi_j)\right]^+ = (d-1)\Big[\sum_{i=1}^{d} \chi_i\Big]^+.$$

Integrating over S^2 and dividing by 4, we have $\mathrm{tc}(q) \geq (d-1)\mathrm{ntc}(q)$. ∎

5.4. **Conditional additivity of net total curvature under taking union.** Observation 3 shows the failure of monotonicity of NTC for subgraphs due to the cancellation phenomena at each vertex. The following subadditivity statement specifies the necessary and sufficient condition for the additivity of net total curvature under taking union of graphs.

Proposition 5. *Given two graphs Γ_1 and $\Gamma_2 \subset \mathbb{R}^n$ with $\Gamma_1 \cap \Gamma_2 = \{p_1, \ldots, p_N\}$, the net total curvature of $\Gamma = \Gamma_1 \cup \Gamma_2$ obeys the sub-additivity law*

$$\begin{aligned} \mathrm{NTC}(\Gamma) &= \mathrm{NTC}(\Gamma_1) + \mathrm{NTC}(\Gamma_2) + \\ &+ \frac{1}{2}\sum_{j=1}^{N} \int_{S^2} [\mathrm{nlm}^+_\Gamma(e, p_j) - \mathrm{nlm}^+_{\Gamma_1}(e, p_j) - \mathrm{nlm}^+_{\Gamma_2}(e, p_j)]\, dA_{S^2} \qquad (7)\\ &\leq \mathrm{NTC}(\Gamma_1) + \mathrm{NTC}(\Gamma_2). \end{aligned}$$

In particular, additivity holds if and only if

$$\mathrm{nlm}_{\Gamma_1}(e, p_j)\, \mathrm{nlm}_{\Gamma_2}(e, p_j) \geq 0$$

for all points p_j of $\Gamma_1 \cap \Gamma_2$ and almost all $e \in S^2$.

Proof. The edges of Γ and vertices other than $p_1, \ldots, p_N$ are edges and vertices of Γ_1 or of Γ_2, so we only need to consider the contribution at the vertices $p_1, \ldots, p_N$ to $\mu(e)$ for $e \in S^2$ (see Definition 6). The sub-additivity follows from the general inequality $(a+b)^+ \leq a^+ + b^+$ for any real numbers a and b. Namely, let $a := \mathrm{nlm}_{\Gamma_1}(e, p_j)$ and $b := \mathrm{nlm}_{\Gamma_2}(e, p_j)$, so that $\mathrm{nlm}_\Gamma(e, p_j) = a + b$, as follows from Lemma 1. Now integrate both sides of the inequality over S^2, sum over $j = 1, \ldots, N$ and apply Theorem 1.

As for the equality case, suppose that $ab \geq 0$. We then note that either $a > 0$ & $b > 0$, or $a < 0$ & $b < 0$, or $a = 0$, or $b = 0$. In all four cases, we have $a^+ + b^+ = (a+b)^+$. Applied with $a = \mathrm{nlm}_{\Gamma_1}(e, p_j)$ and $b = \mathrm{nlm}_{\Gamma_2}(e, p_j)$, assuming that $\mathrm{nlm}_{\Gamma_1}(e, p_j)\mathrm{nlm}_{\Gamma_2}(e, p_j) \geq 0$ holds for all $j = 1, \ldots, N$ and almost all $e \in S^2$, this implies that $\mathrm{NTC}(\Gamma_1 \cup \Gamma_2) = \mathrm{NTC}(\Gamma_1) + \mathrm{NTC}(\Gamma_2)$.

To show that the equality $\mathrm{NTC}(\Gamma_1 \cup \Gamma_2) = \mathrm{NTC}(\Gamma_1) + \mathrm{NTC}(\Gamma_2)$ implies the inequality $\mathrm{nlm}_{\Gamma_1}(e, p_j)\mathrm{nlm}_{\Gamma_2}(e, p_j) \geq 0$ for all $j = 1, \ldots, N$ and for almost all $e \in S^2$, we suppose, to the contrary, that there is a set U of positive measure in S^2, such that for some vertex p_j in $\Gamma_1 \cap \Gamma_2$, whenever e is in U, the inequality $ab < 0$ is satisfied, where $a = \mathrm{nlm}_{\Gamma_1}(e, p_j)$ and $b = \mathrm{nlm}_{\Gamma_2}(e, p_j)$. Then for e in U, a and b are of opposite signs. Let U_1 be the part of U where $a < 0 < b$ holds: we may assume U_1 has positive measure, otherwise exchange Γ_1 with Γ_2. On U_1, we have

$$(a+b)^+ < b^+ = a^+ + b^+.$$

Recall that $a + b = \mathrm{nlm}_\Gamma(e, p_j)$. Hence the inequality between half-integers

$$\mathrm{nlm}^+_\Gamma(e, p_j) < \mathrm{nlm}^+_{\Gamma_1}(e, p_j) + \mathrm{nlm}^+_{\Gamma_2}(e, p_j)$$

is valid on the set of positive measure U_1, which in turn implies that $\mathrm{NTC}(\Gamma_1 \cup \Gamma_2) < \mathrm{NTC}(\Gamma_1) + \mathrm{NTC}(\Gamma_2)$, contradicting the assumption of equality. ■

5.5. **One-point union of graphs.**

Proposition 6. *If the graph Γ is the one-point union of graphs Γ_1 and Γ_2, where the points p_1 chosen in Γ_1 and p_2 chosen in Γ_2 are not topological vertices, then the minimum* NTC *among all mappings is subadditive, and the minimum* NTC *minus 2π is superadditive:*

$$\mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}) - 2\pi \leq \mathrm{NTC}(\{\Gamma\}) \leq \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}).$$

Further, if the points $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ may appear as extreme points on mappings of minimum NTC*, then the minimum net total curvature among all mappings, minus 2π, is additive:*

$$\mathrm{NTC}(\{\Gamma\}) = \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}) - 2\pi.$$

Proof. Write $p \in \Gamma$ for the identified points $p_1 = p_2 = p$.

Choose flat mappings $f_1 : \Gamma_1 \to \mathbb{R}$ and $f_2 : \Gamma_2 \to \mathbb{R}$, adding constants so that the chosen points $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ have $f_1(p_1) = f_2(p_2) = 0$. Further, by Proposition 1, we may assume that f_1 and f_2 are strictly monotone on the edges of Γ_1 resp. Γ_2 containing p_1 resp. p_2. Let $f : \Gamma \to \mathbb{R}$ be defined as f_1 on Γ_1 and as f_2 on Γ_2. Then at the common point of Γ_1 and Γ_2, $f(p) = 0$, and f is continuous. But since f_1 and f_2 are monotone on the edges containing p_1 and

p_2, $\mathrm{nlm}_{\Gamma_1}(p_1) = 0 = \mathrm{nlm}_{\Gamma_2}(p_2)$, so we have $\mathrm{NTC}(\{\Gamma\}) \le \mathrm{NTC}(f) = \mathrm{NTC}(f_1) + \mathrm{NTC}(f_2) = \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\})$ by Proposition 5.

Next, for all $g : \Gamma \to \mathbb{R}$, we shall show that $\mathrm{NTC}(g) \ge \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}) - 2\pi$. Given g, write g_1 resp. g_2 for the restriction of g to Γ_1 resp. Γ_2. Then $\mu_g(e) = \mu_{g_1}(e) - \mathrm{nlm}^+_{g_1}(p_1) + \mu_{g_2}(e) - \mathrm{nlm}^+_{g_2}(p_2) + \mathrm{nlm}^+_g(p)$. Now for any real numbers a and b, the difference $(a + b)^+ - (a^+ + b^+)$ is equal to $\pm a$, $\pm b$ or 0, depending on the various signs. Let $a = \mathrm{nlm}_{g_1}(p_1)$ and $b = \mathrm{nlm}_{g_2}(p_2)$. Then since p_1 and p_2 are not topological vertices of Γ_1 resp. Γ_2, $a, b \in \{-1, 0, +1\}$ and $a + b = \mathrm{nlm}_g(p)$ by Lemma 1. In any case, we have

$$\mathrm{nlm}^+_g(p) - \mathrm{nlm}^+_{g_1}(p_1) - \mathrm{nlm}^+_{g_2}(p_2) \ge -1.$$

Thus, $\mu_g(e) \ge \mu_{g_1}(e) + \mu_{g_2}(e) - 1$, and multiplying by 2π, $\mathrm{NTC}(g) \ge \mathrm{NTC}(g_1) + \mathrm{NTC}(g_2) - 2\pi \ge \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}) - 2\pi$.

Finally, assume p_1 and p_2 are extreme points for flat mappings $f_1 : \Gamma_1 \to \mathbb{R}$ resp. $f_2 : \Gamma_2 \to \mathbb{R}$. We may assume that $f_1(p_1) = 0 = \min f_1(\Gamma_1)$ and $f_2(p_2) = 0 = \max f_2(\Gamma_2)$. Then $\mathrm{nlm}_{f_2}(p_2) = 1$ and $\mathrm{nlm}_{f_1}(p_1) = -1$, and hence using Lemma 1, $\mathrm{nlm}_f(p) = 0$. So $\mu_f(e) = \mu_{f_1}(e) - \mathrm{nlm}^+_{f_1}(p_1) + \mu_{f_2}(e) - \mathrm{nlm}^+{}_{f_2}(p_2) + \mathrm{nlm}^+_f(p) = \mu_{f_1}(e) + \mu_{f_2}(e) - 1$. Multiplying by 2π, we have $\mathrm{NTC}(\{\Gamma\}) \le \mathrm{NTC}(f) = \mathrm{NTC}(\{\Gamma_1\}) + \mathrm{NTC}(\{\Gamma_2\}) - 2\pi$. ■

6. NET TOTAL CURVATURE FOR DEGREE 3

6.1. **Simple description of net total curvature.**

Proposition 7. *For any graph* Γ *and any parameterization* Γ' *of its double,* $\mathrm{NTC}(\Gamma) \le \frac{1}{2}C(\Gamma')$. *If* Γ *is a* trivalent *graph, that is, having vertices of degree at most three, then* $\mathrm{NTC}(\Gamma) = \frac{1}{2}C(\Gamma')$ *for any parameterization* Γ' *which does not immediately repeat any edge of* Γ.

Proof. The first conclusion follows from Corollary 3.

Now consider a trivalent graph Γ. Observe that Γ' would be forced to immediately repeat any edge which ends in a vertex of degree 1; thus, we may assume that Γ has only vertices of degree 2 or 3. Since Γ' covers each edge of Γ twice, we need only show, for every vertex q of Γ, having degree $d = d(q) \in \{2, 3\}$, that

$$2\,\mathrm{ntc}_\Gamma(q) = \sum_{i=1}^{d} \mathrm{c}_{\Gamma'}(q_i), \tag{8}$$

where $q_1, \ldots, q_d$ are the vertices of Γ' over q. If $d = 2$, since Γ' does not immediately repeat any edge of Γ, we have $\mathrm{ntc}_\Gamma(q) = \mathrm{c}_{\Gamma'}(q_1) = \mathrm{c}_{\Gamma'}(q_2)$, so equation (8) clearly holds. For $d = 3$, write both sides of equation (8) as integrals over S^2, using the definition (3) of $\mathrm{ntc}_\Gamma(q)$. Since Γ' does not immediately repeat any edge, the three pairs of tangent vectors $\{T_1^{\Gamma'}(q_j), T_2^{\Gamma'}(q_j)\}$, $1 \le j \le 3$, comprise all three pairs taken from the triple $\{T_1^\Gamma(q), T_2^\Gamma(q), T_3^\Gamma(q)\}$. We need to show that

$$2\int_{S^2} [\chi_1 + \chi_2 + \chi_3]^+ \, dA_{S^2} = \int_{S^2} [\chi_1 + \chi_2]^+ \, dA_{S^2} + \\ + \int_{S^2} [\chi_2 + \chi_3]^+ \, dA_{S^2} + \int_{S^2} [\chi_3 + \chi_1]^+ \, dA_{S^2},$$

where at each direction $e \in S^2$, $\chi_j(e) = \pm 1$ is the sign of $\langle -e, T_j^\Gamma(q)\rangle$. But the integrands are equal at almost every point e of S^2:

$$2\,[\chi_1 + \chi_2 + \chi_3]^+ = [\chi_1 + \chi_2]^+ + [\chi_2 + \chi_3]^+ + [\chi_3 + \chi_1]^+ \,,$$

as may be confirmed by cases: $6 = 6$ if $\chi_1 = \chi_2 = \chi_3 = +1$; $2 = 2$ if exactly one of the χ_i equals -1, and $0 = 0$ in the remaining cases. ■

6.2. **Simple description of net total curvature fails,** $d \geq 4$**.**

Observation 5. *We have seen in Corollary 3 that for graphs with vertices of degree* ≤ 3*, if a parameterization* Γ' *of the double* $\widetilde{\Gamma}$ *of* Γ *does not immediately repeat any edge of* Γ*, then* $\mathrm{NTC}(\Gamma) = \frac{1}{2}C(\Gamma')$*, the total curvature in the usual sense of the link* Γ'*. A natural suggestion would be that for general graphs* Γ*,* $\mathrm{NTC}(\Gamma)$ *might be half the infimum of total curvature of all such parameterizations* Γ' *of the double. However, in some cases, we have the* **strict inequality** $\mathrm{NTC}(\Gamma) < \inf_{\Gamma'} \frac{1}{2}\mathrm{NTC}(\Gamma')$.

In light of Proposition 7, we choose an example of a vertex q of degree four, and consider the local contributions to NTC for $\Gamma = K_{1,4}$ and for Γ', which is the union of four arcs.

Suppose that for a small positive angle α, ($\alpha \leq 1$ radian would suffice) the four unit tangent vectors at q are $T_1 = (1,0,0)$; $T_2 = (0,1,0)$; $T_3 = (-\cos\alpha, 0, \sin\alpha)$; and $T_4 = (0, -\cos\alpha, -\sin\alpha)$. Write the exterior angles as $\theta_{ij} = \pi - \arccos\langle T_i, T_j\rangle$. Then $\inf_{\Gamma'} \frac{1}{2}C(\Gamma') = \theta_{13} + \theta_{24} = 2\alpha$. However, $\mathrm{ntc}(q)$ is strictly less than 2α. This may be seen by writing $\mathrm{ntc}(q)$ as an integral over S^2, according to the definition (3), and noting that cancellation occurs between two of the four lune-shaped sectors. ■

6.3. **Minimum NTC for trivalent graphs.** Using the relation $\mathrm{NTC}(\Gamma) = \frac{1}{2}\mathrm{NTC}(\Gamma')$ between the net total curvature of a given trivalent graph Γ and the total curvature for a non-reversing double cover Γ' of the graph, we can determine the minimum net total curvature of a trivalent graph embedded in $\mathbb{R}^n$, whose value is then related to the Euler characteristic of the graph $\chi(\Gamma) = -k/2$.

First we introduce the following definition.

Definition 11. *For a given graph* Γ *and a mapping* $f : \Gamma \to \mathbb{R}$*, let the* extended bridge number $B(f)$ *be one-half the number of local extrema. Write* $B(\{\Gamma\})$ *for the minimum of* $B(f)$ *among all mappings* $f : \Gamma \to \mathbb{R}$*. For a given isotopy type* $[\Gamma]$ *of embeddings into* $\mathbb{R}^3$*, let* $B([\Gamma])$ *be one-half the minimum number of local extrema for a mapping* $f : \Gamma \to \mathbb{R}$ *in the closure of the isotopy class* $[\Gamma]$.

For an integer $m \geq 3$, let θ_m be the graph with two vertices q^+, q_- and m edges, each of which has q^+ and q^- as its two endpoints. Then $\theta = \theta_3$ has the form of the lower-case Greek letter θ.

Remark 5. *For a knot, the number of local maxima equals the number of local minima. The minimum number of local maxima is called the* bridge number*, and equals the number of local minima. This is consistent with our Definition 11 of the extended bridge number. Of course, for knots, the minimum bridge number among all isotopy classes* $B(\{S^1\}) = 1$*, and only* $B([S^1])$ *is of interest for a specific isotopy class* $[S^1]$*. For certain graphs, the minimum numbers of local maxima and local minima may not occur at the same time for any mapping: see the example of Observation 6 below. For isotopy classes of θ-graphs, Goda* [Go] *has given a definition of an integer-valued bridge index which is similar in spirit to the definition above.*

Theorem 4. *If* Γ *is a trivalent graph, and if* $f_0 : \Gamma \to \mathbb{R}$ *is monotone on topological edges and has the minimum number* $2B(\{\Gamma\})$ *of local extrema, then* $\mathrm{NTC}(f_0) = \mathrm{NTC}(\{\Gamma\}) = \pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big)$*, where k is the number of topological vertices of* Γ*. For a given isotopy class* $[\Gamma]$, $\mathrm{NTC}([\Gamma]) = \pi\Big(2B([\Gamma]) + \frac{k}{2}\Big)$.

Proof. Recall that $\mathrm{NTC}(\{\Gamma\})$ denotes the infimum of $\mathrm{NTC}(f)$ among $f : \Gamma \to \mathbb{R}^3$ or among $f : \Gamma \to \mathbb{R}$, as may be seen from Corollary 7.

We first consider a mapping $f_1 : \Gamma \to \mathbb{R}$ with the property that any local maximum or local minimum points of f_1 are interior points of topological edges. Then all topological vertices v, since they have degree $d(v) = 3$ and $d^\pm(v) \neq 0$, have $\mathrm{nlm}(v) = \pm 1/2$, by Proposition 4. Let Λ be the number of local maximum points of f_1, V the number of local minimum points, λ the number of vertices with $\mathrm{nlm} = +1/2$, and y the number of vertices with $\mathrm{nlm} = -1/2$. Then $\lambda + \mathrm{y} = k$, the total number of vertices, and $\Lambda + V \geq 2B(\{\Gamma\})$. Hence applying Corollary 5,

$$\mu = \frac{1}{2}\sum_v |\mathrm{nlm}(v)| = \frac{1}{2}[\Lambda + V + \frac{\lambda + \mathrm{y}}{2}] \geq B(\{\Gamma\}) + k/4, \tag{9}$$

with equality iff $\Lambda + V = 2B(\{\Gamma\})$.

We next consider any mapping $f_0 : \Gamma \to \mathbb{R}$ in general position: in particular, the critical valuess of f_0 are isolated. In a similar fashion to the proof of Proposition 1, we shall replace f_0 with a mapping whose local extrema are not topological vertices. Specifically, if f_0 assumes a local maximum at any topological vertex v, then, since $d(v) = 3$, $\mathrm{nlm}_{f_0}(v) = 3/2$. f_0 may be isotoped in a small neighborhood of v to $f_1 : \Gamma \to \mathbb{R}$ so that near v, the local maximum occurs at an interior point q of one of the three edges with endpoint v, and thus $\mathrm{nlm}_{f_1}(q) = 1$; while the up-degree $d^+_{f_1}(v) = 1$ and the down-degree $d^-_{f_1}(v) = 2$, so that $\mathrm{nlm}_{f_1}(v)$ is now $\frac{1}{2}$. Thus, $\mu_{f_1}(e) = \mu_{f_0}(e)$. Similarly, if f_0 assumes a local minimum at a topological vertex w, then f_0 may be isotoped in a neighborhood of w to $f_1 : \Gamma \to \mathbb{R}$ so that the local minimum of f_1 near w occurs at an interior point of any of the three edges with endpoint w, and $\mu_{f_1}(e) = \mu_{f_0}(e)$. Then any local extreme points of f_1 are interior points of topological edges. Thus, we have shown that $\mu_{f_0}(e) \geq B(\{\Gamma\}) + k/4$, with equality if f_1 has exactly $2B(\{\Gamma\})$ as its number of local extrema, which holds iff f_0 has the minimum number $2B(\{\Gamma\})$ of local extrema.

Thus $\mathrm{NTC}(\{\Gamma\}) = 2\pi\mu_{f_0}(e) = 2\pi\Big(B(\{\Gamma\}) + k/4\Big) = \pi\Big(2B(\{\Gamma\}) + k/2\Big)$.

Similarly, for a given isotopy class $[\Gamma]$ of embeddings into $\mathbb{R}^3$, we may choose $f_0 : \Gamma \to \mathbb{R}$ in the closure of the isotopy class, deform f_0 to a mapping f_1 in the closure of $[\Gamma]$ having no topological vertices as local extrema and count $\mu_{f_0}(e) = \mu_{f_1}(e) \geq B([\Gamma]) + k/4$, with equality if f_0 has the minimum number $2B([\Gamma])$ of local extrema. This shows that $\mathrm{NTC}([\Gamma]) = \pi\Big(2B([\Gamma]) + k/2\Big)$. ■

Remark 6. *An example geometrically illustrating the lower bound is given by the dual graph Γ^* of the one-skeleton Γ of a triangulation of S^2, with the $\{\infty\}$ not coinciding with any of the vertices of Γ^*. The Koebe-Andreev-Thurston theorem says that there is a circle packing which realizes the vertex set of Γ^* as the set of centers of the circles (see* [S]*). The so realized Γ^*, stereographically projected to $\mathbb{R}^2 \subset \mathbb{R}^3$, attains the lower bound of Theorem 4 with $B(\{\Gamma^*\}) = 1$, namely* $\mathrm{NTC}([\Gamma]) = \pi(2 + \frac{k}{2}) = \pi(2 - \chi(\Gamma^*))$*, where k is the number of vertices.*

Corollary 10. *If Γ is a trivalent graph with k topological vertices, and $f_0 : \Gamma \to \mathbb{R}$ is a mapping in general position, having Λ local maximum points and V local minimum points, then*

$$\mu_{f_0}(e) = \frac{1}{2}(\Lambda + V) + \frac{k}{4} \geq B(\{\Gamma\}) + \frac{k}{4}.$$

Proof. Follows immediately from the proof of Theorem 4: f_0 and f_1 have the same number of local maximum or minimum points. ■

An interesting trivalent graph is L_m, the "ladder of m rungs" obtained from two unit circles in parallel planes by adding m line segments ("rungs") perpendicular to the planes, each joining one vertex on the first circle to another vertex on the second circle. For example, L_4 is the 1-skeleton of the cube in $\mathbb{R}^3$. Note that L_m may be embedded in $\mathbb{R}^2$, and that the bridge number $B(\{L_m\}) = 1$. Since L_m has $2m$ trivalent vertices, we may apply Theorem 4 to compute the minimum NTC for the type of L_m:

Corollary 11. *The minimum net total curvature* $\mathrm{NTC}(\{L_m\})$ *for graphs of the type of L_m equals $\pi(2 + m)$.*

Observation 6. *For certain connected trivalent graphs Γ containing cut points, the minimum extended bridge number $B(\{\Gamma\})$ may be greater than* 1.

Example: Let Γ be the union of three disjoint circles C_1, C_2, C_3 with three edges E_i connecting a point $p_i \in C_i$ with a fourth vertex p_0, which is not in any of the C_i, and which is a *cut point* of Γ: the number of connected components of $\Gamma \backslash p_0$ is greater than for Γ. Given $f : \Gamma \to \mathbb{R}$, after a permutation of $\{1, 2, 3\}$, we may assume there is a minimum point $q_1 \in C_1 \cup E_1$ and a maximum point $q_3 \in C_3 \cup E_3$. If q_1 and q_3 are both in $C_1 \cup E_1$, we may choose C_2 arbitrarily in what follows. Restricted to the closed set $C_2 \cup E_2$, f assumes either a maximum or a minimum at a point $q_2 \neq p_0$. Since $q_2 \neq p_0$, q_2 is also a local maximum or a local minimum for f on Γ. That is, q_1, q_2, q_3 are all local extrema. In the notation of the proof of Theorem 4, we have the number of local extrema $V+\Lambda \geq 3$. Therefore $B(\{\Gamma\}) \geq \frac{3}{2}$, and $\mathrm{NTC}(\{\Gamma\}) \geq \pi(3 + k/2) = 5\pi$.

The reader will be able to construct similar trivalent examples with $B(\{\Gamma\})$ arbitrarily large. ■

In contrast to the results of Theorem 4 and of Theorem 5, below, for trivalent or nearly trivalent graphs, the minimum of NTC for a given graph type cannot be computed merely by counting vertices, but depends in a more subtle way on the topology of the graph:

Observation 7. *When Γ is not trivalent, the minimum* $\mathrm{NTC}(\{\Gamma\})$ *of net total curvature for a connected graph Γ with $B(\{\Gamma\}) = 1$ is not determined by the number of vertices and their degrees.*

Example: We shall construct two planar graphs S_m and R_m having the same number of vertices, all of degree 4.

Choose an integer $m \geq 3$ and take the image of the embedding f_ε of the "sine wave" S_m to be the union of the polar-coordinate graphs $C_\pm \subset \mathbb{R}^2$ of two functions: $r = 1 \pm \varepsilon \sin(m\theta)$. S_m has $4m$ edges; and $2m$ vertices, all of degree 4, at $r = 1$ and $\theta = \pi/m, 2\pi/m, \ldots, 2\pi$. For $0 < \varepsilon < 1$, $f_\varepsilon(S_m) = C_+ \cup C_-$ is the union of two smooth cycles. For small positive ε, C_+ and C_- are convex. The $2m$ vertices all have $\mathrm{nlm}(q) = 0$, so $\mathrm{NTC}(f_\varepsilon) = \mathrm{NTC}(C_+) + \mathrm{NTC}(C_-) = 2\pi + 2\pi$. Therefore $\mathrm{NTC}(\{S_m\}) \leq \mathrm{NTC}(f_\varepsilon) = 4\pi$.

For the other graph type, let the "ring graph" $R_m \subset \mathbb{R}^2$ be constructed by adding m disjoint small circles C_i, each crossing one large circle C at two points v_{2i-1}, v_{2i}, $1 \leq i \leq m$. Then R_m has $4m$ edges. We construct R_m so that the $2m$ vertices $v_1, v_2, \ldots, v_{2m}$, appear in cyclic order around C. Then R_m has the same number $2m$ of vertices as does S_m, all of degree 4. At each vertex v_j, we have $\mathrm{nlm}(v_j) = 0$, so in this embedding, $\mathrm{NTC}(R_m) = 2\pi(m + 1)$. We shall show that $\mathrm{NTC}(f_1) \geq 2\pi m$ for any $f_1 : R_m \to \mathbb{R}^3$. According to Corollary 7, it is enough to show for every $f : R_m \to \mathbb{R}$ that $\mu_f \geq m$. We may assume f is monotone on each topological edge, according to Proposition 1. Depending on the order of $f(v_{2i-2}), f(v_{2i-1})$ and $f(v_{2i})$, $\mathrm{nlm}(v_{2i-1})$ might equal ± 1 or ± 2, but cannot be 0, as follows from Lemma 1, since the unordered pair $\{d^-(v_{2i-1}), d^+(v_{2i-1})\}$ may only be $\{1, 3\}$ or $\{0, 4\}$. Similarly, v_{2i} is connected by three edges to v_{2i-1} and by one edge to v_{2i+1}. For the same reasons, $\mathrm{nlm}(v_{2i})$ might equal ± 1 or ± 2, and cannot $= 0$. So $|\mathrm{nlm}(v_j)| \geq 1$, $1 \leq j \leq 2m$, and thus by Corollary 5, $\mu = \frac{1}{2} \sum_j |\mathrm{nlm}(v_j)| \geq m$. Therefore the minimum of net total curvature $\mathrm{NTC}(\{R_m\}) \geq 2m\pi$, which is greater than $\mathrm{NTC}(\{S_m\}) \leq 4\pi$, since $m \geq 3$.

(A more detailed analysis shows that $\mathrm{NTC}(\{S_m\}) = 4\pi$ and $\mathrm{NTC}(\{R_m\}) = 2\pi(m+1)$.) ■

Finally, we may extend the methods of proof for Theorem 4 to allow **one** vertex of higher degree:

Theorem 5. *If Γ is a graph with one vertex w of degree $d(w) = m \geq 3$, all other vertices being trivalent, and if w shares edges with m distinct trivalent vertices,* *then* $\mathrm{NTC}(\{\Gamma\}) = \pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big)$*, where k is the number of vertices of* Γ *having odd degree. For a given isotopy class* $[\Gamma]$, $\mathrm{NTC}([\Gamma]) \geq \pi\Big(2B([\Gamma]) + \frac{k}{2}\Big)$.

Proof. Consider any mapping $g : \Gamma \to \mathbb{R}$ in general position. If m is even, then $|\mathrm{nlm}_g(w)| \geq 0$; if m is odd, then $|\mathrm{nlm}_g(w)| \geq \frac{1}{2}$, by Proposition 4. If some topological vertex is a local extreme point, then as in the proof of Theorem 4, g may be modified without changing $\mathrm{NTC}(g)$ so that all $\Lambda + V$ local extreme points are interior points of edges, with nlm $= \pm 1$. By Corollary 5, we have $\mu_g(e) = \frac{1}{2}\sum|\mathrm{nlm}(v)| \geq \frac{1}{2}\Big(\Lambda + V + \frac{k}{2}\Big) \geq B(\{\Gamma\}) + \frac{k}{4}$. This shows that

$$\mathrm{NTC}(\{\Gamma\}) \geq \pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big).$$

Now let $f_0 : \Gamma \to \mathbb{R}$ be monotone on topological edges and have the minimum number $2B(\{\Gamma\})$ of local extreme points (see Corollary 1). As in the proof of Theorem 4, f_0 may be modified without changing $\mathrm{NTC}(f_0)$ so that all $2B(\{\Gamma\})$ local extreme points are interior points of edges. f_0 may be further modified so that the distinct vertices $v_1, \ldots, v_m$ which share edges with w are balanced: $f(v_j) < f(w)$ for half of the $j = 1, \ldots, m$, if m is even, or for half of $m + 1$, if m is odd. Having chosen $f(v_j)$, we define f along the (unique) edge from w to v_j to be monotone, for $j = 1, \ldots, m$. Therefore if m is even, then $\mathrm{nlm}_f(w) = 0$; and if m is odd, then $\mathrm{nlm}_f(w) = \frac{1}{2}$, by Lemma 1. We compute $\mu_f(e) = \frac{1}{2}\sum|\mathrm{nlm}(v)| = \frac{1}{2}(\Lambda + V + \frac{k}{2}) = B(\{\Gamma\}) + \frac{k}{4}$. We conclude that $\mathrm{NTC}(\{\Gamma\}) = \pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big)$.

For a given isotopy class $[\Gamma]$, the proof is analogous to the above. Choose a mapping $g : \Gamma \to \mathbb{R}$ in the closure of $[\Gamma]$, and modify g without leaving the closure of the isotopy class. Choose $f : \Gamma \to \mathbb{R}$ which has the minimum number $2B([\Gamma])$ of local extreme points, and modify it so that topological vertices are not local extreme points. In contrast to the proof of Theorem 4, a balanced arrangement of vertices may not be possible in the given isotopy class. In any case, if m is even, then $|\mathrm{nlm}_f(w)| \geq 0$; and if m is odd, $|\mathrm{nlm}_f(w)| \geq \frac{1}{2}$, by Proposition 4. Thus applying Corollary 5, we find $\mathrm{NTC}([\Gamma]) \geq \pi\Big(2B([\Gamma]) + \frac{k}{2}\Big)$. ■

Observation 8. *When all vertices of* Γ *are trivalent except w,* $d(w) \geq 4$*, and when w shares more than one edge with another vertex of* Γ*, then in certain cases,* $\mathrm{NTC}(\{\Gamma\}) > \pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big)$*, where k is the number of vertices of odd degree.*

Example: Choose Γ to be the one-point union of Γ_1, Γ_2 and Γ_3, where $\Gamma_i = \theta = \theta_3$, $i = 1, 2, 3$, and the point w_i chosen from Γ_i is one of its two vertices v_i, w_i. Then the identified point $w = w_1 = w_2 = w_3$ of Γ has $d(w) = 9$, and each of the other three vertices v_1, v_2, v_3 has degree 3.

Choose a flat map $f : \Gamma \to \mathbb{R}$. We may assume that f is monotone on each edge, applying Proposition 1. If $f(v_1) < f(v_2) < f(w) < f(v_3)$, then $d^+(w) = 3$, $d^-(w) = 6$, so $\mathrm{nlm}(w) = \frac{3}{2}$, while v_i is a local extreme point, so $\mathrm{nlm}(v_i) = \pm\frac{3}{2}$, $1 = 1, 2, 3$. This gives $\mu = 3$. The case where $f(v_1) < f(w) < f(v_2) < f(v_3)$ is similar. If w is an extreme point of f, then $\mathrm{nlm}(w) = \pm\frac{9}{2}$ and $\mu \geq \frac{9}{2} > 3$, contradicting flatness of f. This shows that $\mathrm{NTC}(\{\Gamma\}) = \mathrm{NTC}(f) = 6\pi$.

On the other hand, we may show as in Observation 6 that $B(\{\Gamma\}) = \frac{3}{2}$. All four vertices have odd degree, so $k = 4$, and $\pi\Big(2B(\{\Gamma\}) + \frac{k}{2}\Big) = 5\pi$. ■

Let W_m denote the "wheel" of m spokes, consisting of a cycle C containing m vertices $v_1, \ldots, v_m$ (the "rim"), a central vertex w (the "hub") not on C, and edges E_i (the "spokes") connecting w to v_i, $1 \le i \le m$.

Corollary 12. *The minimum net total curvature* $\mathrm{NTC}(\{W_m\})$ *for graphs in* $\mathbb{R}^3$ *homeomorphic to* W_m *equals* $\pi(2 + \lceil \frac{m}{2} \rceil)$.

Proof. We have one "hub" vertex w with $d(w) = m$, and all other vertices have degree 3. Observe that the bridge number $B(\{W_m\}) = 1$. According to Theorem 5, we have $\mathrm{NTC}(\{W_m\}) = \pi\Big(2B(\{W_m\}) + \frac{k}{2}\Big)$, where k is the number of vertices of odd degree: $k = m$ if m is even, or $k = m + 1$ if m is odd: $k = 2\lceil \frac{m}{2} \rceil$. Thus $\mathrm{NTC}(\{W_m\}) = \pi\Big(2 + \lceil \frac{m}{2} \rceil\Big)$. ■

7. LOWER BOUNDS OF NET TOTAL CURVATURE

The *width* of an isotopy class $[\Gamma]$ of embeddings of a graph Γ into $\mathbb{R}^3$ is the minimum among representatives of the class of the maximum number of points of the graph meeting a family of parallel planes. More precisely, we write $\mathrm{width}([\Gamma]) := \min_{f:\Gamma\to\mathbb{R}^3 | f\in[\Gamma]} \min_{e\in S^2} \max_{s\in\mathbb{R}} \#(e, s)$. For any homeomorphism type $\{\Gamma\}$ define $\mathrm{width}(\{\Gamma\})$ to be the minimum over isotopy types.

Theorem 6. *Let* Γ *be a graph, and consider an isotopy class* $[\Gamma]$ *of embeddings* $f : \Gamma \to \mathbb{R}^3$. *Then*

$$\mathrm{NTC}([\Gamma]) \ge \pi \,\mathrm{width}([\Gamma]).$$

As a consequence, $\mathrm{NTC}(\{\Gamma\}) \ge \pi\, \mathrm{width}(\{\Gamma\})$. *Moreover, if for some* $e \in S^2$*, an embedding* $f : \Gamma \to \mathbb{R}^3$ *and* $s_0 \in \mathbb{R}$*, the integers* $\#(e, s)$ *are increasing in* s *for* $s < s_0$ *and decreasing for* $s > s_0$*, then* $\mathrm{NTC}([\Gamma]) = \#(e, s_0)\,\pi$.

Proof. Choose an embedding $g : \Gamma \to \mathbb{R}^3$ in the given isotopy class, with $\max_{s\in\mathbb{R}} \#(e, s) = \mathrm{width}([\Gamma])$. There exist $e \in S^2$ and $s_0 \in \mathbb{R}$ with $\#(e, s_0) = \max_{s\in\mathbb{R}} \#(e, s) = \mathrm{width}([\Gamma])$. Replace e if necessary by a nearby point in S^2 so that the values $g(v_i)$, $i = 1, \ldots, m$ are distinct. Next do cylindrical shrinking: without changing $\#(e, s)$ for $s \in \mathbb{R}$, shrink the image of g in directions orthogonal to e by a factor $\delta > 0$ to obtain a family $\{g_\delta\}$ from the same isotopy class $[\Gamma]$, with $\mathrm{NTC}(g_\delta) \to \mathrm{NTC}(g_0)$, where we may identify $g_0 : \Gamma \to \mathbb{R}e \subset \mathbb{R}^3$ with $p_e \circ g = p_e \circ g_\delta : \Gamma \to \mathbb{R}$. But

$$\mathrm{NTC}(p_e \circ g) = \frac{1}{2}\int_{S^2} \mu(u)\, dA_{S^2}(u) = 2\pi\mu(e),$$

since for $p_u \circ p_e \circ g$, the local maximum and minimum points are the same as for $p_e \circ g$ if $\langle e, u\rangle > 0$ and reversed if $\langle e, u\rangle < 0$ (recall that $\mu(-e) = \mu(e)$).

We write the topological vertices and the local extrema of g_0 as $v_1, \ldots, v_m$. Let the indexing be chosen so that $g_0(v_i) < g_0(v_{i+1})$, $i = 1, \ldots, m - 1$. Now estimate

$\mu(e)$ from below: using Lemma 1,

$$\mu(e) = \sum_{i=1}^{m} \mathrm{nlm}^{+}_{g_0}(v_i) \geq \sum_{i=k+1}^{m} \mathrm{nlm}_{g_0}(v_i) = \frac{1}{2}\#(e,s) \tag{10}$$

for any s, $g_0(v_k) < s < g_0(v_{k+1})$. This shows that $\mu(e) \geq \frac{1}{2}\mathrm{width}([\Gamma])$, and therefore

$$\mathrm{NTC}(g) \geq \mathrm{NTC}(g_0) = 2\pi\,\mu(e) \geq \pi\,\mathrm{width}([\Gamma]).$$

Now suppose that the integers $\#(e,s)$ are increasing in s for $s < s_0$ and decreasing for $s > s_0$. Then for $g_0(v_i) > s_0$, we have $\mathrm{nlm}(g_0(v_i)) \geq 0$ by Lemma 1, and the inequality (10) becomes equality at $s = s_0$. ■

Lemma 5. *For an integer ℓ, the minimum width of the complete graph $K_{2\ell}$ on 2ℓ vertices is* $\mathrm{width}(\{K_{2\ell}\}) = \ell^2$*; for $2\ell + 1$ vertices,* $\mathrm{width}(\{K_{2\ell+1}\}) = \ell(\ell + 1)$.

Proof. Write E_{ij} for the edge of K_m joining v_i to v_j, $1 \leq i < j \leq m$, and suppose $g : K_m \to \mathbb{R}$ has distinct values at the vertices: $g(v_1) < g(v_2) < \cdots < g(v_m)$.

Then for any $g(v_k) < s < g(v_k + 1)$, there are $k(m - k)$ edges E_{ij} with $i \leq k < j$; each of these edges has at least one interior point mapping to s, which shows that $\#(e,s) \geq k(m - k)$. If m is even: $m = 2\ell$, these lower bounds have the maximum value ℓ^2 when $k = \ell$. If m is odd: $m = 2\ell+1$, these lower bounds have the maximum value $\ell(\ell + 1)$ when $k = \ell$ or $k = \ell + 1$. This shows that the width of $K_{2\ell} \geq \ell^2$ and the width of $K_{2\ell+1} \geq \ell(\ell + 1)$. On the other hand, equality holds for the piecewise linear embedding of K_m into $\mathbb{R}$ with vertices in general position and straight edges E_{ij}, which shows that $\mathrm{width}(\{K_{2\ell}\}) = \ell^2$ and $\mathrm{width}(\{K_{2\ell+1}\}) = \ell(\ell + 1)$. ■

Proposition 8. *For all $g : K_m \to \mathbb{R}$,* $\mathrm{NTC}(g) \geq \pi\,\ell^2$ *if $m = 2\ell$ is even; and* $\mathrm{NTC}(g) \geq \pi\,\ell(\ell + 1)$ *if $m = 2\ell + 1$ is odd. Equality holds for an embedding of K_m into $\mathbb{R}$ with vertices in general position and monotone on each edge; therefore* $\mathrm{NTC}(\{K_{2\ell}\}) = \pi\,\ell^2$*, and* $\mathrm{NTC}(\{K_{2\ell+1}\}) = \pi\,\ell(\ell + 1)$.

Proof. The lower bound on $\mathrm{NTC}(\{K_m\})$ follows from Theorem 6 and Lemma 5.

Now suppose $g : K_m \to \mathbb{R}$ is monotone on each edge, and number the vertices of K_m so that for all i, $g(v_i) < g(v_{i+1})$. Then as in the proof of Lemma 5, $\#(e,s) = k(m - k)$ for $g(v_k) < s < g(v_{k+1})$. These cardinalities are increasing for $0 \leq k \leq \ell$ and decreasing for $\ell + 1 < k < m$. Thus, if $g(v_\ell) < s_0 < g(v_{\ell+1})$, then by Theorem 6, $\mathrm{NTC}([\Gamma]) = \#(e,s_0)\,\pi = \ell(m - \ell)\,\pi$, as claimed. ■

Let $K_{m,n}$ be the complete bipartite graph with $m + n$ vertices divided into two sets: v_i, $1 \leq i \leq m$ and w_j, $1 \leq j \leq n$, having one edge E_{ij} joining v_i to w_j, for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proposition 9. $\mathrm{NTC}(\{K_{m,n}\}) = \lceil \frac{mn}{2} \rceil\,\pi$.

Proof. $K_{m,n}$ has vertices $v_1, \ldots, v_m$ of degree $d(v_i) = n$ and vertices $w_1, \ldots, w_n$ of degree $d(w_j) = m$. Consider a mapping $g : K_{m,n} \to \mathbb{R}$ in general position, so that the $m+n$ vertices of $K_{m,n}$ have distinct images. We wish to show $\mu(e) = \mu_g(e) \geq \frac{mn}{4}$, if m or n is even, or $\frac{mn+1}{4}$, if both m and n are odd.

For this purpose, according to Proposition 1, we may first reduce $\mu(e)$ or leave it unchanged by replacing g with a mapping (also called g) which is monotone on each edge E_{ij} of $K_{m,n}$. The values of $\mathrm{nlm}(w_j)$ and of $\mathrm{nlm}(v_i)$ are now determined by the order of the vertex images $g(v_1), \ldots, g(v_m), g(w_1), \ldots, g(w_n)$. Since $K_{m,n}$ is symmetric under permutations of $\{v_1, \ldots, v_m\}$ and permutations of $\{w_1, \ldots, w_n\}$, we shall assume that $g(v_i) < g(v_{i+1})$, $i = 1, \ldots, m-1$ and $g(w_j) < g(w_{j+1})$, $j = 1, \ldots, n-1$. For $i = 1, \ldots, m$ we write k_i for the largest index j such that $g(w_j) < g(v_i)$. Then $0 \le k_1 \le \cdots \le k_m \le n$, and these integers determine $\mu(e)$. According to Lemma 1, $\mathrm{nlm}(v_i) = k_i - \frac{n}{2}, i = 1, \ldots, m$. For $j \le k_1$ and for $j \ge k_m + 1$, we have $\mathrm{nlm}(w_j) = \pm\frac{m}{2}$; for $k_1 < j \le k_2$ and for $k_{m-1} < j \le k_m$, we find $\mathrm{nlm}(w_j) = \pm\left(\frac{m}{2} - 1\right)$; and so on until we find $\mathrm{nlm}(w_j) = 0$ on the middle interval $k_p < j \le k_{p+1}$, if $m = 2p$ is even; or, if $m = 2p + 1$ is odd, $\mathrm{nlm}(w_j) = -\frac{1}{2}$ for $k_p < j \le k_{p+1}$ and $\mathrm{nlm}(w_j) = +\frac{1}{2}$ for the other middle interval $k_{p+1} < j \le k_{p+2}$. Thus according to Lemma 1 and Corollary 5, if $m = 2p$ is **even**,

$$
\begin{aligned}
2\mu(e) &= \sum_{i=1}^{m} |\mathrm{nlm}(v_i)| + \sum_{j=1}^{n} |\mathrm{nlm}(w_j)| = \sum_{i=1}^{m} |k_i - \frac{n}{2}| + (k_1 + n - k_m)\frac{m}{2} \\
&+ (k_2 - k_1 + k_m - k_{m-1})\Big[\frac{m}{2} - 1\Big] + \ldots \\
&+ (k_p - k_{p-1} + k_{p+2} - k_{p+1})\Big[\frac{m}{2} - (p-1)\Big] + (k_{p+1} - k_p)\Big[0\Big] \qquad (11) \\
&= \sum_{i=1}^{m} \Big|k_i - \frac{n}{2}\Big| + \frac{mn}{2} + \sum_{i=1}^{p} k_i - \sum_{i=p+1}^{m} k_i \\
&= \frac{mn}{2} + \sum_{i=1}^{p} \Big[|k_i - \frac{n}{2}| + (k_i - \frac{n}{2})\Big] + \sum_{i=p+1}^{m} \Big[|k_i - \frac{n}{2}| - (k_i - \frac{n}{2})\Big].
\end{aligned}
$$

Note that formula (11) assumes its minimum value $2\mu(e) = \frac{mn}{2}$ when $k_1 \le \cdots \le k_p \le \frac{n}{2} \le k_{p+1} \le \cdots \le k_m$.

If $m = 2p + 1$ is **odd**, then

$$
\begin{aligned}
2\mu(e) &= \sum_{i=1}^{m} |k_i - \frac{n}{2}| + (k_1 + n - k_m)\frac{m}{2} + (k_2 - k_1 + k_m - k_{m-1})\Big[\frac{m}{2} - 1\Big] + \ldots \\
&+ (k_{p+3} - k_{p+2})\Big[\frac{m}{2} - (p-1)\Big] + (k_{p+2} - k_p)\Big[\frac{1}{2}\Big] = \\
&= \sum_{i=1}^{m} |k_i - \frac{n}{2}| + \frac{mn}{2} + \sum_{i=1}^{p} k_i - \sum_{i=p+2}^{m} k_i \qquad (12) \\
&= \frac{mn}{2} + \sum_{i=1}^{p} \Big[|k_i - \frac{n}{2}| + (k_i - \frac{n}{2})\Big] + \sum_{i=p+2}^{m} \Big[|k_i - \frac{n}{2}| - (k_i - \frac{n}{2})\Big] + |k_{p+1} - \frac{n}{2}|.
\end{aligned}
$$

Observe that formula (12) has the minimum value $2\mu(e) = \frac{mn}{2}$ when n is even and $k_1 \le \cdots \le k_p \le \frac{n}{2} = k_{p+1} \le \cdots \le k_m$. If n as well as m is odd, then the last term $|k_{p+1} - \frac{n}{2}| \geq \frac{1}{2}$, and the minimum value of $2\mu(e)$ is $\frac{mn+1}{2}$, attained iff $k_1 \leq \cdots \leq k_p \leq \frac{n}{2} \leq k_{p+2} \leq \cdots \leq k_m$.

This shows that for either parity of m or of n, $\mu(e) \geq \frac{mn}{4}$. If n and m are both odd, we have the stronger inequality $\mu(e) \geq \frac{mn+1}{4}$. We may summarize these conclusions as $2\mu(e) \geq \lceil\frac{mn}{2}\rceil$, and therefore as in the proof of Corollary 7, $\mathrm{NTC}(\{K_{m,n}\}) \geq \lceil\frac{mn}{2}\rceil\pi$, as we wished to show.

By abuse of notation, write the formula (11) or (12) as $\mu(k_1, \ldots, k_m)$.

To show the inequality in the opposite direction, we need to find a mapping $f : K_{m,n} \to \mathbb{R}$ with $\mathrm{NTC}(f) = \frac{mn\pi}{2}$ (m or n even) or $\mathrm{NTC}(f) = \frac{(mn+1)\pi}{2}$ (m and n odd). The above computation suggests choosing f with $f(v_1), \ldots, f(v_m)$ together in the middle of the images of the w_j. Write $n = 2\ell$ if n is even, or $n = 2\ell + 1$ if n is odd. Choose values $f(w_1) < \cdots < f(w_\ell) < f(v_1) < \cdots < f(v_m) < f(w_{\ell+1}) < \cdots < f(w_n)$, and extend f monotonically to each of the mn edges E_{ij}. From formulas (11) and (12), we have $\mu_f(e) = \mu(\ell, \ldots, \ell) = \frac{mn}{4}$, if m or n is even; or $\mu_f(e) = \mu(\ell, \ldots, \ell) = \frac{mn+1}{4}$, if m and n are odd. ■

Recall that θ_m is the graph with two vertices q^+, q_- and m edges.

Corollary 13. $\mathrm{NTC}(\{\theta_m\}) = m\pi$.

Proof. θ_m is homeomorphic to the complete bipartite graph $K_{m,2}$, and by the proof of Proposition 9, we find $\mu(e) \geq \frac{m}{2}$ for a.a. $e \in S^2$, and hence $\mathrm{NTC}(\{K_{m,2}\}) = m\pi$. ■

8. FÁRY-MILNOR TYPE ISOTOPY CLASSIFICATION

Recall the Fáry-Milnor theorem, which states that if the total curvature of a Jordan curve Γ in $\mathbb{R}^3$ is less than or equal to 4π, then Γ is unknotted. As we have demonstrated above, there are a collection of graphs whose values of the minimum total net curvatures are known. It is natural to hope when the net total curvature is small, in the sense of being in a specific interval to the right of the minimal value, that the isotopy type of the graph is restricted, as is the case for knots: $\Gamma = S^1$. The following proposition and corollaries, however, tell us that results of the Fáry-Milnor type **cannot** be expected to hold for more general graphs.

Proposition 10. *If Γ is a graph in $\mathbb{R}^3$ and if $C \subset \Gamma$ is a cycle, such that for some $e \in S^2$, $p_e \circ C$ has at least two local maximum points, then for each positive integer q, there is a nonisotopic embedding $\widetilde{\Gamma}_q$ of Γ in which C is replaced by a knot not isotopic to C, with* $\mathrm{NTC}(\widetilde{\Gamma}_q)$ *as close as desired to* $\mathrm{NTC}(p_e \circ \Gamma)$.

Proof. It follows from Corollary 7 that the one-dimensional graph $p_e \circ \Gamma$ may be replaced by an embedding $\widehat{\Gamma}$ into a small neighborhood of the line $\mathbb{R}e$ in $\mathbb{R}^3$, with arbitrarily small change in its net total curvature. Since $p_e \circ C$ has at least two local maximum points, there is an interval of $\mathbb{R}$ over which $p_e \circ C$ contains an interval which is the image of four oriented intervals J_1, J_2, J_3, J_4 appearing in that cyclic order around the oriented cycle C. Consider a plane presentation of Γ by orthogonal projection into a generic plane containing the line $\mathbb{R}e$. Choose an integer $q \in \mathbb{Z}$, $|q| \geq 3$. We modify $\widehat{\Gamma}$ by wrapping its interval J_1 q times around J_3 and returning, passing over any other edges of Γ, including J_2 and J_4, which it encounters along the way. The new graph in $\mathbb{R}^3$ is called $\widetilde{\Gamma}_q$. Then, if C was the unknot, the cycle $\widetilde{C}_q$ which has replaced it is a $(2, q)$-torus knot (see [L]). In any case, $\widetilde{C}_q$ is not isotopic to C, and therefore $\widetilde{\Gamma}_q$ is not isotopic to Γ.

As in the proof of Theorem 6, let $g_\delta : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by cylindrical shrinking, so that g_1 is the identity and $g_0 = p_e$. Then $p_e \circ \widetilde{\Gamma}_q = g_0(\widetilde{\Gamma}_q)$, and for $\delta > 0$, $g_\delta(\widetilde{\Gamma}_q)$ is isotopic to $\widetilde{\Gamma}_q$. But $\mathrm{NTC}(g_\delta) \to \mathrm{NTC}(g_0)$ as $\delta \to 0$. ■

Corollary 14. *If $e = e_0 \in S^{n-1}$ minimizes $\mathrm{NTC}(p_e \circ \Gamma)$, and there is a cycle $C \subset \Gamma$ so that $p_{e_0} \circ C$ has two (or more) local maximum points, then there is a sequence of nonisotopic embeddings $\widetilde{\Gamma}_q$ of Γ with $\mathrm{NTC}(\widetilde{\Gamma}_q)$ less than, or as close as desired, to $\mathrm{NTC}(\Gamma)$, in which C is replaced by a $(2, q)$-torus knot.*

Corollary 15. *If Γ is an embedding of K_m into $\mathbb{R}^3$, linear on each topological edge of K_m, $m \geq 4$, then there is a sequence of nonisotopic embeddings $\widetilde{\Gamma}_q$ of Γ with $\mathrm{NTC}([\widetilde{\Gamma}_q])$ as close as desired to $\mathrm{NTC}([\Gamma])$, in which an unknotted cycle C of Γ is replaced by a $(2, q)$-torus knot.*

Proof. According to Corollary 14, we only need to construct an isotopy of K_m with the minimum value of NTC, such that there is a cycle C so that $p_e \circ C$ has two local maximum points, where $\mu(e)$ is a minimum among $e \in S^2$.

Choose $g : K_m \to \mathbb{R}$ which is monotone on each edge of K_m, and has distinct values at vertices. Then according to Proposition 8, we have $\mathrm{NTC}(g) = \mathrm{NTC}(\{K_m\})$. Number the vertices $v_1, \ldots, v_m$ so that $g(v_1) < g(v_2) < \cdots < g(v_m)$. Write E_{ji} for the edge E_{ij} with the reverse orientation, $i \neq j$. Then the cycle C formed in sequence from E_{13}, E_{32}, E_{24} and E_{41} has local maximum points at v_3 and v_4, and covers the interval $\big(g(v_2), g(v_3)\big) \subset \mathbb{R}$ four times. Since C is formed out of four straight edges, it is unknotted. The procedure of Corollary 14 replaces C with a $(2, q)$-torus knot, with an arbitrarily small increase in NTC. ■

Note that Corollary 14 gives a set of conditions for those graph types where a Fáry-Milnor type isotopy classification might hold. In particular, we consider one of the simpler homeomorphism types of graphs, the **theta graph**, $\theta = \theta_3 = K_{3,2}$ (cf. description following Definition 11). The **standard theta graph** is the isotopy class in $\mathbb{R}^3$ of a plane circle plus a diameter. We have seen in Corollary 13 that the minimum of net total curvature for a theta graph is 3π. On the other hand note that in the range $3\pi \leq \mathrm{NTC}(\Gamma) < 4\pi$, for e in a set of positive measure of S^2, $p_e(\Gamma)$ cannot have two local maximum points. In Theorem 7 below, we shall show that a theta graph Γ with $\mathrm{NTC}(\Gamma) < 4\pi$ is isotopically standard.

We may observe that there are nonstandard theta graphs in $\mathbb{R}^3$. For example, the union of two edges might form a knot. Moreover, as S. Kinoshita has shown, there are θ-graphs in $\mathbb{R}^3$, not isotopic to a planar graph, such that each of the three cycles formed by deleting one edge is unknotted [Ki].

We begin with a well-known property of knots, whose proof we give for the sake of completeness.

Lemma 6. *Let $C \subset \mathbb{R}^3$ be homeomorphic to S^1, and* **not** *a convex planar curve. Then there is a nonempty open set of planes $P \subset \mathbb{R}^3$ which each meet C in at least four points.*

Proof. For $e \in S^2$ and $t \in \mathbb{R}$ write the plane $P^e_t = \{x \in \mathbb{R}^3 : \langle e, x\rangle = t\}$.

If C is not planar, then there exist four non-coplanar points p_1, p_2, p_3, p_4, numbered in order around C. Note that no three of the points can be collinear. Let an oriented plane P_0 be chosen to contain p_1 and p_3 and rotated until both p_2 and p_4 are above P_0 strictly. Write e_1 for the unit normal vector to P_0 on the side where p_2 and p_3 lie, so that $P_0 = P^{e_1}_{t_0=0}$. .Then the set $P_t \cap C$ contains at least four points, for $t_0 = 0 < t < \delta_1$, with some $\delta_1 > 0$, since each plane $P_t = P^{e_1}_t$ meets each of the four open arcs between the points p_1, p_2, p_3, p_4. This conclusion remains true, for some $0 < \delta < \delta_1$, when the normal vector e_1 to P_0 is replaced by any nearby $e \in S^2$, and t is replaced by any $0 < t < \delta$.

If C is planar but nonconvex, then there exists a plane $P_0 = P^{e_1}_0$, transverse to the plane containing C, which supports C and touches C at two distinct points, but does not include the arc of C between these two points. Consider disjoint open arcs of C on either side of these two points and including points not in P_0. Then for $0 < t < \delta \ll 1$, the set $P_t \cap C$ contains at least four points, since the planes $P_t = P^{e_1}_t$ meet each of the four disjoint arcs. Here once again e_1 may be replaced by any nearby unit vector e, and the plane P^e_t will meet C in at least four points, for t in a nonempty open interval $t_1 < t < t_1 + \delta$. ■

Using the notion of net total curvature, we may extend the theorems of Fenchel [Fen] as well as of Fáry-Milnor ([Fa],[M]), for curves homeomorpic to S^1, to graphs homeomorphic to the theta graph. An analogous result is given by Taniyama in [T], who showed that the minimum of TC for polygonal θ-graphs is 4π, and that any θ-graph Γ with $\mathrm{TC}(\Gamma) < 5\pi$ is isotopically standard,

Theorem 7. *Suppose $f : \theta \to \mathbb{R}^3$ is a continuous embedding, $\Gamma = f(\theta)$. Then* $\mathrm{NTC}(\Gamma) \geq 3\pi$. *If* $\mathrm{NTC}(\Gamma) < 4\pi$, *then Γ is isotopic in $\mathbb{R}^3$ to the planar theta graph. Moreover,* $\mathrm{NTC}(\Gamma) = 3\pi$ *iff the graph is a planar convex curve plus a straight chord.*

Proof. We consider first the case when $f : \theta \to \mathbb{R}^3$ is piecewise C^2.

(1) We have shown the **lower bound** 3π for $\mathrm{NTC}(f)$, where $f : \theta \to \mathbb{R}^n$ is any piecewise C^2 mapping, since $\theta = \theta_3$ is one case of Corollary 13, with $m = 3$.

(2) We show next that if there is a cycle C in a graph Γ (a subgraph homeomorphic to S^1) which satisfies the conclusion of Lemma 6, then $\mu(e) \geq 2$ for e in a nonempty open set of S^2. Namely, for $t_0 < t < t_0 + \delta$, a family of planes P^e_t meets C, and therefore meets Γ, in at least four points. This is equivalent to saying that the cardinality $\#(e, t) \geq 4$. This implies, by Corollary 4, that $\sum\{\mathrm{nlm}(e, q) : p_e(q) > t_0\} \geq 2$. Thus, since $\mathrm{nlm}^+(e, q) \geq \mathrm{nlm}(e, q)$, using Definition 6, we have $\mu(e) \geq 2$.

Now consider the **equality** case of a theta graph Γ with $\mathrm{NTC}(\Gamma) = 3\pi$. As we have seen in the proof of Proposition 9 with $m = 3$ and $n = 2$, the multiplicity $\mu(e) \geq \frac{3}{2} = \frac{mn}{4}$ for a.a. $e \in S^2$, while the integral of $\mu(e)$ over S^2 equals $2\,\mathrm{NTC}(\Gamma) = 6\pi$ by Theorem 1, implying $\mu(e) = 3/2$ a.e. on S^2. Thus, the conclusion of Lemma 6 is impossible for any cycle C in Γ. By Lemma 6, all cycles C of Γ must be planar and convex.

Now Γ consists of three arcs a_1, a_2 and a_3, with common endpoints q^+ and q^-. As we have just shown, the three Jordan curves $\Gamma_1 := a_2 \cup a_3$, $\Gamma_2 := a_3 \cup a_1$ and $\Gamma_3 := a_1 \cup a_2$ are each planar and convex. It follows that Γ_1, Γ_2 and Γ_3 lie in a common plane. In terms of the topology of this plane, one of the three arcs a_1, a_2 and a_3 lies in the middle between the other two. But the middle arc, say a_2, must be a line segment, as it needs to be a shared piece of two curves Γ_1 and Γ_3 bounding disjoint convex open sets in the plane. The conclusion is that Γ is a planar, convex Jordan curve Γ_2, plus a straight chord a_2, whenever $\mathrm{NTC}(\Gamma) = 3\pi$.

(3) We next turn our attention to the **upper bound** of NTC, to imply that a θ-graph is isotopically standard: we shall assume that $g : \theta \to \mathbb{R}^3$ is an embedding in general position with $\mathrm{NTC}(g) < 4\pi$, and write $\Gamma = g(\theta)$. By Theorem 1, since S^2 has area 4π, the average of $\mu(e)$ over S^2 is less than 2, and it follows that there exists a set of positive measure of $e_0 \in S^2$ with $\mu(e_0) < 2$. Since $\mu(e_0)$ is a half-integer, and since $\mu(e) \geq 3/2$, as we have shown in part **(1)** of this proof, we have $\mu(e_0) = 3/2$ exactly.

From Corollary 10 applied to $p_{e_0} \circ g : \theta \to \mathbb{R}$, we find $\mu_g(e_0) = \frac{1}{2}(\Lambda + V) + \frac{k}{4}$, where Λ is the number of local maximum points, V is the number of local minimum points and $k = 2$ is the number of vertices, both of degree 3. Thus, $\frac{3}{2} = \frac{1}{2}(\Lambda + V) + \frac{1}{2}$, so that $\Lambda + V = 2$. This implies that the local maximum/minimum points are unique, and must be the unique global maximum/minimum points $p_{\max}$ and $p_{\min}$ (which may be one of the two vertices $q^\pm$). Then $p_{e_0} \circ g$ is monotone along edges except at the points $p_{\max}$, $p_{\min}$ and $q^\pm$.

Introduce Euclidean coordinates (x, y, z) for $\mathbb{R}^3$ so that e_0 is in the increasing z-direction. Write $t_{\max} = p_{e_0} \circ g(p_{\max}) = \langle e_0, p_{\max} \rangle$ and $t_{\min} = \langle e_0, p_{\min} \rangle$ for the maximum and minimum values of z along $g(\theta)$. Write $t^\pm$ for the value of z at $g(q^\pm)$, where we may assume $t_{\min} \leq t^- < t^+ \leq t_{\max}$.

We construct a "model" standard θ-curve $\widehat{\Gamma}$ in the (x, z)-plane, as follows. $\widehat{\Gamma}$ will consist of a circle C plus the straight chord of C, joining $\widehat{q^-}$ to $\widehat{q^+}$ (points to be chosen). Choose C so that the maximum and minimum values of z on C equal $t_{\max}$ and $t_{\min}$. Write $\widehat{p}_{\max}$ resp. $\widehat{p}_{\min}$ for the maximum and minimum points of z along C. Choose $\widehat{q^+}$ as a point on C where $z = t^+$. There may be two nonequivalent choices for $\widehat{q^-}$ as a point on C where $z = t^-$: we choose so that $\widehat{p}_{\max}$ and $\widehat{p}_{\min}$ are in the same or different topological edge of $\widehat{\Gamma}$, where $p_{\max}$ and $p_{\min}$ are in the same or different topological edge, resp., of Γ. Note that there is a homeomorphism from $g(\theta)$ to $\widehat{\Gamma}$ which preserves z.

We now proceed to extend this homeomorphism to an isotopy. For $t \in \mathbb{R}$, write P_t for the plane $\{z = t\}$. As in the proof of Proposition 3, there is a continuous 1-parameter family of homeomorphisms $\Phi_t : P_t \to P_t$ such that $\Phi_t(\Gamma \cap P_t) = \widehat{\Gamma} \cap P_t$;

Φ_t is the identity outside a compact subset of P_t; and Φ_t is isotopic to the identity of P_t, uniformly with respect to t. Defining $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi(x, y, z) := \Phi_z(x, y)$, we have an isotopy of Γ with the model graph $\widehat{\Gamma}$.

(4) Finally, consider an embedding $g : \theta \to \mathbb{R}^3$ which is only **continuous**, and write $\Gamma = g(\theta)$.

It follows from Theorem 3 that for any θ-graph Γ of finite net total curvature, there is a Γ-approximating polygonal θ-graph P isotopic to Γ, with $\mathrm{NTC}(P) \leq \mathrm{NTC}(\Gamma)$ and as close as desired to $\mathrm{NTC}(\Gamma)$.

If a θ-graph Γ would have $\mathrm{NTC}(\Gamma) < 3\pi$, then the Γ-approximating polygonal graph P would also have $\mathrm{NTC}(P) < 3\pi$, in contradiction to what we have shown for piecewise C^2 theta graphs in part **(1)** above. This shows that $\mathrm{NTC}(\Gamma) \geq 3\pi$.

If equality $\mathrm{NTC}(\Gamma) = 3\pi$ holds, then $\mathrm{NTC}(P) \leq \mathrm{NTC}(\Gamma) = 3\pi$, so that by the equality case part **(2)** above, $\mathrm{NTC}(P)$ must equal 3π, and P must be a convex planar curve plus a chord. But this holds for *all* Γ-approximating polygonal graphs P, implying that Γ itself must be a convex planar curve plus a chord.

Finally, If $\mathrm{NTC}(\Gamma) < 4\pi$, then $\mathrm{NTC}(P) < 4\pi$, implying by part **(3)** above that P is isotopic to the standard θ-graph. But Γ is isotopic to P, and hence is isotopically standard. ∎

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